Conformal blocks, related to conformally invariant Ramond states of a free scalar field

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(Submitted 9 July 1986)
Zh. Eksp. Teor. Fiz. **92**, 34–45 (January 1987)

The space of Ramond states of a two-dimensional scalar field of zero mass contains a set of conformally invariant Ramond fields with dimensions $(2k+1)^2/16$. Four-point conformal blocks of these invariant operators are discussed. It is shown that these blocks can be evaluated by starting with the block of primary fields of dimension 1/16 (Ramond vacuum).

1. INTRODUCTION

The conformal block plays a major part in two-dimensional conformally invariant quantum field theory. The evaluation of the conformal block was originally considered in Ref. 1, which began the active development of this theory that has taken place in recent years. Conformal field theory has led to striking advances in the description of second-order phase transitions in two-dimensional systems, and can serve as a convenient tool in the quantum theory of relativistic strings.[1–6]

In two-dimensional conformal theory, the four-point conformal block plays the part of the partial amplitude, and serves as a "building block" when one attempts to construct the four-point Green's function in the confromal bootstrap method.[1,7] In the intermediate channel of the four-point function, it sums the contributions of all states belonging to the irreducible representation of the Virasoro algebra that corresponds to the infinite-dimensional group of space-time symmetries of the two-dimensional conformal field theory. At present, the explicit form of the general conformal block (corresponding to arbitrary parameters of the conformal theory, namely, the dimensions of the invariant field, and the central charge of the Virasoro algebra) is still unknown. Algorithms for its evaluation in the form of power series are considered in Ref. 8. Examples of explicit implementation of the conformal block program can be found in Refs. 3, 4, and 9.

An explicit expression for the conformal block in conformal theory with central charge $c=1$ and conformal field dimension $\delta_0=1/16$ was obtained in Ref. 9, where it is shown that this block is related to the correlation functions of fields that correspond to the Ramond states of a free scalar field. This space implements the higher order representation of the Ramond algebra of a Heisenberg free scalar field. The principal vector of the representation has the dimension 1/16 with respect to the corresponding conformal algebra (see Ref. 9 and Section 2 of the present paper). This space is reducible under the conformal Virasoro algebra and contains an infinite set of conformally invariant states with dimensions $\delta_k=(2k+1)^2/16$, $k=0,1,2,\ldots$.

In the present paper, we shall consider the explicit evaluation of conformal blocks containing conformal fields σ_k, $k=1,2,\ldots$, corresponding to these "higher invariant states."

In general, the conformal block as a function of the dimension of the intermediate channel Δ is found to have poles at points corresponding to the degeneracy of the Verma modulus of the conformal algebra on the corresponding invariant field of dimension Δ. It is interesting that these poles are absent in the case of the block found in Ref. 9, which corresponds to correlations between the lowest Ramond field σ_0, and the block is an entire function of Δ. The particular feature of the blocks of fields σ_k considered here is the finite number of poles in Δ.

The paper is arranged as follows. In Section 2, we examine the space of Ramond states of a free scalar field and investigate the structure of conformally invariant subspaces. We also derive a number of operator-algebra relations for the Ramond fields. These relations are used in Section 3 to obtain an exact four-point block of the four lowest fields σ_0. An algorithm is constructed for evaluating the blocks containing the fields σ_k with $k>0$, and some of these blocks are evaluated. The structure of the singularities of these blocks in the intermediate dimension Δ is discussed. In Section 4, we consider another construction that can be used to evaluate these blocks as the correlation functions of free fields on a type 1 Riemann surface (torus) suitably coupled to the four-point correlation function of the Ramond fields.

2. RAMOND STATES OF A FREE SCALAR FIELD

We shall consider the operator algebra of a free scalar field in two-dimensional Euclidean space with complex coordinates

$$z=x^1+ix^2,\quad \bar z=x^1-ix^2. \tag{2.1}$$

The holomorphic part $\Phi(z)$ of the free scalar field is normalized by the vacuum correlation function

$$\langle\Phi(z)\Phi(z')\rangle=-{}^1/{}_2\log(z-z') \tag{2.2}$$

and may be looked upon as part of the real scalar field

$$\varphi(z,\ z')=\Phi(z)+\bar\Phi(\bar z) \tag{2.3}$$

with the action

$$A(\varphi)=\frac{1}{4\pi}\int(\partial_\mu\varphi)^2\,d^2x. \tag{2.4}$$

The holomorphic current

$$I(z)=i\frac{\partial\Phi(z)}{\partial z} \tag{2.5}$$

is a conformal field of dimension 1 and forms an Abelian current algebra:

$$I(z)I(z')=\frac{1}{2(z-z')^2}+O(1). \tag{2.6}$$

 0038-5646/87/010019-06$04.00

The conformal energy-momentum tensor in this theory is quadratic in the field $I(z)$:

$$T(z)=:I^2(z):, \tag{2.7}$$

where the regularization of the singular product (2.7) of the two fields $I(z)$ at a given point must be performed so that $T(z)$ is the next term in the operator expansion (2.6), i.e., the following local relation must be satisfied:

$$I(z)I(z')=\frac{1}{2(z-z')^2}+T(z')+O(z-z'). \tag{2.8}$$

This energy-momentum tensor satisfies the local operator expansion

$$T(z)T(z')=\frac{1}{2(z-z')^4}+\frac{2T(z')}{(z-z')^2}+\frac{T'(z')}{z-z'}+O(1) \tag{2.9}$$

and, consequently, generates a conformal algebra with central charge $c=1$.

We shall examine the following two states of the free current (2.5).

(1) Single-valued states corresponding to the point $z=z_0$ of single-valued character for the current operator $I(z)$ in the complex plane, around which the current can be expanded into the Laurent series (we assume that $z_0=0$):

$$I(z)=\sum_n I_n z^{-n-1}. \tag{2.10}$$

According to (2.6), we have the following operator algebra for the I_n:

$$[I_n, I_m]=\frac{n}{2}\delta_{n+m}. \tag{2.11}$$

The operator

$$I_0=\oint_C I(z)\frac{dz}{2\pi i} \tag{2.12}$$

is conserved and will be called the charge of the state within the contour C. We note that the conformally-invariant single-valued state with charge p corresponds to the conformal field

$$V_p=:\exp[2ip\Phi(z)]:, \tag{2.13}$$

which has the conformal dimension p^2. Of particular interest among these fields are the currents

$$J_\pm=:\exp[\pm 2i\Phi(z)]:, \tag{2.14}$$

which have conformal dimension 1 and, together with the current $J_0=I(z)$, form the Kac-Moody $SL(2)$ algebra with central charge $k=1$ (Refs. 10 and 11):

$$J_\alpha(z)J_\beta(z')=\frac{q_{\alpha\beta}}{2(z-z')^2}+\frac{f_{\alpha\beta}{}^\gamma J_\gamma(z')}{z-z'}+O(1), \tag{2.15}$$

where $q_{\alpha\beta}$ and $f^\gamma_{\alpha\beta}$ are, respectively, the Killing form and the structural constants of the $SL(2)$ algebra.

(2) The Ramond states corresponding to a point of two-valued character for $I(z)$, such that the current changes sign when one circuit around it is completed. In this case,

$$I(z)=\sum_n I_{n-1/2}z^{-n-1/2} \tag{2.16}$$

and the corresponding algebra is

$$[I_{m-1/2}, I_{n+1/2}]=\frac{m-1/2}{2}\delta_{m+n}. \tag{2.17}$$

The lowest Ramond state σ_0 (Ramond vacuum) corresponds to the principal vector of the highest-order representation of this algebra

$$I_{1/2}\sigma_0=I_{3/2}\sigma_0=I_{5/2}\sigma_0=\ldots=0. \tag{2.18}$$

The Verma modulus formed by the action of the operators $I_{n-\frac{1}{2}}$ with $n\leqslant 0$ is a space of the Ramond states of the current $I(z)$. The energy-momentum tensor constructed in accordance with the recipe given by (2.8) can be written in the following form:

$$T=\sum_n L_n z^{-n-2}, \tag{2.19}$$

$$L_n=\sum_k I_{k-1/2}I_{n-k+1/2}, \quad n\neq 0,$$

$$L_0=1/16+2\sum_{k>0}I_{-k+1/2}I_{k-1/2}. \tag{2.20}$$

It is clear from these formulas that the vacuum state σ_0 is a conformal field of dimension 1/16.

The conformally invariant states in Ramond space are determined by the conditions

$$L_n\sigma_k=0 \quad \text{for} \quad n>0. \tag{2.21}$$

Thus, by solving these relations for the first few levels, we find that the states

$$\sigma_1=2I_{-1/2}\sigma_0, \quad \sigma_2=2/3(I_{-3/2}-4I_{-1/2}^3)\sigma_0,$$

$$\sigma_3=4/45(9I_{-5/2}I_{-1/2}-5I^2_{-3/2}-20I_{-3/2}I^3_{-1/2}+16I^6_{-1/2})\sigma_0 \tag{2.22}$$

are conformally invariant fields with dimensions $\delta_1=9/16$, $\delta_2=25/16$, and $\delta_3=49/16$, respectively. For higher levels, direct evaluation becomes more complicated. To determine the conformal field in the space of the Ramond states, consider the character of this space:

$$\chi_R(t)=\mathrm{Tr}\, t^{2L_0}=t^{1/8}\prod_{k=1}^{\infty}\frac{1}{(1-t^{2k-1})}. \tag{2.23}$$

Assuming that the Ramond space contains only the nondegenerate representations of the Virasoro algebra, we can now write down the trace over the space of representation with dimension δ_i:

$$\chi_c(t)=\mathrm{Tr}\, t^{2L_0-2\delta i}=\prod_{k=1}^{\infty}\frac{1}{1-t^{2k}}. \tag{2.24}$$

Hence,

$$\frac{\chi_R(t)}{\chi_c(t)}=t^{1/8}\prod_{k=1}^{\infty}\frac{1-t^{2k}}{1-t^{2k-1}}=\sum_{k=0}^{\infty}t^{(2k+1)^2/8}. \tag{2.25}$$

We thus see that there is an infinite series of conformal Ramond fields σ_k, $k=0,1,2,\ldots$, corresponding to dimensions $\delta_k=(2k+1)^2/16$.

It is interesting to consider the nature of this Ramond series from the standpoint of representations of the Kac-Moody algebra (2.15) in the Ramond sector. The expansion given by (2.16) can now be integrated:

$$\Phi(z)=\Phi_0+i\Sigma\frac{I_{n-1/2}z^{1/2-n}}{n-1/2}. \tag{2.26}$$

Since, in the Ramond sector,

$$\langle(\Phi(z)-\Phi_0)(\Phi(z')-\Phi_0)\rangle=-\frac{1}{2}\log\frac{z^{1/2}-(z')^{1/2}}{z^{1/2}+(z')^{1/2}}, \tag{2.27}$$

we arrive at the following representation of the operators (2.14):

$$J_{\pm}=\frac{\exp(\pm 2i\Phi_0)}{4z}:\exp[\pm 2i(\Phi(z)-\Phi_0)]:, \qquad (2.28)$$

where the usual ordering with respect to the Ramond vacuum σ_0 is implied (the creation operators $I_{k-\frac{1}{2}}$ with $k\leqslant 0$ are placed on the left). We can now introduce the real currents

$$J_1=\frac{1}{4z}:\sin 2(\Phi(z)-\Phi_0):,$$
$$J_2=\frac{1}{4z}:\cos 2(\Phi(z)-\Phi_0):, \qquad (2.29)$$

which satisfy the same local operator expansions (2.8) as the current $I(z)$ [this is clear from the global $SL(2)$ symmetry of the algebra (2.15)]. The field $J_1(z)$ is then a two-valued field for the Ramond states, whereas $J_2(z)$ has a single-valued character and satisfies the relation

$$J_2(z)\sigma_0(0)=\frac{1}{4z}\sigma_0(0)+O(1). \qquad (2.30)$$

This means that the Ramond vacuum σ_0 has a charge of 1/4 with respect to this current (in agreement with $\delta_0=1/16$). If we derive this current from another scalar field $\chi(z)$

$$J_2(z)=i\frac{\partial}{\partial z}\chi(z), \qquad (2.31)$$

the Ramond vacuum σ_0 assumes the form

$$\sigma_0=:\exp(i\chi(z)/2):. \qquad (2.32)$$

The field $\chi(z)$ is related to the original field $\Phi(z)$ by a nonlinear transformation. We now use this field to write the following obvious relations:

$$K_{\pm}(z)=I_0(z)\pm iJ_1(z)=:\exp[\pm 2i\chi(z)]:. \qquad (2.33)$$

The higher-order Ramond fields $\sigma_k(z)$ appear when these currents are combined with the lowest Ramond field. Thus,

$$K_-(z):\exp\left(\frac{i\chi(0)}{2}\right):=z^{-1/2}:\exp\left[-\frac{3i}{2}\chi(0)\right]:+\ldots,$$
$$K_+(z):\exp\left(\frac{i\chi(0)}{2}\right):=z^{1/2}:\exp\left[\frac{5i}{2}\chi(0)\right]:+\ldots. \qquad (2.34)$$

This gives us the representation

$$\sigma_{2n}=:\exp i(2n+{}^1/_2)\chi:,$$
$$\sigma_{2n+1}=:\exp i(-{}^3/_2-2n)\chi:. \qquad (2.35)$$

The currents

$$K_{\pm}(z)=I(z)\pm\frac{i}{4z}:\sin 2\Phi(z): \qquad (2.36)$$

are then the shift operators for the invariant Ramond states:

$$K_+(z)\sigma_{2n}(0)=z^{(4n+1)/2}\sigma_{2n+2}(0)+\ldots,$$
$$K_-(z)\sigma_{2n-1}(0)=z^{2n+3/2}\sigma_{2n+3}(0)+\ldots. \qquad (2.37)$$

These operators can be used in their explicit form (2.36) to evaluate the invariant Ramond states in the basis formed by the operators $I_{n-\frac{1}{2}}$ with $n\leqslant 0$.

3. CONFORMAL BLOCKS OF INVARIANT RAMOND FIELDS

In this Section, we consider four-point conformal blocks of the fields σ_k. We shall show that the relationships of the operator algebra enable us to evaluate these blocks in a consistent manner, beginning with the block of four lowestRamond fields σ_0. We shall use the notation

$$G_{k_1k_2k_3k_4}(x)=\langle\sigma_{k_1}(x_1)\sigma_{k_2}(x_2)\,|_\Delta\sigma_{k_3}(x_3)\sigma_{k_4}(x_4)\rangle, \qquad (3.1)$$

and will assume that, in the channel containing the operators $\sigma_{k1}(x_1)$ and $\sigma_{k2}(x_2)$, we have isolated the contribution of the conformal representation of an invariant operator of dimension Δ. We note that the product of Ramond operators, $\sigma_{k1}(x_1)\sigma_{k2}(x_2)$, is a single-valued state for the free-field current $I(z)$. The conformally invariant operator for the channel can therefore be taken in the form $\exp[+2ip\Phi(z)]$: with one of the signs, and $\Delta=p^2$. We know that, for general values of the charge $p\neq n/2$ (where n is an integer), the space of the irreducible representation of the current algebra (2.11) coincides with the space of the representation of the conformal algebra. This means that the conformal blocks (3.1) can be looked upon as current blocks that sum the contribution of the representation of the current algebra.

The isolation of the charge p in the channel of the four-point function (3.1) means that

$$\oint_C \Gamma_{k_1,k_2,k_3,k_4}(z,x)\frac{dz}{2\pi i}=pG_{k_1k_2k_3k_4}(x), \qquad (3.2)$$

where

$$\Gamma_{k_1k_2k_3k_4}(z,x)=\langle I(z)\sigma_{k_1}(x_1)\sigma_{k_2}(x_2)\sigma_{k_3}(x_3)\sigma_{k_4}(x_4)\rangle. \qquad (3.3)$$

From now on, it will be convenient to use projective invariance and place three of the four points x_i, $i=1,2,3,4$ in fixed positions: $x_1=0, x_2=x, x_3=1$, and $x_4=\infty$; the four-point blocks (3.1) are analytic functions of the single complex variable x. Moreover, we shall normalize the blocks (3.1) by the condition:

$$G_{k_1k_2k_3k_4}(x)=x^{\Delta-[(2k_1+1)^2+(2k_2+1)^2]/16}(1+O(x)). \qquad (3.4)$$

The following explicit expressions for the block of lowest fields was obtained in Ref. 9:

$$G_{0000}(x)=(16q)^\Delta[x(1-x)]^{-1/8}\theta_3^{-1}(q), \qquad (3.5)$$

where

$$\theta_3(q)=\sum_{n=-\infty}^{\infty}q^{n^2}, \qquad (3.6)$$

$$q=\exp(i\pi\tau),\qquad \tau=iK'(x)/K(x), \qquad (3.7)$$

$$K(x)=\frac{1}{2}\int_0^1\frac{dt}{[t(1-t)(1-xt)]^{1/2}},\qquad K'(x)=K(1-x) \qquad (3.8)$$

is the complete elliptic integral of the second kind.

To evaluate the blocks with the participation of the next invariant field σ_1, we use the operator-algebra relations that follow from (2.16)–(2.20):

$$I(z)\sigma_0(0)=\frac{1}{2\sqrt{z}}\sigma_1(0)+O(z^{1/2}),$$
$$I(z)\sigma_1(0)=\frac{1}{2z^{3/2}}\sigma_0(0)+\frac{2}{z^{1/2}}\sigma_0'(0)+O(z^{1/2}). \qquad (3.9)$$

This structure of the expansions establishes the character of the asymptotic behavior of the functions $\Gamma_{0000}(z,x)$ as z tends to the Ramond points $0, x, 1$, and ∞. Accordingly, we can write

$$\Gamma_{0000}(z,x)=\frac{A_{0000}(x)}{[z(z-x)(z-1)]^{1/2}}. \qquad (3.10)$$

Apart from a constant factor that can be determined from the normalization condition (3.4), the quantity A_{0000} can be related to blocks containing the field σ_1 if we use (3.9):

$$A_{0000}(x) \sim x^{1/2} G_{1000}(x) \sim [x(1-x)]^{1/2} G_{0100}(x). \tag{3.11}$$

At the same time, the block condition (3.2) leads to

$$A_{0000}(x) \sim K(x) G_{0000}(x). \tag{3.12}$$

Thus, using the normalization condition and the well-known relation

$$K(x) = \frac{\pi}{2} \theta_3^2(q), \tag{3.13}$$

we obtain

$$G_{0100}(x) = (16q)^{\Delta} [x(1-x)]^{-5/8} \theta_3^{-3}(q). \tag{3.14}$$

The remaining blocks containing the single operator σ_1 are related to (3.14) by the cross-symmetry relations. By analogy with (3.10),

$$\Gamma_{0100}(x) = [z(z-x)(z-1)]^{1/2} \left(\frac{A_{0100}(x)}{z-x} + B_{0100}(x) \right). \tag{3.15}$$

The operator relations given by (3.9) determine the functions A_{0100} and B_{0100} as $z \to x$:

$$\Gamma_{0100}(z,x) \sim \left[\frac{x(1-x)}{z(z-x)(z-1)} \right]^{1/2} \left[\frac{G_{0000}(x)}{z-x} + 4(x(1-x))^{-1/8} \frac{\partial}{\partial x} \left((x(1-x))^{1/8} G_{0000}(x) \right) \right]. \tag{3.16}$$

As $z \to 0$ and $z \to 1$, these relations yield (normalization has been carried out)

$$G_{0110}(x) = \frac{(16q)^{\Delta}}{x^{5/8}(1-x)^{9/8}\theta_3^5} \left(1 + \frac{x}{4\Delta} \left(\theta_3^4 - (1-x) \frac{d}{dx} \theta_3^4 \right) \right), \tag{3.17}$$

$$G_{1100}(x) = \frac{(16q)^{\Delta}}{x^{9/8}(1-x)^{5/8}\theta_3^5} \times \left(1 + \frac{1}{4\Delta - 1} \left(1 - (1-x)\theta_3^4 - x(1-x) \frac{d}{dx} \theta_3^4 \right) \right). \tag{3.18}$$

As functions of Δ, these blocks contain one pole each at $\Delta = 0$ and $\Delta = 1/4$, respectively.

We note that the block condition (3.2) for $\Gamma_{0100}(z,x)$ imposes additional restrictions on the functions A_{0100} and B_{0100} which, together with the relations used here, determine unambiguously the form of the original block of fields σ_0 (3.5) (see Ref. 9).

It is shown in Ref. 8 (see also Section 4 below) that the general conformal block of four invariant fields of dimensions δ_1, δ_2, δ_3, and δ_4 with intermediate dimension Δ (in the channel containing operators of dimension δ_1 and δ_2) can be written in the following form in conformal theory with central charge c:

$$G_{\delta_1,\delta_2,\delta_3,\delta_4}(x) = (16q)^{\Delta-(c-1)/24} \theta_3^{(c-1)/2-4(\delta_1+\delta_2+\delta_3+\delta_4)} \cdot x^{(c-1)/24-\delta_1-\delta_2} (1-x)^{(c-1)/24-\delta_2-\delta_3} H_{\delta_1,\delta_2,\delta_3,\delta_4}(\Delta, q), \tag{3.19}$$

where $H\delta_1,\delta_2,\delta_3,\delta_4(\Delta,q) = 1 + 0(1/\Delta)$ for $\Delta \to \infty$. Accordingly, we now introduce functions $H_{k_1k_2k_3k_4}(q)$ that are related to the blocks (3.1) by

$$G_{k_1k_2k_3k_4}(x) = (16q)^{\Delta} x^{-\delta_1-\delta_2} (1-x)^{-\delta_2-\delta_3} \theta_3^{-4(\delta_1+\delta_2+\delta_3+\delta_4)} H_{k_1k_2k_3k_4}(q). \tag{3.20}$$

In this case, these quantities are meromorphic functions of Δ with a finite number of poles (they are rational functions), and all their poles can lie at the points $\Delta = n^2/4$, $n = 0,1,2,\ldots$, in accordance with the degeneracy structure of the representations of the conformal algebra for $c = 1$. For the block (3.14), we have

$$H_{0100}(q) = 1, \tag{3.21}$$

and the corresponding functions for block (3.17) and (3.18) can be represented by the ratios of two theta-series:

$$H_{0110}(q) = \frac{1}{\Delta} \left(\sum_{n=-\infty}^{\infty} (-)^n (\Delta - n^2) q^{n^2} \right) \Big/ \left(\sum_{n=-\infty}^{\infty} (-)^n q^{n^2} \right), \tag{3.22}$$

$$H_{1100}(q) = \left(\sum_{n=-\infty}^{\infty} \frac{\Delta - (n+1/2)^2}{\Delta - 1/4} q^{(n+1/2)^2} \right) \Big/ \left(\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \right). \tag{3.23}$$

In precisely the same way, if we consider the operator algebra for the function $\Gamma_{1100}(z,x)$, we obtain a block with the three operators σ_1:

$$H_{1110}(q) = \left(\sum_{n=-\infty}^{\infty} \frac{\Delta - (n+1/2)^2}{\Delta - 1/4} (-)^n (n+1/2) q^{(n+1/2)^2} \right) \times \left(\sum_{n=-\infty}^{\infty} (-)^n (n+1/2) q^{(n+1/2)^2} \right)^{-1}. \tag{3.24}$$

Finally, by "associating" the current $I(z)$ with the fourth operator σ_0 in this block, we obtain the symmetric block $G_{1111}(x)$:

$$H_{1111}(q) = \left[\Delta^2 - \frac{\Delta}{2} \left(\theta_3^4 (1-2x) + 3x(1-x) \frac{d\theta_3^4}{dx} \right) + \frac{1}{16} \left(\theta_3^8 - 4x(1-x)\theta_3^4 \frac{d}{dx} x(1-x) \frac{d\theta_3^4}{dx} + 5x^2(1-x)^2 \left(\frac{d\theta_3^4}{dx} \right)^2 + 2x(1-x)(1-2x)\theta_3^4 \frac{d\theta_3^4}{dx} \right) \right] \Big/ \left(\Delta - \frac{1}{4} \right)^2. \tag{3.25}$$

This block has one double pole at $\Delta = 1/4$.

In principle, the same algorithm can be used to evaluate blocks containing the next invariant Ramond operators σ_2, σ_3, etc. However, the evaluation becomes much more complicated. For example, it is clear from the explicit expressions (2.22) that the operator σ_2 can be obtained by "associating" a particular combination of one and three currents $I(z)$ with the Ramond vacuum σ_0. To evaluate blocks with the operator σ_3, we have to "associate" six current operators, and so on. However, the evaluation can sometimes be simplified. For example, for the block G_{0200} we can use the operator expansion that follows from (2.17) and (2.20):

$$I(z)\sigma_2(0) = {}^1/_2 z^{-5/2} \sigma_0(0) - 2z^{-3/2} \sigma_0'(0) - {}^8/_3 z^{-1/2} \sigma_0''(0) + {}^5/_3 z^{-1/2} L_{-2}\sigma_0(0) + O(z^{1/2}). \tag{3.26}$$

The general form of the function $\Gamma_{0200}(z,x)$ is

$$\Gamma_{0200}(z,x) = [z(z-1)(z-x)]^{-1/2} \times \left[\frac{A_{0200}(x)}{(z-x)^2} + \frac{B_{0200}(x)}{z-x} + C_{0200}(x) \right]. \tag{3.27}$$

The expansion given by (3.26) can be used to express the functions A_{0200}, B_{0200}, and C_{0200} in this expression in terms of the function G_{0000} and its derivatives. The correlation function with the operator $L_{-2}\sigma_0$ and all the functions containing the conformal "descendants" of invariant fields are then related by a differential operator to the block of invariant fields.[1] Integration with respect to z then gives the answer for the block G_{0200}, as in (3.2):

$$H_{0200}(q)=(4\Delta-1)^{-1}\left(4\Delta-(1-2x)\theta_3^4-3x(1-x)\frac{d\theta_3^4}{dx}\right) \tag{3.28}$$

with one pole at $\Delta = 1/4$. The functions G_{1200} and G_{0210} now have two poles each, at $\Delta = 0.1$ and $\Delta = 0.1/4$, respectively.

The process of "associating" a large number of current operators to obtain blocks with the higher Ramond fields can be made easier by considering the multicurrent expectation

$$\langle I(z_1)\ldots I(z_n)\sigma_0(x_1)\ldots\sigma_0(x_4)\rangle/\langle\sigma_0(x_1)\ldots\sigma_0(x_4)\rangle.$$

The current $I(z)$ is a free field, so that such correlators can be decomposed in accordance with Wick's rule into the product of paired current correlation functions. We shall show in the next Section how these blocks can be related to particular free-current correlation functions on a torus that is the Riemann surface of the elliptic curve

$$y^2=z(1-z)(z-x). \tag{3.29}$$

4. RELATION TO CORRELATION FUNCTIONS ON A TORUS

The quantity $H_{\delta_1\delta_2\delta_3\delta_4}(\Delta,q)$ in the representation (3.19) of a conformal block (see Ref. 8) can be related to the correlation function of conformal fields on a torus with particular boundary conditions. For the conformal block

$$G_{\delta_1\delta_2\delta_3\delta_4}(x)=\langle V_{\delta_1}(x_1)V_{\delta_2}(x_2)V_{\delta_3}(x_3)V_{\delta_4}(x_4)\rangle$$

the expectation value of the energy-momentum tensor has the following form (in accordance with the conformal Ward identities):[1]

$$\langle\langle T(z)\rangle\rangle=\langle T(z)V_{\delta_1}(x_1)V_{\delta_2}(x_2)V_{\delta_3}(x_3)V_{\delta_4}(x_4)\rangle G^{-1}_{\delta_1\delta_2\delta_3\delta_4}(x)$$

$$=\frac{\delta_1}{z^2}+\frac{\delta_2}{(z-x)^2}+\frac{\delta_3}{(z-1)^2}+\frac{\delta_1+\delta_2+\delta_3+\delta_4}{z(1-z)}+\frac{x(1-x)\mathrm{C}(x)}{z(z-x)(1-z)}, \tag{4.1}$$

where the coefficient $C(x)$ determines the x dependence of the block:

$$\partial\log G_{\delta_1\delta_2\delta_3\delta_4}(x)/\partial x=\mathrm{C}. \tag{4.2}$$

The elliptic substitution

$$4K\xi=\int_0^z\frac{dt}{[t(1-t)(1-tx)]^{1/2}},\qquad z=\mathrm{sn}^2\,2K\xi \tag{4.3}$$

maps the plane z onto a parallelogram on the plane with generators 1 and $\tau/2$ [$K(x)$ and τ are given by (3.7) and (3.8)]. Under the conformal substitution, the energy-momentum tensor transforms as follows:[1]

$$T(\xi)=\left(\frac{\partial z}{\partial\xi}\right)^2T(z)+\frac{c}{12}\{z,\xi\}, \tag{4.4}$$

where

$$\{z,\xi\}=\frac{z_{\xi\xi\xi}}{z_\xi}-\frac{3}{2}\left(\frac{z_{\xi\xi}}{z_\xi}\right)^2 \tag{4.5}$$

is the Schwartz derivative. The result is that, on the ξ plane,

$$(2K)^{-2}\langle\langle T(\xi)\rangle\rangle=\frac{\tilde\delta_1}{\mathrm{sn}^2\,2K\xi}+\frac{\tilde\delta_2}{\mathrm{cn}^2\,2K\xi}+\frac{\tilde\delta_3}{dn^2\,2K\xi}$$
$$\times-\tilde\delta_4\,dn^2 2K\xi+T_0, \tag{4.6}$$

where $\tilde\delta_i = 4\delta_i - c/8$, and

$$T_0=\frac{c}{12}(2x-1)-\tilde\delta_1+\tilde\delta_2(2x-1)+\tilde\delta_3+\tilde\delta_4(1-x)-4x(1-x)\mathrm{C}. \tag{4.7}$$

Comparison of (4.6) and (3.19) shows that the function

$$Z(\tau)=(16q)^{\Delta-(c-1)/24}h^{-1/2}(q)H(\Delta,q), \tag{4.8}$$

where

$$h(q)=\prod_{k=1}^{\infty}(1-q^{2k}) \tag{4.9}$$

satisfies the relation

$$\frac{\partial}{\partial\tau}\log Z(\tau)=\frac{1}{2}\int_0^1\langle\langle T(\xi)\rangle\rangle\frac{d\xi}{2\pi i}. \tag{4.10}$$

This function can therefore be interpreted as the correlation function of the four conformal fields with dimensions $\tilde\delta_i$, $i = 1,2,3,4$ [placed at the point 0, $\frac{1}{2}$, $\tau/2$, and $(1+\tau)/2$] in conformal field theory on a torus (which appears when the plane ξ is factored on a lattice with periods 1 and τ), with the additional symmetry condition $T(\xi) = T(-\xi)$.

In our case, the current operator

$$I(\xi)=\frac{dz}{d\xi}I(z) \tag{4.11}$$

satisfies the symmetry conditions

$$I(\xi+1)=I(\xi+\tau)=I(\xi),\quad I(\xi)=I(-\xi) \tag{4.12}$$

and the same local operator algebra conditions as the current $I(z)$:

$$I(\xi)I(\xi')=\frac{1}{2(\xi-\xi')^2}+O(1),\quad \xi\to\xi',$$
$$I(\xi)I(\xi')=\frac{1}{2(\xi+\xi')^2}+O(1),\quad \xi\to-\xi'. \tag{4.13}$$

The expansion of this current around $\xi = 0$ has the form

$$I(\xi)=2\sum I_{n-1/2}\xi^{-2n} \tag{4.14}$$

with the same operators $I_{n-\frac{1}{2}}$ as in Section 2.

When we evaluate the correlation functions for the current $I(\xi)$ subject to (4.12), we shall assume that the block conditions are satisfied, i.e., the charge flowing through the closed circuit corresponding to C in (3.2) is fixed and equal to p. On the ξ plane, this circuit can be represented by a segment of unit length, parallel to the real axis. Hence,

$$\int_\alpha^{\alpha+1}I(\xi)\frac{d\xi}{2\pi i}=p. \tag{4.15}$$

In particular, it follows from these conditions that

$$\langle I(\xi)\rangle=I_0=2\pi ip. \tag{4.16}$$

The current $I(\xi)$ can therefore be written in the form

$$I(\xi)=2\pi ip+j(\xi), \tag{4.17}$$

where the current $j(\xi)$ satisfies the same relations (4.12) and (4.13) and has zero expectation value. Since the current $I(\xi)$ is a free (Gaussian) field, the many-point correlation functions of the current $j(\xi)$ can be evaluated by Wick's rule and expressed as products of two-point functions:

$$R(\xi,\xi')=\langle j(\xi)j(\xi')\rangle. \tag{4.18}$$

The two-point function $R(\xi,\xi')$ is uniquely determined by the above symmetry conditions:

$$R(\xi,\xi')=-\frac{1}{2}\frac{d^2}{d\xi^2}\log\theta_1(\xi-\xi'/q)\theta_1(\xi+\xi'/q), \tag{4.19}$$

$$\theta_1(\xi/q)=i\sum_{n=-\infty}^{\infty}(-)^n q^{(n+1/2)^2}e^{i\pi(2n+1)\xi}. \tag{4.20}$$

The use of these rules produces a substantial simplification of the procedure of "association" of different combinations of current operators $I(\xi)$ with Ramond points which, in this representation, correspond to the points 0, $\frac{1}{2}$, $\tau/2$, and $(1+\tau)/2$ on a torus. Thus, for the blocks (3.14), (3.17), (3.18), (3.24), and (3.25), we have (without taking normalization into account)

$$H_{0100}=\langle I(0)\rangle=2\pi ip, \tag{4.21}$$

$$H_{1100}=\langle I(0)I(1/2)\rangle=-4\pi p^2+R(0,1/2)$$
$$=-4\pi^2\left(p^2-q\frac{\partial}{\partial q}\log\theta_2(q)\right), \tag{4.22}$$

$$H_{0110}=\langle I(0)I(\tau/2)\rangle=-4\pi^2\left(p^2-q\frac{\partial}{\partial q}\log\theta_0(q)\right), \tag{4.23}$$

$$H_{1110}=\langle I(0)I(1/2)I(\tau/2)\rangle=-8\pi^2 ip\left(p^2-q\frac{\partial}{\partial q}\theta_1'(q)\right) \tag{4.24}$$

and, finally,

$$H_{1111}=\left\langle I(0)I(1/2)I\left(\frac{\tau}{2}\right)I\left(\frac{1+\tau}{2}\right)\right\rangle$$
$$=(2\pi)^4\left(p^4-2q\frac{\partial}{\partial q}\theta_1'(q)+R_1^2+R_2^2+R_3^2\right), \tag{4.25}$$

where

$$\theta_0(q)=\sum_{n=-\infty}^{\infty}(-)^n q^{n^2},\quad \theta_2(q)=\sum_{n=-\infty}^{\infty}q^{(n+1/2)^2},$$
$$\theta_1'(q)=\frac{\partial}{\partial\xi}\theta_1\left(\frac{\xi}{q}\right)\Big|_{\xi=0} \tag{4.26}$$

$$R_1=\frac{1}{4\pi^2}R(0,1/2),\quad R_2=\frac{1}{4\pi^2}R\left(0,\frac{\tau}{2}\right),$$
$$R_3=\frac{1}{4\pi^2}R\left(0,\frac{1+\tau}{2}\right). \tag{4.27}$$

It is readily verifed that, when the normalization conditions are taken into account, all these expressions become identical with the block formulas written out in Section 2. The explicit representations (2.22) for the higher-order Ramond fields can be used to derive the corresponding local fields that are constructed nonlinearly from the current $i(\xi)$, where the "associations" of these fields with the vertices 0, $\frac{1}{2}$, $\tau/2$, and $(1+\tau)/2$ gives the blocks containing these higher fields. For example, the following operators correspond to the fields σ_2 and σ_3:

$$1/6\, I''(\xi)-2{:}I^3(\xi){:} \tag{4.28}$$

and

$$\frac{1}{360}(3:II'''':-10:(I'')^2:-20:I''I^3:+8:I^6:). \tag{4.29}$$

Here, the normal ordering symbol signifies that singular contributions of the form $1/(\xi-\xi')^2$, $1/(\xi-\xi')^4$, etc., are subtracted when we take the limit of equal arguments in these operators. For example:

$$:I^3(\xi):=\lim_{\substack{\xi'\to\xi\\ \xi''\to\xi}}\left[I(\xi)I(\xi')I(\xi'')-\frac{I(\xi)}{2(\xi-\xi')^2}\right.$$
$$\left.-\frac{I(\xi')}{2(\xi-\xi'')^2}-\frac{I(\xi'')}{2(\xi-\xi')^2}\right]. \tag{4.30}$$

In particular, for the block (3.28), we obtain

$$H_{0200}=\frac{1}{6}\langle I''(0)-2{:}I^3(0){:}\rangle=-\frac{8\pi^3p}{3}(p^2-\langle{:}j^2(0){:}\rangle). \tag{4.31}$$

Since

$$\langle{:}j^2(\xi){:}\rangle=\lim_{\xi'\to\xi}\left[R(\xi'-\xi)-\frac{1}{2(\xi'-\xi)^2}\right], \tag{4.32}$$

we find, as in the last Section,

$$H_{0200}=\left(p^2-\frac{1}{4}\right)^{-1}\left(p^2-q\frac{\partial}{\partial q}\log\theta_1'(q)\right). \tag{4.33}$$

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Translated by S. Chomet