Dynamics of the instability of a smectic *A* in a magnetic field

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The behavior of the layer structure of a smectic A in a magnetic field parallel to the layers is investigated. It is shown that at a certain critical field the layer structure becomes unstable. The instability (Helfrich instability) leads to the formation of regions with periodic modulation of the layers. The modulation amplitude and the shape and velocity of propagation of the regions is found. The relation between the time of formation of the domains and the initial conditions is investigated.

It is well known that in a homeotropically oriented smectic A, with positive diamagnetic anisotropy χ_a, in a magnetic field H parallel to the smectic planes a Helfrich deformation, consisting in the formation of periodic bending of the smectic layers, can arise.[1] The reason for the appearance of this modulated structure is the instability of smectics against disturbance of their macroscopic uniformity. The usual description of this effect consists in finding the characteristic period of the finite structure and the magnitude of the critical field by investigating the free energy averaged over the volume of the samples. In such a description there is no possibility of drawing any conclusions about the amplitude of the modulation or its relation to the initial disturbance, or about the time dependence of the establishment of the final structure. Therefore, a study of the nonlinear dynamics of the process, making it possible to give at least qualitative answers to these questions, is of interest.

To describe the nonlinear dynamics of a smectic A, following Refs. 2 and 3 we use the variable $W(\mathbf{r}, t)$ describing the layer structure ($W=$ const specifies the position of a layer of molecules). The free-energy density of the smectic A is written in the form of an expansion in the gradients of this function; the leading terms of this expansion have the form[1)]

$$F_W={}^1/_8 B[l^2(\nabla W)^2-1]^2+{}^1/_2 Kl^2(\nabla^2 W)^2. \tag{1}$$

Here l is the equilibrium distance between neighboring smectic layers, and B and K are elastic moduli. In equilibrium, $W_0=z/l$ and describes a system of layers perpendicular to the z axis.

Away from equilibrium it is convenient to write the variable W in the form

$$W=(z+u)/l,$$

where u is the displacement of the layers along the z axis. Below we shall consider the geometry when the smectic liquid crystal lies between two bounding surfaces at $z=0$ and $z=d$, and the magnetic field is in the direction of the x axis and is uniform over the whole sample. Thus, the problem becomes effectively two-dimensional. The magnetic field is taken into account by adding to (1) a term $-\frac{1}{2}\chi_a(\mathbf{n}\cdot\mathbf{H})^2$, where the director $\mathbf{n}$ is defined by the relation

$$\mathbf{n}=\nabla W/|\nabla W|.$$

For simplicity, in writing the hydrodynamic equations we shall neglect thermal conduction. In addition, we shall neglect permeation effects; this is possible when the characteristic wave vectors of the problem are not large: $q^2 \ll (\eta\nu)^{-1} \sim 10^{14}\,\text{cm}^{-2}$ (η is the viscosity coefficient and ν is the permeation coefficient). This is the case we shall consider below.

With allowance for dissipation, the equations of motion for $\mathbf{v}$ and W have the form[3]

$$\begin{gathered}\partial\rho/\partial t+\operatorname{div}\mathbf{j}=0,\\ \partial j_i/\partial t+\nabla_k T_{ik}-\chi_a H^2 l^2 W_i W_{xx}=\nabla_k \partial R/\partial(\nabla_k v_i),\\ \partial W/\partial t+\mathbf{v}\nabla W=0,\end{gathered} \tag{2}$$

where $\mathbf{j}=\rho\mathbf{v}$ is the momentum density, ρ is the density, T_{ik} is the stress tensor, and R is the dissipative function. With the assumptions made, the stress tensor is defined by

$$\begin{aligned}T_{ik}=P\delta_{ik}+\rho v_i v_k+{}^1/_2 Bl^2[l^2(\nabla W)^2-1]\nabla_i W\nabla_k W\\ -Kl^2\nabla_i W\nabla_k(\nabla^2 W)+Kl^2\nabla^2 W\nabla_i\nabla_k W,\end{aligned} \tag{3}$$

(where P is the pressure, including a part due to the smectic variable), and the dissipative function is defined by

$$R={}^1/_2\eta_{iklm}\nabla_k v_i\nabla_l v_m, \tag{4}$$

where the viscosity tensor has the form

$$\begin{gathered}\eta_{iklm}=\eta_1(\delta_{il}^{\perp}\delta_{km}^{\perp}+\delta_{kl}^{\perp}\delta_{im}^{\perp})\\ +(2\eta_2-\eta_1)\delta_{ik}^{\perp}\delta_{lm}^{\perp}+2\eta_3(\delta_{ik}^{\parallel}\delta_{lm}^{\perp}+\delta_{lm}^{\parallel}\delta_{ik}^{\perp})\\ +{}^1/_2\eta_4(\delta_{il}^{\parallel}\delta_{km}^{\perp}+\delta_{kl}^{\parallel}\delta_{im}^{\perp}+\delta_{im}^{\parallel}\delta_{kl}^{\perp}+\delta_{km}^{\parallel}\delta_{il}^{\perp})+2\eta_5\delta_{ik}^{\parallel}\delta_{lm}^{\parallel},\\ \delta_{ik}^{\parallel}=\nabla_i W\nabla_k W/|\nabla W|^2,\qquad \delta_{ik}^{\perp}=\delta_{ik}-\delta_{ik}^{\parallel}.\end{gathered}$$

Substituting the expressions (3) and (4) into (2), changing to the variable u, and retaining the most important terms nonlinear in u (henceforth it is assumed that $\nabla_i u \ll 1$), we obtain as a result

$$\begin{gathered}\partial\rho/\partial t+\operatorname{div}\rho\mathbf{v}=0,\qquad \partial u/\partial t+v_z+v_x u_x=0,\\ \rho\frac{\partial v_x}{\partial t}+\rho v_i\nabla_i v_x+\frac{\partial P}{\partial x}+B\Big[2u_x u_{xx}(u_x^2+u_z)\\ +u_x u_{zz}+u_{xz}\Big(\frac{5u_x^2}{2}+u_z\Big)\Big]-Ku_x\nabla^2(\nabla^2 u)\\ -\chi_a H^2 u_x u_{xx}=(2\eta_2+\eta_1)\frac{\partial^2 v_x}{\partial x^2}+\Big(2\eta_3+\frac{\eta_4}{2}\Big)\frac{\partial^2 v_z}{\partial x\,\partial z}\\ +\frac{\eta_4}{2}\frac{\partial^2 v_x}{\partial z^2}+\alpha_1\frac{\partial}{\partial x}\Big(\frac{\partial v_x}{\partial z}+u_x\frac{\partial v_z}{\partial x}\Big)+\alpha_2\frac{\partial}{\partial z}\Big(u_x\frac{\partial v_z}{\partial z}\Big),\\ \rho\frac{\partial v_z}{\partial t}+\rho v_i\nabla_i v_z+\frac{\partial P}{\partial z}+B\Big[u_{xx}\Big(\frac{3}{2}u_x^2+u_z\Big)\\ +u_{zz}+2u_x u_{xz}\Big]-K\nabla^2(\nabla^2 u)-\chi_a H^2 u_{xx}\end{gathered}$$

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$$=2\eta_5\frac{\partial^2 v_z}{\partial z^2}+\left(2\eta_3+\frac{\eta_4}{2}\right)\frac{\partial^2 v_x}{\partial x\,\partial z}+\frac{\eta_4}{2}\frac{\partial^2 v_z}{\partial x^2} \qquad (5)$$

$$+\alpha_1\frac{\partial}{\partial x}\left(u_x\frac{\partial v_x}{\partial x}\right)+\alpha_2\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial x}+u_x\frac{\partial v_x}{\partial z}\right),$$

$$\alpha_1=\eta_4+2\eta_3-2\eta_2-\eta_1, \qquad \alpha_2=2\eta_5-2\eta_3-\eta_4.$$

In the linear approximation, in the absence of a magnetic field, Eqs. (5) describe two modes of the acoustic type.[1] One of them is due to oscillations of the density ρ and is ordinary longitudinal sound, propagating with a velocity c_0 determined by the compressibility $(\partial P/\partial\rho)_S$. The other mode (second sound) is connected with the displacement of the layers, propagates with velocity $c_2=(B/\rho)^{1/2}\ll c_0$, and is associated with the condition of incompressibility. As was shown in Ref. 3, allowance for terms that are nonlinear in the smectic variable u can substantially alter the form of the spectrum. Whereas for ordinary sound these changes reduce to small corrections to the sound velocity c_0 and to the damping constant, for second sound they give qualitatively different behavior in the dependence on the geometry of the experiment.

Since the aim of the present work is to study the dynamics of the smectic variable u, it is natural to expect that, in the presence of a magnetic field, terms that are nonlinear in this variable will have a substantial effect on the result. Eliminating the pressure P from Eqs. (5) and assuming the smectic to be incompressible (the validity of this assumption will be demonstrated later), we obtain for the variables **v** and u the following system of equations:

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0, \qquad \frac{\partial u}{\partial t}+v_z+v_xu_x=0,$$

$$\rho\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\rho v_z\Delta v_x-\rho v_x\Delta v_z$$

$$-K[u_{xz}\Delta(\Delta u)+u_x\Delta(\Delta u_z)-\Delta(\Delta u_x)]-\chi_aH^2(u_xu_{xxz}-u_{xxx})$$

$$-(\alpha_1-\alpha_2-\eta_4)\frac{\partial^3 v_z}{\partial x\,\partial z^2}+\frac{\eta_4}{2}\left(\frac{\partial^3 v_z}{\partial x^3}-\frac{\partial^3 v_x}{\partial z^3}\right) \qquad (6)$$

$$+\alpha_1\frac{\partial^3 v_z}{\partial z^3}+\alpha_2\frac{\partial^3 v_z}{\partial x^2\,\partial z}$$

$$-B[u_{xxx}({}^3/_2u_x^2+u_z)+(3u_{xx}-2u_{zz})(u_xu_{xx}+u_{xz})+u_{xzz}+2u_xu_{xxz}$$
$$-u_xu_{zzz}-5u_xu_{xz}^2]=0.$$

From the first equation of the system (6) we obtain the estimate $v_x\sim v_zL_x/L_z$, where L_x and L_z are the characteristic lengths over which the velocity varies. Substitution of this expression into the second equation gives for its last term the estimate $v_xu_x\sim v_zu_z$, which, since u_z is small, enables us to neglect this term in the subsequent calculations.

In the third equation of the system (6) we can neglect the terms that are nonlinear in the velocity if one of the inequalities $v_i\ll L_i/\tau_x$, $\eta_m/\rho L_i$ holds where τ_x is the characteristic time and η_m the corresponding viscosity coefficient of this equation. As will be seen from the final result, these conditions are practically always fulfilled. Taking into account the remarks made above, we can eliminate v_x and v_z from the last equation of the system (6) and obtain as a result a closed nonlinear equation for the smectic variable u:

$$\rho\frac{\partial^2\Delta u}{\partial t^2}-K\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}(u_x\Delta^2u)-\Delta^2u_x\right]$$

$$-\chi_aH^2\frac{\partial}{\partial x}(u_xu_{xxz}-u_{xxx})-B\frac{\partial^2}{\partial x^2}[u_{xx}({}^3/_2u_x^2+u_z)+u_{zz}+2u_xu_{xz}]$$

$$+B\frac{\partial^2}{\partial x\,\partial z}(u_xu_{zz}+{}^3/_2u_x^2u_{xz})=(\alpha_2-\alpha_1)\frac{\partial u_{xxzz}}{\partial t}+\frac{\eta_4}{2}\frac{\partial\Delta^2u}{\partial t}$$

$$+\alpha_1\frac{\partial u_{xzzz}}{\partial t}+\alpha_2\frac{\partial u_{xxxz}}{\partial t}. \qquad (7)$$

In principle, the solution of this equation gives the answer to the problem posed, for quite arbitrary initial conditions. It is clear, however, that it is not possible to find the solution in explicit form. Therefore, the aim of the paper from now on will be to investigate the conditions under which Eq. (7) can have an analytic solution that can be realized under actual experimental conditions.

We shall consider the case when, at the initial time, the smectic is stationary and the position of the layers corresponds to the equilibrium state $u=0$. After the field is switched on, motion due to particular additional initial conditions (fluctuations, defects, and boundary distortions) can arise in the system. (We note that the solution $u=0$ satisifes Eq. (7), and, therefore, the realization of this solution is stable against disturbances.) However, at times close to the initial time, the deviation of the quantity u from its equilibrium value is small, and so its dynamics will be described by Eq. (7) if we linearize in u:

$$\rho\frac{\partial^2}{\partial t^2}(\Delta u)$$

$$-\frac{\partial}{\partial t}\left[(\alpha_2-\alpha_1)u_{xxzz}+\frac{\eta_4}{2}\Delta(\Delta u)+\alpha_1u_{xzzz}+\alpha_2u_{xxxz}\right]$$

$$-Bu_{xxzz}+\chi_aH^2u_{xxxx}+K\Delta(\Delta u_{xx})=0. \qquad (8)$$

In order to investigate the stability of the initial state $u=0$ after the magnetic field has been switched on, we consider disturbances having the form

$$\delta u\propto\exp(ikx+iqz+\omega t).$$

Equation (8) leads to the following dispersion relation for $\omega\equiv\omega(k,q,H)$:

$$\rho\omega^2(1+\gamma^2)+\omega[(\alpha_2-\alpha_1)\gamma^2+{}^1/_2\eta_4(1+\gamma^2)^2$$
$$+\alpha_1\gamma^3+\alpha_2\gamma]+B\gamma^2-\chi_aH^2+Kk^2(1+\gamma^2)^2=0, \qquad (9)$$

where $\gamma=q/k$.

We consider first the case $\gamma^2\ll 1$. Then

$$\omega=\{-{}^1/_2\eta_4\pm[{}^1/_4\eta_4^2+4\rho(\chi_aH^2-Kk^2-B\gamma^2)]^{1/2}\}/2\rho.$$

It is obvious that for $\chi_aH^2-Kk^2-B\gamma^2<0$, i.e., for small fields, $\omega<0$ and the initial state is stable against disturbances with any wave vectors. At a certain field $H=H_c$ the quantity ω vanishes for the first time and the system loses stability. Simple calculations give

$$H_c^2=2Kq/\chi_a\lambda, \qquad \lambda=(K/B)^{1/2}\sim l. \qquad (10)$$

Here $k=k_c=(q/\lambda)^{1/2}$. This implies that the system becomes unstable only for perturbations with wave vector k_c. For values of H slightly greater than H_c the region of instability spreads to a narrow band of wave vectors k close to k_c. It can be seen from (10) that the magnitude of the critical field H_c increases with increase of q. Since the system under consideration is bounded in the z direction, and the quantitities v and u vanish on the bounding surfaces, the wave vector q can have only the values $q = m\pi/d (m = 1,2,...)$.

Thus, the corresponding critical fields for the harmonics are

$$H_{cm}=H_{c0}m^{1/2}, \quad H_{c0}=(2K\pi/\chi_a\lambda d)^{1/2}.$$

Substituting into this expression the values of the physical constants and the experimentally achievable value $d \sim 1$ cm, we obtain the estimate $H_{c0} \sim 10^4$ Oe. Thus, the magnitude of the minimum critical field is found to be rather large and so there exists a wide range of magnetic fields $H \sim H_{c0}$ for which the instability is determined entirely by the first harmonic q. This is the case (weakly supercritical) that will be considered below.

Analysis of the case $\gamma^2 \gtrsim 1$ shows that the corresponding critical fields should be of the order of 10^6–10^7 Oe, which is scarcely achievable in experiment.

Thus, let $H \gtrsim H_{c0}$. In this case, as was indicated above, if in the system there exist disturbances with $k \sim k_c = \pi/\lambda d$, they will grow, and as a result a region of quasi-periodic structure, with wavelength close to $2\pi/k_c$, should be formed. The character of the structure and the dynamics of its formation with increase of u should be determined by the nonlinear equation (7). However, in the weakly supercritical case the solution can be obtained by a rather general method using the so-called amplitude equation, which was introduced in Ref. 4 and has been applied by many authors to describe a broad class of phenomena (see Refs. 5–7 and the literature cited therein).

Equation (7), in the dimensionless variables

$$\xi=(\pi/\lambda d)^{1/2}x, \quad \eta=(\pi/d)z,$$
$$u=\lambda\varphi, \quad \tau=(2\pi K/\lambda d\eta_4)t$$

and with allowance for the fact that only the first harmonic q is important, takes the form

$$-\frac{4\rho K}{\eta_4^2}\frac{\partial^2\varphi}{\partial\tau^2}+\frac{\partial^3\varphi}{\partial\tau\,\partial\xi^2}+\frac{\partial^2\varphi}{\partial\eta^2}-\frac{2H^2}{H_c^2}\frac{\partial^2\varphi}{\partial\xi^2}$$
$$-\frac{\partial^4\varphi}{\partial\xi^4}+\left[\frac{\partial^2\varphi}{\partial\xi^2}\left(\frac{3}{2}\varphi_\xi^2+\varphi_\eta\right)+2\varphi_\xi\varphi_{\xi\eta}\right]=0. \tag{11}$$

In these variables $k_c = 1$, and the width of the band of wave vectors for which the system is unstable is $\Delta k = 2\nu^{1/2}$ ($\nu = (H - H_c)/H_c$ is the supercriticality parameter), while $\omega(k_c) = \nu$ and is a maximum for the wave vectors in this band. In these same variables the solution of Eq. (8) can be represented in the form

$$\varphi=[A(\xi,\ \tau)e^{i\xi}+A^*(\xi,\ \tau)e^{-i\xi}]\sin\eta, \tag{12}$$

where A is a function that depends only weakly on ξ and τ ($\partial A/\partial\xi, \partial A/\tau \ll 1$) on account of the smallness of Δk and ω.

Proceeding to the study of Eq. (11), it is necessary to take into account the interaction of the different linear modes that arises from the nonlinearity. As a result of this interaction, higher and higher harmonics in ξ and τ appear, for which the first harmonic is the "source." Their amplitudes can be determined by successive solution of an infinite system of equations. However, as follows from analysis of the structure of the nonlinearity, for small amplitudes A of the fundamental harmonic it is sufficient to take into account only a few of the lowest harmonics, since the others will give a contribution of higher order in the amplitude A. The assumption that the amplitude A is small is confirmed by the result of the calculations.

Solving the necessary equations for the lowest harmonics, we finally obtain

$$\varphi=(Ae^{i\xi}+A^*e^{-i\xi})\sin\eta+A_0f_0(\eta)+(A_2e^{2i\xi}+A_2^*e^{-2i\xi})f_2(\eta),$$
$$A_0=|A|^2/4,\quad f_0=\sin 2\eta;\quad A_2=-A^2/8,\quad f_2=\sin 2\eta. \tag{13}$$

Substitution of the expression (13) into (11) gives for the amplitude A the equation

$$\partial A/\partial\tau=4\nu A+4\partial^2A/\partial\xi^2-A|A|^2/2, \tag{14}$$

which, in the literature, is customarily called the amplitude equation. By the scale changes

$$X=\nu^{1/2}\xi,\quad T=4\nu\tau,\quad A=2^{3/2}\nu^{1/2}Z$$

Eq. (14) is brought to the more convenient form

$$\partial Z/\partial T=Z+\partial^2Z/\partial X^2-Z|Z|^2. \tag{15}$$

It is easy to see that the solution $Z = 0$ of this equation is unstable in the region of wave vectors $p^2 \leqslant 1$ (which corresponds to $1 - \nu^{1/2} \leqslant k \leqslant 1 + \nu^{1/2}$ in the previous variables). As the time increases the system arrives at a stationary state. The stationary solutions of Eq. (15) have the form

$$Z_{st}=(1-p^2)^{1/2}e^{ipX}$$

and exist for all p in the region $p^2 \leqslant 1$. It is clear that in the limit of large times the system should arrive at a stationary state corresponding to the minimum of the free energy and to $|Z| = 1$. It can be shown that such solutions are stable.

For the case of arbitrary initial conditions, when Z fluctuates about $Z = 0$ over the whole sample, it does not appear to be possible to write down the temporal dynamics of the nonlinear stage of the development of the instability. However, in the case of initial perturbations (fluctuations, defects) localized in a small region of the sample, or of the appearance in a small region of the sample (for whatever reasons) of disturbances larger in amplitude than in the rest of the sample, it is possible to describe the development at large times of the periodic structure being formed.

For this type of initial conditions a fundamental role in the formation of the periodic structure is played by solutions of Eq. (15) that have the forms of a traveling wave, propagating with velocity c in the region of the unstable ($Z = 0$) phase and carrying the latter over into the stable phase (with solution $|Z| = 1$). Such solutions $Z(X, T) \equiv Z(X - cT)$ satisfy the equation

$$d^2Z/dy^2+c\,dZ/dy+Z-Z|Z|^2=0, \tag{16}$$

where $y = X - cT$. Analysis of this ordinary differential equation shows that solutions bounded on the entire X axis have the form $Z(X, T) = Z_c \exp(i\theta)$, where θ is a constant phase ($\partial\theta/\partial X = 0$), and the real function Z_c satisfies the equation

$$d^2Z_c/dy^2+c\,dZ_c/dy+Z_c-Z_c^3=0, \tag{17}$$

which can be interpreted as the equation of motion of a particle with friction in the potential

$$U=Z_c^2/2-Z_c^4/4.$$

The particle trajectory of interest to us "leaves" the point

$Z_c = 1$ with zero velocity at $y = -\infty$, and, with the friction, falls into the potential-well minimum at $Z_c = 0$. In the neighborhood of $Z_c = 0$ (i.e., as $y \to \infty$), Z_c satisfies the linearized equation (17), whose solution of interest to us is

$$Z_c = Z_{c0} e^{\varepsilon y}, \tag{18}$$

where ε is a root of the equation $\varepsilon^2 + c\varepsilon + 1 = 0$. As y decreases, in the region $y \sim \varepsilon^{-1} \ln Z_{c0}$ the function Z_c changes from zero to unity over a scale $\sim |\varepsilon|^{-1}$.

Solutions of this type exist for any velocity c, but, as will be seen in the following, of all the solutions only a single solution, with $c = 2$, is realized. To convince ourselves of this, we return to the analysis of the linear stage of the development of the instability. The general solution of the linearized equation (15) has the form

$$Z(X,T) = \int Z_p \exp[ipX + (1-p^2)T]\,dp, \tag{19}$$

where Z_p is the Fourier transform of the initial condition $Z(X, T)$ at $T = 0$. We note that if $Z(X, 0)$ is concentrated on a finite interval, then Z_p is an entire function of the complex variable p.

Setting $X = X_0 + vT$, we now consider the expression (19) as $T \to \infty$. The asymptotic form of (19) at large T is determined by the saddle point $p = iv/2$ and has the form

$$Z(X,\ T) \approx (\pi/T)^{1/2} Z_p(iv/2) \exp(-vX_0/2) \exp[T(1-v^2/4)]. \tag{20}$$

Here we have used the fact that Z_p can be continued into the complex plane.

It can be seen from (20) that for $v > 2$ the quantity $Z(X, T)$ decreases exponentially with increase of T, while for $v < 2$ it increases without bound. For $v = 2$,

$$Z(X,\ T) = (\pi/T)^{1/2} Z_p(i) \exp(-X_0) = (\pi/T)^{1/2} Z_p(i) \exp(2T - X). \tag{21}$$

In the entire region $X > 2T$ the quantity $Z(X, T) \to 0$. In the region $X < 2T$ the solution constructed using the linear theory increases and becomes inapplicable (the nonlinear equation must be used). Thus, the transition from the unstable state $Z = 0$ to a stationary stable state with bounded Z occurs only for $v = 2$. In this case, $Z(X, Y)$ is, with logarithmic accuracy, a function only of X_0, i.e., does not depend on the time.

Comparison of (21) with the solution of Eq. (16) for $c = 2$ shows that these solutions can be matched with the same logarithmic accuracy. For the matching it is sufficient to set the constant in formula (18) approximately equal to $T^{-1/2}$, which is equivalent to a correction $\Delta c = -\frac{1}{2} d \ln T / dT = -1/2T$ to the velocity c. (At large T the solution reaches the regime with $c = 2$). The phase θ of the solution of Eq. (16) is given by the quantity $\arg Z_p(i) = \theta$.

The condition for applicability of the method of steepest descent has in the present case the form $T \gg X_0^2$ (equivalently, $X - X_0 \gg 2X_0^2$) and reflects the fact that a solution of the type described above is formed in a region remote from the coordinate X_0 of the initial disturbance, and after long times.

Thus, it has been shown that, at long times, from a small disturbance localized on a finite interval a region of periodic structure with amplitude $Z = 1$ is formed. The boundaries of this region are oppositely traveling waves, with the same velocities equal to ± 2, and the shape of the front is determined by Eq. (17). The characteristic size of the front is ~ 1. The coordinates of the fronts,

$$X_f \approx \pm [2T - \ln(\pi^{1/2} |Z_p(i)|)]$$

and the constant phase $\theta = \arg Z_p(i)$ of the periodic structure are determined by the form of the initial condition $Z(X, 0)$.

In order that such a pattern be observed, it is obviously necessary that additional conditions be fulfilled. First, the size L of the sample should be such that the front has time to be formed. Secondly, other disturbances (noise) present in the sample should remain small in the time taken by the front to traverse a distance of the order of the size of the sample. For an estimate of the characteristic time of the development of a disturbance we use Eq. (15) in the linearized form. Its solutions have the form $Z = Z_p(0) \exp(ipX + \gamma_p T)$, where $\gamma_p = 1 - p^2$. The mode with $p = 0$ grows fastest, and so the characteristic time of the growth of a disturbance to a size ~ 1 is $T_0 \sim -\ln|Z_0(0)|$. Assuming that the perturbation is localized in the center of the sample, we obtain for the first condition:

$$cT_0 = -2 \ln |Z_0(0)| < L/2$$

(in times $\sim T_0$ the front does not succeed in reaching the boundary of the sample). To fulfill the second condition we must have

$$T_0 + L/c < T_n = -\ln |Z_{0n}(0)|,$$

where T_n is the characteristic growth time of the noise, and therefore

$$L/2 - \ln |Z_0(0)| < -\ln |Z_{0n}(0)|,$$

where $Z_{0n}(0)$ are the initial conditions for the noise. These conditions impose restrictions both on the size of the sample and on the character of the disturbances.

In dimensional variables the amplitude of the modulation of the periodic structure formed is $u_0 = \lambda(2\nu)^{1/2}$, and, since $\lambda \sim l$, for fields just above the critical value it is smaller than the distance between neighboring layers (i.e., defects of the layer structure are not formed). Both in the region of the periodic structure and in the region of the wave front, $u_z \sim \nu^{1/2}(\lambda/d) \ll 1$ and $u_x \sim \nu(\lambda/d)^{1/2} \ll 1$ for $\lambda < d$. Thus, for samples that are not too thin, the conditions of the derivation of Eqs. (5) are fulfilled.

The flow velocities in the nonstationary region ($v_z \sim \nu^{3/2} K/\eta d$ and $v_x \sim v_z(\lambda/d)^{1/2}$) satisfy, as is easily verified, the conditions $v_i \ll L_i/\tau_x$, $\eta/\rho L_i$, and this justifies the neglect of the terms nonlinear in the velocities in the hydrodynamic equations.

The velocity of the wave front is $c = (16K/\eta)(\pi\nu/\lambda d)^{1/2}$, which is considerably smaller than the sound velocity, ensuring the applicability of the incompressibility condition. Thus, the results obtained agree with all the assumptions made earlier.

As was indicated above, for an estimate of the characteristic times to establish the solution it is necessary to know the concrete form of the initial conditions of the problem. Thermal fluctuations, of order $\sim T(KB)^{-1/2} \ln(d/l)$, though small in magnitude, can turn out to be important near defects that localize them. Distortions at the side boundaries of the sample can play an important role as initial disturbances. However, the form of the initial disturbances affects only the stage in which a region of periodic structure sets in, and the subsequent behavior of this region will be described by the general characteristics obtained in this paper.

[1)]The inclusion of terms of the type $K'(\nabla_i W \nabla^2_{ik} W)^2$ and $K''(\nabla_i \nabla_k W \nabla_i \nabla_k W)$ in the energy does not essentially alter the result, since the contributions they make to the stress tensor either are small and do not change its structure or effectively produce a slight change in the numerical value of the modulus K.

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