Detonation on a timelike front for relativistic systems

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In collisions of high-energy heavy ions, in the early universe, and in other relativistic systems, phase transitions can take place on a three-dimensional hypersurface whose points are not causally connected. It is shown that the derivation of the equation for the relativistic shock (or detonation) adiabat proposed by Taub can be generalized for discontinuity surfaces having a timelike normal vector; the form of the equation is universal, i.e., the same for spacelike and timelike discontinuities. In the considered physical example of implosion induced by radiative heat transfer there is a transition from a spacelike to a timelike front.

The relativistic hydrodynamic theory of shock waves[1] has found elegant applications in cosmology[2–5] and in reactions with relativistic heavy ions.[6–10] A shock wave usually occurs when there is rapid compression of matter under the influence of an external force. Similar discontinuities can also arise spontaneously in an expanding system if a first-order phase transition occurs.[1)] In all these cases, shock waves propagate with a velocity less than the velocity of light. Therefore, the world lines of the points of the surface of the shock front form a spacelike hypersurface in Minkowski space (Fig. 1a), i.e., a hypersurface with a spacelike normal vector. The conservation laws on this surface lead to the Rankine-Hugoniot-Taub equation,[1] which relates the properties of the fluid (pressure, fluid velocity, densities of the conserved charges) on the two sides of the discontinuity.

It was noted recently[11] that under certain conditions there may occur in a system a rapid phase transition leading to a timelike discontinuity surface. Such a situation occurs when the system undergoes a rapid and homogeneous rarefaction and there forms a set of bubbles at different spatial points causally unconnected to each other. As an example of this we can mention the inflationary universe model. In this case, the spacelike phase boundary becomes after smoothing a timelike surface Σ (see Fig. 1b). The thickness τ of this transition region will be determined by the rates of formation and growth of the bubbles. If τ is sufficiently small compared with the characteristic time scale of the considered process, then it can be assumed that the phase transition takes place through a structureless timelike surface. The aim of the present paper is to give a general derivation of the Rankine-Hugoniot-Taub equation valid for both spacelike and timelike surfaces. The obtained result will be illustrated by a simple example.

We denote the vector of the normal to the surface Σ by Λ^μ. It is normalized as follows:

$$\Lambda_\mu \Lambda^\mu = \begin{cases} +1 \text{ for timelike } \Sigma, \\ -1 \text{ for spacelike } \Sigma. \end{cases} \tag{1}$$

The state of the system is characterized by the energy-momentum tensor

$$T^{\mu\nu} = w u^\mu u^\nu - p g^{\mu\nu}, \tag{2}$$

where we have introduced the enthalpy density $w = e + p$, the sum of the energy density e and the pressure p; $u^\mu = (\gamma, \gamma \mathbf{v})$ is the 4-velocity of the fluid, normalized such that $u^\mu u_\mu = 1$; and $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor. If there is a discontinuity surface, the index 1 will identify the dynamical characteristics of the fluid, for example, Q, on one side of the discontinuity (Q_1), and the index 2 those on the other side (Q_2). Then the jump of Q across the discontinuity will be expressed as

$$[Q] = Q_2 - Q_1. \tag{3}$$

In this notation, the conservation laws across the discontinuity surface have the form

$$[R^\mu] = [T^{\mu\nu} \Lambda_\nu] = 0 \tag{4}$$

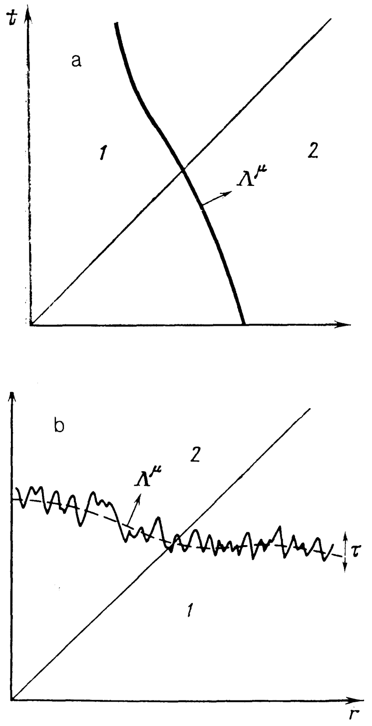

FIG. 1. Comparison of spacelike (a) and timelike (b) discontinuity surfaces characterized by normal vector Λ^μ (1 and 2 are the phases of the matter before and after the discontinuity, and the thin straight line is the generator of the light cone). A timelike discontinuity can occur in the case of the spontaneous production and growth of bubbles, which are united after a time τ (continuous curve). If this complicated surface is smoothed, then a timelike front (broken curve) is obtained. Note that in a viscous fluid spacelike discontinuities also have finite width. Thus, the substructure of the timelike front (b) can develop only if the distance between the bubbles is greater than the width of the spacelike discontinuities.

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(conservation of the energy and momentum), and

$$[j]=[n^\mu\Lambda_\mu]=0 \tag{5}$$

(conservation of the particle number). Systems in which the particle number is not conserved are considered in Appendix 1. The expression (5) can be written down for any independent conserved charge with corresponding 4-current of this charge. Under the assumption that the flux is related to the conserved charge, we can introduce an invariant scalar density $n = n^\mu u_\mu$, and also the quantity $x = w/n^2$,[2,3,6] which plays the part of the specific volume V in nonrelativistic theory. In fact, in the nonrelativistic limit $x = mV$ (m is the mass of a particle).

To derive the equation of the detonation adiabat, which relates only thermodynamic quantities, it is necessary to eliminate the 4-velocity from Eqs. (4) and (5). Since (4) is a vectorial equation, it can be decomposed into two independent equations. For this it is necessary to take its projection onto the direction of the normal to the surface Σ:

$$[R^\mu]\Lambda_\mu=0, \tag{6}$$

and also to project it onto the surface itself by means of the projection operator $P^{\mu\nu} = g^{\mu\nu} - \Lambda^\mu\Lambda^\nu/\Lambda^\alpha\Lambda_\alpha$:

$$[G^\mu]=[P^{\mu\nu}R_\nu]=0. \tag{7}$$

From Eq. (7) it is possible to obtain a scalar equation which shows that the length of the vector G^μ is conserved,

$$[G^\mu G_\mu]=0, \tag{8}$$

and also an equation that ensures an unchanging direction of the projection of the vector G^μ:

$$[G^\mu/|G^\mu|]=0.$$

After manipulations, we can obtain from Eqs. (5) and (6) the following expression for the flux:

$$j^2=[p]\Lambda^\mu\Lambda_\mu/[x]. \tag{9}$$

From Eqs. (5) and (8), we find

$$j^2=[wx]/[x^2]\Lambda_\mu\Lambda^\mu. \tag{10}$$

From these two relations, we obtain directly Taub's well-known adiabat equation:

$$[p](x_1+x_2)=[wx]. \tag{11}$$

For known p_1 and x_1 and for known equation of state Eq. (11) determines the connection between p_2 and x_2. Note that the vector Λ^μ of the normal to the discontinuity surface occurs in this equation only in the combination $(\Lambda^\mu\Lambda_\mu)^2$, which is equal to unity. Therefore, Taub's adiabat equation has the same form for spacelike and timelike discontinuity surfaces. However, there is an important difference between these two cases.

This is demonstrated by Fig. 2, which shows shock and detonation adiabats on the (p,x) plane. The initial state is indicated by the point 1, the final state by the point 2. In the case of an ordinary shock wave, the adiabat determined by Eq. (11) passes through the point 1, since the final state of the matter is described by the same equation of state as the initial state. If in the final state there has been a change of the equation of state due to a chemical reaction or a phase transi-

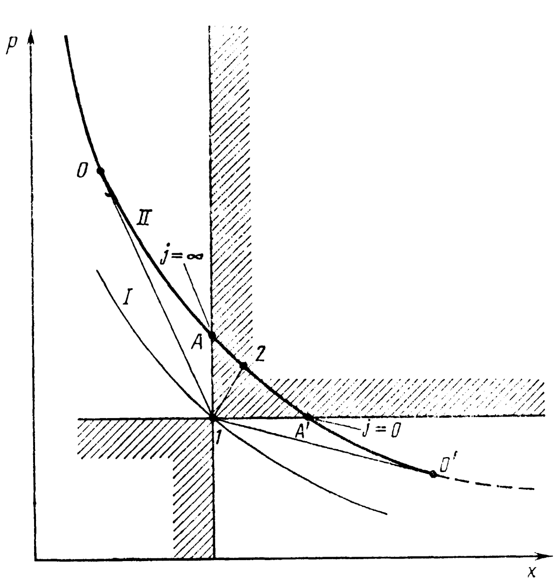

FIG. 2. Shock (I) and detonation (II) adiabats on the (p,x) plane (p is the pressure and $x = w/n^2$ is the generalized specific volume). The point 1 is the initial point; the final state 2 must lie on the adiabat. The inclination of the straight line joining the initial and final points is determined by the current through the discontinuity surface: $j^2 = \pm[p]/[x]$ for spacelike ($+$) and timelike ($-$) discontinuities, respectively. The segment AA′ of the detonation adiabat corresponds to timelike discontinuities; O and O′ are the Chapman-Jouguet points on the detonation adiabat. In the presence of enhancement due to boundary conditions, rapid combustion and condensation (segment OA) can pass smoothly through the point A to timelike detonation.

tion, the point 2 lies on a curve that does not pass through the initial point 1. Usually, this curve is called the detonation adiabat. Figure 2 shows the detonation adiabat for an exothermic process. If the process is endothermic, the corresponding curve passes below the point 1. The segment OA on this curve corresponds to detonation, and the segment O′A′ to slow burning (deflagration). The hatched part of the plane, where $[p]/[x] < 0$, was previously assumed to be unphysical[9,3] on the basis of nonrelativistic analogies.[12] According to Ref. 9, in this region the flux j becomes imaginary for a spacelike discontinuity surface, when $\Lambda_\mu\Lambda^\mu = -1$. However, this region can be reached for a real value of the flux j in the case of a timelike surface of the detonation front. We note that the existence of a shock front with timelike normal is impossible since the shock adiabat cannot lie in the timelike region (hatched region in Fig. 2). This is readily seen by recalling that by means of a Lorentz transformation one can go over to a system in which the vector of the normal to the timelike surface has the form $\Lambda^\mu = (1,0,0,0)$. In this system there is an abrupt change of p and x but the density $N = n\gamma$ measured in this system remains unchanged, since $[j] = [n\gamma] = [N] = 0$. This is possible only in the case when a phase transition or chemical reaction occurs in the system, i.e., there is detonation.

However, spontaneous detonation can occur only if the condition of increase of the entropy is satisfied. Since at the point 1 the Poisson adiabat is parallel to the shock adiabat, states with higher entropy lie above the point 1. Thus, only the upper quarter of the timelike detonation region can be reached in a physical process. In other words, only exothermic detonation can occur on a timelike front. In this case, when the exothermic process has a threshold (with respect to the pressure or the temperature), the physically realized section of the timelike detonation adiabat must begin at a certain point within the segment AA′.

In Appendix 2 we describe a simple schematic model that demonstrates a continuous transition from spacelike to timelike detonation in an implosion process induced by radiation. Suppose a spherical core of unit radius ($R = 1, c = 1$) is surrounded by a rapidly igniting shell. If at $t = 0$ this shell is ignited from all sides, then some of the released energy will be radiated inward and heat the core. Neglecting the opacity and compression of the core, we can readily calculate the constant-temperature contours:

$$T(r,t) \propto \begin{cases} 0, & t<1-r, \\ \frac{t}{r}\left(\ln\frac{t}{1-r}-1\right)+ & \\ +\frac{1-r}{t}, & 1-r<t<1+r, \\ \frac{t}{r}\ln\frac{1+r}{1-r}-2, & 1+r>t. \end{cases}$$

Suppose that in the core an exothermic transition occurs if the temperature reaches the critical value T_c. Then this transition takes place on the surface $T(r,t) = T_c$. If the heating of the core to the temperature T_c takes place rapidly ($t \approx 2.5$), then an appreciable part of this surface ($r \lesssim 0.5$, $2.3 \lesssim t \lesssim 2.5$) corresponds to timelike detonation (Fig. 3). If the heating is slower, the timelike detonation region is concentrated in a smaller central zone.

Summarizing what we have said, we must emphasize that the general treatment of an arbitrary discontinuity in a relativistic fluid has made it possible to extend the Rankine-Hugoniot-Taub equation to a new region not hitherto considered. Moreover, on the basis of nonrelativistic analogies this region has hitherto been regarded as unphysical. The inclusion in the treatment of timelike detonation closes the relativistic theory of rapid combustion and condensation.

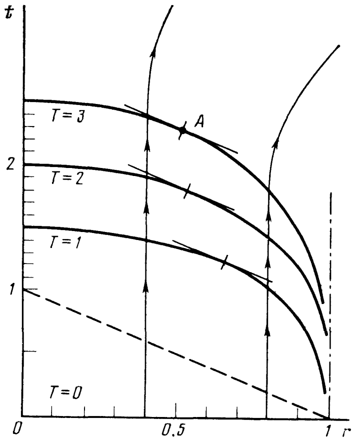

FIG. 3. Acceleration of discontinuity due to radiation can lead to a smooth transition from a spacelike to a timelike front at the point A, where the front propagates with luminal velocity. This is possible because the fluid does not move together with the front. The world lines of the fluid particles (the lines with arrows) remain spacelike, i.e., their velocity is less than c. The broken line is the light cone, t is measured in units of R/c, r in units of R, and T in units of $2\pi CQ/C_V$.

The formalism developed makes it possible to understand more clearly the conceptual unity of problems of relativistic hydrodynamics and make their mathematical description more transparent. This is illustrated by the concrete example in Fig. 3.

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APPENDIX 1. DETONATION AND DEFLAGRATION FRONTS FOR THE QCD PLASMA AND HADRONIC MATTER WITH ZERO BARYON CHARGE

Notation:

$$u^\mu=(\gamma, \gamma\mathbf{v}), \quad u_\mu=(\gamma, -\gamma\mathbf{v}),$$
$$u^\mu u_\mu=+1, \quad g^{\mu\nu}=\mathrm{diag}\,(1, -1, -1, -1).$$

The vector Λ^μ of the normal to the discontinuity surface is normalized in such a way that $\Lambda^\mu \Lambda_\mu = \pm 1$ for a timelike (+) and spacelike (−) discontinuity surface. In the local rest frame $\Lambda^\mu = (1,0,0,0)$ for timelike discontinuity and $\Lambda^\mu = (0,\hat{e})$ for spacelike discontinuity.

1. Projection parallel to Λ^μ:

$$[T^{\mu\nu}\Lambda_\mu\Lambda_\nu]=[w(u^\mu\Lambda_\mu)^2-p\Lambda_\mu\Lambda^\mu]=0. \tag{A1}$$

2. Projection perpendicular to Λ^μ. We introduce

$$G_\tau=T^{\mu\nu}\Lambda_\nu\Delta_{\tau\mu}, \quad \Delta^{\tau\mu}=g^{\tau\mu}-\Lambda^\tau\Lambda^\mu/\Lambda^\sigma\Lambda_\sigma$$

(G_τ is orthogonal to Λ_τ),

$$G^\mu=\{wu_\tau(u_\nu\Lambda^\nu)-p\Lambda_\tau\}\{g^{\tau\mu}-\Lambda^\tau\Lambda^\mu/\Lambda^\sigma\Lambda_\sigma\}$$
$$=w(u^\tau\Lambda_\tau)u^\mu-w(u_\nu\Lambda^\nu)^2\Lambda^\mu/\Lambda^\sigma\Lambda_\sigma.$$

It also follows from the condition $[G^\mu]=0$ that $[G^\mu G_\mu]=0$. This leads to the expression

$$[w^2(u_\nu\Lambda^\nu)^2-w^2(u_\nu\Lambda^\nu)^4/\Lambda^\sigma\Lambda_\sigma]=0. \tag{A2}$$

We introduce the notation $Q = w(u_\nu\Lambda^\nu)^2$ and $N = \Lambda^\nu\Lambda_\nu$. Then Eqs. (A1) and (A2) can be rewritten in the form

$$[Q]=N[p], \quad [Q^2]=N[wp].$$

Eliminating N, we have

$$[p]\,(Q_2+Q_1)=[wQ],$$
$$Q_1=-N(p_2-p_1)\,(e_2+p_1)/(e_2-p_2-e_1+p_1). \tag{A3}$$

Therefore,

$$(u^\mu\Lambda_\mu)_1{}^2=N(p_2-p_1)\,(e_2+p_1)/(e_2-p_2-e_1+p_1)\,(e_1+p_1). \tag{A4}$$

On the other hand,

$$(u^\mu\Lambda_\mu)_1{}^2=\begin{cases}\gamma_1^2 v_1^2\cos^2\theta_1, & \text{spacelike discontinuity,}\\ \gamma_1^2, & \text{timelike discontinuity.}\end{cases} \tag{A5}$$

In Ref. 9, only spacelike discontinuities with $\theta_1 = 0$ were considered. From (A4) and (A5) we obtain an expression for the velocity of the oncoming flow in the rest frame of the front (for $\theta_1 = \theta_2 = 0$):

$$v_1{}^2=(p_1-p_2)\,(e_2+p_1)/(e_1-e_2)\,(e_1+p_2). \tag{A6}$$

In the case of timelike detonation, the velocity v_1' is

$$v_1'^2=v_1^{-2}.$$

The relative velocity of the incoming and outgoing flows for both spacelike and timelike fronts is

$$v_{12}{}^2=v_{12}'^2=(p_1-p_2)\,(e_1-e_2)/(e_1+p_2)\,(e_2+p_1). \tag{A7}$$

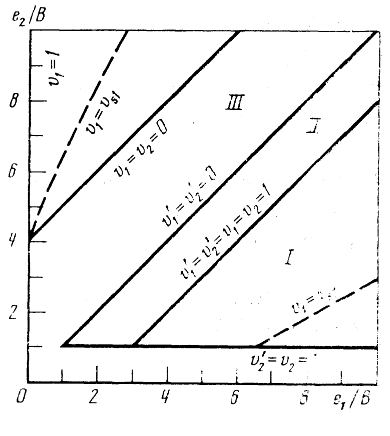

FIG. 4. Kinematic regions in which the continuity equations are satisfied for physical values of the flow velocity for the transition between quark matter with energy density e_2 and hadronic matter with energy density e_1 (B is the constant in the quark bag model). In the figure, I is the region of spacelike detonation, II is the region of timelike detonation, and III is the unphysical region. Because of the possibility of timelike detonation the unphysical region is here smaller than the one obtained in Ref. 9 (Fig. 4).

If we denote the entropy flux by $S^\mu = su^\mu$, then the condition of increase of the entropy can be written as

$$[S^\mu\Lambda_\mu]\geqslant 0 \quad \text{or} \quad s_2/s_1\geqslant (T_2/T_1)(e_2+p_1)/(e_1+p_2) \qquad \text{(A8)}$$

for both cases of detonation in the process $1\to 2$. If in Eqs. (A6)–(A8) we substitute the same equation of state and make the same analysis as in Ref. 9, then we can find the region in which timelike detonation is possible (see Fig. 4).

APPENDIX 2. IMPLOSION INDUCED BY RADIATION

We consider a physical volume filled with matter and fairly transparent for radiation. This matter undergoes an exothermic transition if its temperature exceeds T_c. We assume that the core is surrounded by a shell of explosive that ignites rapidly and whose radiation leads to heating of the core. We ignore the expansion of the shell inward, and also the expansion of the core, i.e., we shall assume that its radius is unchanged, $R = \text{const}$. Suppose the shell is ignited at the time $t_0 = 0$ simultaneously at all points. In what follows, the length is measured in units of R and the time t in units of R/c.

Let Q be the heat that the shell radiates in unit time through unit surface. If it is assumed that in the matter of the core a constant fraction C of this heat is absorbed, then at distance r from the center of the core we have (we ignore the opacity of the core)

$$\frac{dQ}{dt} = CQ\int_0^t d\tau\cdot 2\pi \times\int_{-1}^{1} d\cos\theta\,(1+r^2-2r\cos\theta)^{-1}\delta[\tau-(1+r^2-2r\cos\theta)^{1/2}]$$

$$= C\cdot 2\pi Q r^{-1}\int_{1-r}^{a}\frac{d\tau}{\tau} = \frac{2\pi CQ}{r}\ln\tau\Big|_{1-r}^{a}, \qquad \text{(A9)}$$

where

$$a=\begin{cases}1-r, & t\leqslant 1-r,\\ t, & 1-r<t<1+r,\\ 1+r, & t>1+r.\end{cases}$$

Thus, the heat absorbed in unit time is

$$\frac{dQ}{dt}=\frac{2\pi CQ}{r}\begin{cases}\ln[(1+r)/(1-r)], & t>1+r,\\ \ln\dfrac{t}{1-r}, & 1-r<t<1+r,\\ 0, & t<1-r.\end{cases} \qquad \text{(A10)}$$

Ignoring the compression, assuming that the specific heat C_V is constant, and using the fact that $dT\approx dQ/C_V$, we obtain

$$T(r,t)\approx\frac{1}{C_V}\int_0^t dt\,\frac{dQ}{dt}$$

$$=\frac{2\pi CQ}{C_V r}\begin{cases} t\ln\dfrac{1+r}{1-r}-2r, & t>1+r,\\ t\left(\ln\dfrac{t}{1-r}-1\right)-1-r, & 1-r<t<1+r,\\ 0, & t<1-r\end{cases} \qquad \text{(A11)}$$

(thus, if $t>1+r$, then $T(r=0,t)\propto t-1$). The discontinuity surface is determined by the contour $T(r,t) = T_c$. The tangent to this contour at $t>1+r$ is given by the expression

$$\left(\frac{\partial r}{\partial t}\right)_{T_c}=\left(\frac{\partial T}{\partial t}\right)_{T_c}\Big/\left(\frac{\partial T}{\partial r}\right)_{T_c}$$

$$=\ln\left(\frac{1+r}{1-r}\right)\Big/ t\left(\frac{2}{1-r}-\frac{1}{r}\ln\frac{1+r}{1-r}\right). \qquad \text{(A12)}$$

The point (t_c, r_c), at which the spacelike and timelike parts of the surface converge, is determined by the condition $(\partial r/\partial t)T_c = 1$, whence

$$t_c=\left\{\left[(1-r_c)\ln\left(\frac{1+r_c}{1-r_c}\right)^{1/2}\right]^{-1}-\frac{1}{r_c}\right\}^{-1}.$$

For example, for $r_c = 0.5$ we have $t_c = 2.34$ and $T_c = 3.142(2\pi CQ/C_V)$. The center of the core is heated to T_c during the time $t = 2.57$. The line $t = t_c(r)$ separates the spacelike and timelike parts of the discontinuity surface $T(r,t) = T_c$. The discontinuity is formed at $r = R$ at the time $t = 0$ and then propagates inward. This process initially proceeds slowly, but it is then accelerated by the radiative heat transfer and at $r_c = t_c^{-1}(t_c(r))$ goes over smoothly into a timelike discontinuity (see Fig. 3). If $T_c = 4\pi QC/C_V$, then this occurs approximately at $r_c = 0.5$–0.6. A similar gradual transition from spacelike to timelike detonation can be realized in the late stages of ultrarelativistic nuclear collisions. If we include radiative heat transfer in the scenario described in Ref. 11, then the transition from spacelike to timelike deflagration will be smooth. However, this question requires a more detailed numerical analysis.

[1)]Such discontinuities can appear in the process of matter condensation in a rarefaction wave[8] or in the case of deflagration and detonation of expanding bubbles in a supercooled metastable medium.[3,9] Here, we call any discontinuity surface a shock front. The difference between these discontinuities and ordinary shock waves will be discussed below.

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