Neutrino oscillations in an inhomogeneous medium: adiabatic regime

S. P. Mikheev and A. Yu. Smirnov

Nuclear Research Institute, USSR Academy of Sciences
(Submitted 6 June 1986)
Zh. Eksp. Teor. Fiz. **92**, 404–416 (February 1987)

The properties of a ν beam can undergo the strongest changes in the adiabatic regime over a wide range of energies. The adiabaticity condition is discussed; the relations describing the ν oscillations in the adiabatic regime are derived; the divergence of neutrino packets in matter are considered. A geometric representation of the ν oscillations is described.

1. INTRODUCTION

Vacuum neutrino oscillations[1] become modified in matter.[2,3] The effect of the medium is due to elastic scattering of the neutrino by "zero" angle from the electrons and nuclei, and reduces to the appearance of various refractive indices for the waves describing the motion of neutrinos (ν_e, ν_μ, ...).

The influence of the motion is resonant.[4] The dependence of the effective-mixing parameter $-\sin^2 2\theta_m$ on the density of the medium (ρ)or on the neutrino energy (E_ν) has a Breit-Wigner form: at definite ρ or E the mixing becomes maximal ($\sin^2 2\theta_m = 1$), $\sin^2 2\theta_m$ decreases with increasing distance from the resonant value of ρ or E, and the width of the peak ($\Delta\rho_R$ or ΔE_R) is proportional to the vacuum mixing $\sin 2\theta$. The meaning of the resonance is the following.

1. At resonance, the frequency characterizing the external medium is equal to the natural frequency of the system (mixed neutrinos). The quantity $l_0 \propto (G_F\rho)^{-1}$, the proper length for the medium, is equal at resonance to the neutrino vacuum oscillation length $l_\nu = 4\pi E/\Delta m^2$: $l_0 = l_\nu\,(\theta \ll 1)$ (Ref. 4).

2. The influence of the medium can be described with the aid of a potential that changes the effective masses of the oscillating neutrinos.[5] The effective masses of the neutrinos are equal at resonance. In other words, the natural frequencies of the "weakly coupled" oscillators are equal, and the oscillations of one of them can be completely transferred to another.

3. At resonance the mixing is a maximum: $\theta_m = 45°$, and if neutrinos that are eigenstates of weak interactions are created in a medium of density $\rho = \rho_E$, their oscilllations have maximum length. If a continuous energy spectrum is generated in a medium of constant density, the neutrino oscillations will be resonantly amplified in the energy range $(E_R - \Delta E_R)$ to $(E_R + \Delta E_R)$.

The manifestations of resonance (of the resonance condition) in inhomogeneous media depend on the character of the density variation. If the density varies slowly enough, the oscillation regime is adiabatic.[4,6,7] Practically total conversion of one type of neutrino into another is then possible in a wide energy interval. In media with variable density, the depth of the oscillations is no longer uniquely connected with the mixing angle θ_m. Conditions are possible, and are quite realistic, under which, even in a resonant layer ($\rho = \rho_R$), this depth is close to zero ("nonoscillatory neutrino conversion").[6,7]

The medium exerts substantial effects if its thickness is large: $d \approx m_N/G_F \approx 3.5\cdot 10^9$ g/cm^3 (there m_N is the nucleon mass and G_F is the Fermi constant. These effects are used in neutrino astrophysics and geophysics. Neutrino oscillations were considered in this context in the sun,[4,5,8–10] in cores and envelopes of collapsing stars,[6,7] in the earth,[11,7] and in the early universe.

The present paper is devoted to neutrino oscillations in the adiabatic regime. The adiabaticity conditions were formulated in Refs. 4,7, and 9, where the principal characteristics of the neutrino oscillations in the adiabatic regime were presented. We derive here the previously published results and obtain new relations concerning the adiabatic regime. A discussion of the evolution of the eigenstates of neutrinos in matter is given in Sec. 2 and is followed (in Sec. 3) by formulation of the adiabatic condition and an explanation of its meaning. Relations describing the oscillations in the adiabatic regime are derived in Sec. 4. In Sec. 5 are considered next effects of divergence of neutrino packets; a graphic representation of neutrino oscillations is described in Sec. 6. Realizations of the adiabatic condition in specific objects (sun, envelopes, supernovas) are the subject of Sec. 7.

2. EVOLUTION OF NEUTRINO EIGENSTATES IN MATTER

Consider the mixing of two neutrinos, e.g., ν_e and ν_m:

$$\nu_e = c\nu_1 + s\nu_2, \qquad \nu_\mu = c\nu_2 - s\nu_1.$$

Here $c = \cos\theta$, $s = \sin\theta$ (θ is the mixing angle in vacuum), ν_1 and ν_2 are states with definite masses m_1 and m_2.

The equations of evolution in a medium for wave functions $\nu_e(t)$ and $\nu_\mu(t)$ (Ref. 2) can be written in the form

$$i\frac{d}{dt}\boldsymbol{\nu}_\alpha = \hat{M}\boldsymbol{\nu}_\alpha, \tag{1}$$

where $\boldsymbol{\nu}_\alpha^{\rm T} = (\nu_e, \nu_\mu)$, and $\hat{M}$ is the evolution matrix:

$$\hat{M} = \begin{vmatrix} M_e & \overline{M} \\ \overline{M} & M_\mu \end{vmatrix}. \tag{2}$$

The matrix elements in $\hat{M}$ satisfy the relations

$$\begin{aligned} M_e + M_\mu &= (m_1^2 + m_2^2)/2k + \Sigma_e + \Sigma_\mu, \\ M \equiv M_e - M_\mu &= (\Delta m^2/2k)\cos 2\theta + \Sigma_e - \Sigma_\mu, \\ 2\overline{M} &= -\sin 2\theta(\Delta m^2/2k), \end{aligned} \tag{3}$$

where k is the neutrino momentum, $\Delta m^2 = m_1^2 - m_2^2$, Σ_α ($\alpha = e,\mu$) are quantities that take the effect of the matter into account[2]:

$$\Sigma_\alpha = \sum_i \frac{f_{\alpha i}(0)\,n_i}{k}$$

 0038-5646/87/020230-07$04.00

($f_{\alpha i}(0)$ is the amplitude of forward scattering of ν_α by the ith component of the medium, and n_i is the density of this component).

Equations (1) are in effect Schrödinger equations from which are omitted factors that influence equally the phase velocities of ν_e and ν_m. We define the effective density of the medium as[4]

$$\rho_{\rm eff}=\frac{m_N}{2^{1/2}G_F}(\Sigma_e-\Sigma_\mu)=m_N\sum_i\frac{\Delta f_i(0)n_i}{2^{1/2}G_Fk},$$

G_F is the Fermi constant, m_N is the nucleon mass, and $\Delta f_i(0)=f_{ei}(0)-f_{\mu i}(0)$. Recall that $f\propto Gk$, so that $\rho_{\rm eff}\propto m_N n_i$. For simplicity, we omit hereafter the subscript "eff." The expression for M can be rewritten with the aid of ρ in the form

$$M=\frac{\Delta m^2}{2k}\cos 2\theta\left(1-\frac{\rho}{\rho_R}\right), \tag{4}$$

where

$$\rho_R=\frac{m_N\Delta m^2\cos 2\theta}{2\cdot 2^{1/2}G_Fk} \tag{5}$$

is the resonant density.[4]

We introduce now the eigenstates. The states $\mathbf{v}_m^{\rm T}=(\nu_{1m},\nu_{2m})$ that diagonalize the evolution density $\hat{M}$ in (1) will be called the eigenstates of the neutrinos in the medium. By definition,

$$\mathbf{v}_\alpha=\hat{S}\mathbf{v}_m, \tag{6}$$

where the matrix

$$\hat{S}=\begin{pmatrix}\cos\theta_m & \sin\theta_m\\ -\sin\theta_m & \cos\theta_m\end{pmatrix}$$

is defined by the condition

$$\hat{S}^{-1}\hat{M}\hat{S}=\hat{M}^d. \tag{7}$$

Here $\hat{M}^d$ is a diagonal matrix: $M^d=\mathrm{diag}(M_1^d,M_2^d)$. In the matrix $\hat{S}$, the angle θ_m that establishes according to (6) the connection between the eigenstates of the weak interactions $\nu_\alpha=(\nu_e,\nu_\mu)$ and the eigenstates in the medium, is called the angle of mixing in the medium. Taking (2)–(5) into account we get

$$M_{1,2}^d=\tfrac{1}{2}[M_e+M_\mu\mp(M^2+4\overline{M}^2)^{1/2}], \tag{8}$$

$$\sin 2\theta_m=2\overline{M}/(M^2+4\overline{M}^2)^{1/2}. \tag{9}$$

Relations (3) and (4) allow us to rewrite (9) in the form

$$\sin 2\theta_m=\mathrm{tg}\,2\theta\,[(1-\rho/\rho_R)^2+\mathrm{tg}^2\,2\theta]^{-1/2}. \tag{10}$$

According to (6) we have $\mathbf{v}_m=\hat{S}^{-1}\mathbf{v}_\alpha$ and the angle θ_m determines the "flavor," i.e., the ν_e, ν_m makeup of the eigenstates of the neutrinos in the medium.

Two important conclusions follow from (10).

1. θ_m varies in a medium of variable density, and consequently the "flavor" of the eigenstates ν_{im} changes. When ρ is decreased from $\rho\gg\rho_R$ to $\rho\ll\rho_R$ the value of θ_m decreases from $\pi/2$ to θ. If the angle θ is small, the "flavor" ν_{im} is almost completely altered. If, for example, ν_{im} consisted at $\rho\gg\rho_R$ mainly of ν_μ, at $\rho\ll\rho_R$ the "electronic flavor" will predominate ν_{im}, viz, $\nu_{1m}\approx\nu_e$.

2. The dependence of $\sin^2 2\theta_m$ on ρ has a resonant character,[4] with a maximum $\sin^2 2\theta_m=1$ at $\rho=\rho_R$; the half-width of the resonance at half maximum is $\Delta\rho_R=\rho_R\tan 2\theta$.

Let us find the evolution equations for the eigenstates of neutrinos in matter. It follows from (6) and (1) that

$$i\frac{d}{dt}\mathbf{v}_m=\left(\hat{M}^d-i\hat{S}^{-1}\frac{\partial\hat{S}}{dt}\right)\mathbf{v}_m. \tag{11}$$

Using the expressions for $\sin 2\theta_m$ as a function of ρ [see (10)], we get

$$\hat{S}^{-1}\frac{d\hat{S}}{dt}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{d\theta_m}{dt}, \tag{12}$$

where

$$\frac{d\theta_m}{dt}=\frac{1}{2\rho_R}\frac{d\rho}{dt}\frac{\mathrm{tg}\,2\theta}{(1-\rho/\rho_R)^2+\mathrm{tg}^2\,2\theta}. \tag{13}$$

With allowance for (12), the evolution equations (11) for the eigenstates ν_{1m} and ν_{2m} (11) in matter take the form

$$i\frac{d}{dt}\mathbf{v}_m=\begin{vmatrix}M_1^d & -i\dot\theta_m\\ i\dot\theta_m & M_2^d\end{vmatrix}\mathbf{v}_m. \tag{14}$$

In a medium of constant density the solution of (14) is trivial: 1) $\dot\theta_m=0$; the ν_{im} diagonalize the entire system of equations (14) and consequently are eigenstates of the Hamiltonian

$$|\nu_{im}(t)\rangle=|\nu_{im}\rangle\exp[-iM_i^dt];$$

2) $\theta=$ const, meaning that the "flavor" $|\nu_{im}\rangle$ does not change in the course of the evolution. The $\nu_e\leftrightarrow\nu_\mu$ oscillations have then the usual form with an average depth determined by θ_m ($A_P=\sin^2 2\theta_m$, $\overline{P}=1-\sin^2 2\theta_m/2$), and with a length

$$l_m=2\pi/(M_1^d-M_2^d)=2\pi/(M^2+4\overline{M}^2)^{1/2}.$$

In a medium with variable density, the evolution matrix $\mathbf{v}_m$ is not diagonal, and the $\nu_m(t)$ are not conserved: transitions $\nu_{1m}\leftrightarrow\nu_{2m}$ are realized during the propagation.

3. ADIABATICITY CONDITION

Assume that the density changes in the course of propagation: $\dot\theta_m\neq 0$. Let us find the condition under which the off-diagonal elements in the matrix (14) can be neglected. We call this hereafter the adiabaticity condition. Since $\dot\theta_m\propto d\rho/dt$, the adiabaticity presupposes a rather slow variation of the density.

The off-diagonal terms in (14) have little effect on the evolution of the states ν_{im} if the "frequency" connected with $\dot\theta_m$ is much lower than the "frequencies" generated by the diagonal terms. The lowest of them is given by the difference

$$|M^d|=|M_1^d-M_2^d|=(M^2+4\overline{M}^2)^{1/2}=2\pi/l_m, \tag{15}$$

meaning that the adiabaticity condition is written in the form

$$|\dot\theta_m|\ll|M^d|=2\pi/l_m.$$

This relation can be made more accurate. From the system (14) we obtain the following equation for the probability $P_1=|\nu_{1m}(t)|^2$ of the $\nu_{1m}\to\nu_{1m}$ transition:

$$M^d\dddot{P}_1-\dot{M}^d\ddot{P}_1+(M^d)^3\dot{P}_1+4M^d\dot\theta_m^2\dot{P}_1-2\dot\theta_m^2\dot{M}^d(2P_1-1)=0 \tag{16}$$

[M^d is defined in (15)]; the terms containing $\ddot\theta_m$ are left out. According to (16), the change of P_1, due to the time dependence of θ_m, is equal to $\Delta P_1 = \dot{\theta}_m^2/(M^d)^2$, and therefore the adiabaticity condition takes the form

$$\dot{\theta}_m^2 \ll (M^d)^2 = 4\pi^2/l_m^2. \qquad (17)$$

The ratio $\dot{\theta}_m^2/(M^d)^2$ determines the accuracy of the adiabatic-approximation results.

The inequality (17) must hold at all instants of time, but it is most critical in the resonance layer from $(\rho_R - \Delta\rho_R)$ to $(\rho_R + \Delta\rho_R)$. In fact, on the one hand $\dot{\theta}_m$ contains besides $d\rho/dt$ (13) a factor that increases resonantly as $\rho \to \rho_R$, and on the other hand l_m is a maximum in the resonance layer: $l_m^R = l_\nu/\sin 2\theta$. In addition, it is precisely in the resonance layer that the strongest changes of the composition of the neutrino beam takes place.[4,6,7] Therefore, even if the condition (17) is violated outside the resonance layer, this will not lead to a substantial change of the results. At $\rho = \rho_R$ it follows from (13) that

$$\dot{\theta}_m^R = \frac{1}{2\rho_R \operatorname{tg} 2\theta}\frac{d\rho}{dt} = \frac{1}{2\Delta\rho_R}\frac{d\rho}{dt}.$$

Substituting $\dot{\theta}_m^R$ in (17), we obtain the adiabaticity condition for the resonance layer:

$$(2\Delta r_R)^2 \gg (l_m^R)^2/4\pi^2, \qquad (17a)$$

where

$$\Delta r_R = (d\rho/dt)^{-1}\Delta\rho_R$$

is the spatial width of the resonance layer. At $2\Delta r_R \geqslant l_m^R$ (Refs. 4, 6, 7) the inequality (17a) holds. Hence the gist of the adiabaticity condition: the density of the medium must vary so slowly that at least one oscillation length in the substance decreases in the layer.

4. NEUTRINO OSCILLATIONS IN THE ADIABATIC REGIME

We say that the oscillations occur in the adiabatic regime if the inequality (17) holds. The eigenstates ν_{im} of the neutrinos in the medium diagonalize in this case the system (14) and are thus approximately the eigenstates of the Hamiltonian. From (14) we get

$$|\nu_{im}(t)\rangle = |\nu_{im}\rangle \exp\left[-i\int_0^t dt'\, M_i^d(t')\right]. \qquad (18)$$

The features of the adiabatic regime are the following: 1) the eigenstates ν_{im} evolve independently and the transitions $\nu_{1m} \leftrightarrow \nu_{2m}$ can be neglected. 2) The "flavors" (ν_e, ν_μ composition) of the eigenstates ν_{im} change. We emphasize that in contrast to the case of constant ρ, when the (ν_e, ν_μ) composition of ν_{im} is constant, the "flavors" ν_{im} vary here in accordance with the variation of θ_m, and hence of the density of the medium. This leads to substantial changes of the ν-beam properties.

Consider the evolution of the eigenstates of weak interactions. Let electronic neutrinos be produced in a layer of density ρ_0:

$$\nu(t=0) = \nu_e, \qquad \rho(t=0) = \rho_0,$$

ρ_0 and the resonant density ρ_R, the latter fixed by the values of k, θ, Δm^2 (5), determine in accordance with (10) the mixing angle θ_m^0 in the medium at the instant $t=0$. This angle specifies the admixtures of the eigenstantes in the initial state of the neutrino:

$$\nu(t=0) = \nu_e = \cos\theta_m^0 \nu_{1m}(0) + \sin\theta_m^0 \nu_{2m}(0). \qquad (19)$$

Since the ν_{im} evolve independently in the adiabatic regime and the transitions $\nu_{1m} \leftrightarrow \nu_{2m}$ can be neglected, the admixtures ν_{1m} and ν_{2m} in a given neutrino state do not change with time. They are equal to the ν_{im} admixtures (19) at the initial instant. For an arbitrary instant it follows hence that

$$|\nu(t)\rangle = \cos\theta_m^0 |\nu_{1m}(t)\rangle + \sin\theta_m^0 |\nu_{2m}(t)\rangle. \qquad (20)$$

Substituting the time dependences of $\nu_{im}(t)$ known from (18) in (20), we obtain the evolution of the neutrino state

$$|\nu(t)\rangle = \cos\theta_m^0 \exp\left(-i\int_0^t M_1^d\, dt'\right)|\nu_{1m}\rangle + \sin\theta_m^0 \exp\left(-i\int_0^t M_2^d\, dt'\right)|\nu_{2m}\rangle. \qquad (21)$$

We determine now the probability $P(t)$ of observing ν_e in a state $\nu(t)$ at an instant t. To this end we must take into account the change of the "flavors" of the eigenstates. By definition,

$$\langle\nu_e|\nu_{1m}(t)\rangle = \cos\theta_m(t), \qquad \langle\nu_e|\nu_{2m}(t)\rangle = \sin\theta_m(t), \qquad (22)$$

where $\theta_m(t)$ is the mixing angle in the matter at the instant t. Taking (22) into account, we find the probability amplitude (accurate to a common phase factor)

$$\langle\nu_e|\nu(t)\rangle = \cos\theta_m^0\cos\theta_m + \sin\theta_m^0\sin\theta_m \exp\left[-i\int_0^t dt'\, M^d\right],$$

where $M^d \equiv M_1^d - M_2^d$. This gives the sought probability

$$P(t) = \bar{P}(t) + \frac{1}{2}A_P(t)\cos\int_0^t dt'\, M^d(t'), \qquad (23)$$

where the mean value is

$$\bar{P}(t) = \cos^2\theta_m^0\cos^2\theta_m + \sin^2\theta_m^0\sin^2\theta_m \qquad (24)$$

and the oscillation depth is

$$A_P(t) = \sin 2\theta_m^0 \sin 2\theta_m(t). \qquad (25)$$

The quasiperiod of the $P(t)$ oscillations is, according to (23),

$$T \approx 2\pi/M^d.$$

The probability of the $\nu_e \leftrightarrow \nu_e$ transition in the adiabatic regime is thus a quasiperiodic function that oscillates about a mean value $\bar{P}(t)$ with amplitude $A_P(t)$.

We consider now the properties of the oscillations in the adiabatic regime. First, the solution obtained for the probability is universal. The mean value $\bar{P}$ and the oscillation depth A_P are functions of one variable θ_m and of one parameter—the value of θ_m at the initial instant. The value of θ_m is determined by the density at the given instant of time. Therefore A_P and $\bar{P}(t)$ are functions of the local ρ, independent of the specific density distribution. The values of A_P are the same in layers having the same ρ; they are also independent of the oscillation period. The dependences of A and $\overline{P}$ on Δm^2, θ, E, and t are contained in θ_m and θ_m^0.

It is convenient to introduce in place of θ_m the dimensionless variable n which is directly related to the density[6]:

$$n=(\rho-\rho_R)/\Delta\rho_R. \quad (26)$$

Obviously, n is the deviation of the given density from the resonant value and is expressed in units of the half-width of the resonance layer ($n=0$ at resonance). The connection between n and θ_m follows from (10):

$$\sin 2\theta_m=(n^2+1)^{-1/2}. \quad (27)$$

The initial condition is fixed by the value of n_0, viz., $\sin 2\theta_m^0=(n_0^2+1)^{-1/2}$. Substituting (27) in (24) and (25) we get

$$\overline{P}=\frac{1}{2}\{1+n_0 n[(n_0^2+1)(n^2+1)]^{-1/2}\}, \quad (28)$$

$$A_P=[(n_0^2+1)(n^2+1)]^{-1/2}. \quad (29)$$

which coincide with the values published in Refs. 6 and 7, and are universal functions of n (Fig. 1). Let us examine their properties. The probability $\overline{P}$ is a monotonic function of n: it decreases monotonically with n at $n_0>0$ and increases monotonically at $n_0<0$. At $n_0=0$ we have $\overline{P}=1/2$ (creation at resonance). In the resonance layer itself $\overline{P}=1/2$ independently of the value of n_0. The oscillation depth is a maximum at resonance ($\rho=\rho_R$): $A_P^R=(n_0^2+1)^{-1/2}$ (Ref. 6) and decreases with increasing deviation of ρ from ρ_R. We emphasize that the oscillations with maximum depth A_P occur only in the resonance layer and only at $n_0=0$, i.e., when the initial density is equal to the resonance value.

As $|n_0|\to\infty$ the amplitude of the oscillations increases: $|A_P \propto 1/n_0$, and the neutrino propagation turns into a nonoscillatory conversion of one type of neutrino into the other.[6,7] The dependence of the average probability P on n converges to the asymptotic value $\overline{P}_a=\frac{1}{2}[1+n(n^2+1)]^{-1/2}$. The limit $|n_0|\to\infty$ corresponds to neutrino creation either in layers of high density, $\rho\gg\rho_R$ or at $\rho\approx 0$ and very small $\sin^2 2\theta$.

Let us determine the values of $\overline{P}$ and A_P at the exit from the object. In the adiabatic regime, according to (28) and (29), the depth of the oscillations and the average probability are determined only by the values of n at the neutron-generation point and at the boundary (n_0 and n_f). The values of A_P and $\overline{P}$ are independent of the density distribution, i.e., of the values of n at the intermediate instants of time. If both densitites ρ_0 and ρ_f are larger or smaller than ρ_R, then n_0 and n_f are of equal sign and $P_f>1/2$ in accordance with (28). If one of the densities ρ_0 and ρ_f is larger than ρ_R and the other is smaller, then n_0 and n_f are of opposite sign and $P_f<1/2$. The averaged suppression effect is >2 if the neutrinos cross the resonance layer. For actual applications, interest attaches to the case when the neutrinos are produced in dense layers and then go out into a region with $\rho\approx 0$. In this case $n_f=-1/\tan 2\theta$ and the oscillation parameters at the exit take the form

$$\overline{P}_f=\tfrac{1}{2}[1-n_0\cos 2\theta/(n_0^2+1)^{1/2}],\ A_P^f=\sin 2\theta(n_0^2+1)^{-1/2}. \quad (30)$$

The effect at the exit is determined only by the mixing angle in vacuum and by the initial condition n_0. The smaller θ and the larger n_0, the stronger the suppression of the flux of neutrinos of the initial type. At small and $n_0\to\infty$ we have $\overline{P}_f\approx 1/(4n_0^2)+\sin^2\theta\to\sin^2\theta$ (Refs. 4 and 6). The value $\overline{P}_f=\sin^2\theta$ corresponds to a nonoscillatory transition.

Note that after passing through a layer of matter, the oscillations in vacuum are described by parameters (30) that differ from the usual vacuum values. The reason is that what emerges from the object is a coherent neutrino state ν_f that does not coincide with the weak interaction eigenstate ν_e or ν_m. In particular, there for a nonoscillatory transition $\nu_f\approx\nu_i$ (ν_i—is a finite state with definite mass) there are no oscillations in the vacuum.

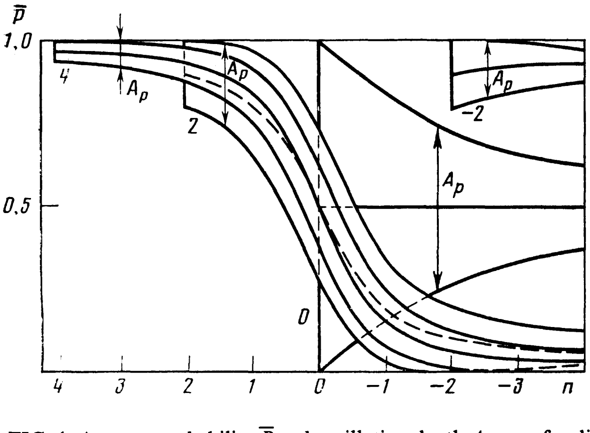

FIG. 1. Average probability $\overline{P}$ and oscillation depth A_P vs n for different initial conditions n_0 (numbers at the curves).

5. GRAPHIC DESCRIPTION OF OSCILLATIONS

The oscillations can be represented by a definite geometric picture that clearly duplicate the results above. Consider first the case of vacuum or of a medium of constant density. Assume that the initially generated neutrinos are ν_e: $\nu(0)=\nu_e$. The neutrino state at an arbitrary instant t can then be represented in the form

$$|\nu(t)\rangle=\cos\theta_m|\nu_{1m}\rangle+\sin\theta_m\exp[-i2\pi t/l_m]\,|\nu_{2m}\rangle \quad (31)$$

($\theta_m=\theta$, $\nu_{im}=\nu_i$, $l_m=l_\nu$ in vacuum). We have left out of (31) a common factor $\exp[-i(E_1 t-p_1 x)]$ that has no effect on the oscillations, so that the coefficient preceding $|\nu_{im}\rangle$ is independent of the time and is real. We introduce the orthonormal basis of vectors where $\{\mathbf{v}_m\}=\{\mathbf{v}_{1m}^R, \mathbf{v}_{2m}^R, \mathbf{v}_{2m}^I\}$, where $\mathbf{v}_{1m}^R$, $\mathbf{v}_{2m}^R$ and $\mathbf{v}_{2m}^I$ correspond to the real parts of the ν_{1m} and ν_{2m} wave functions and to the imaginary part of the ν_{2m} wave function. We set in correspondence with the arbitrary neutrino state $\nu(t)$ a unit vector $\mathbf{v}(t)$ in the basis $\{\mathbf{v}_m\}$ such that the components of $\mathbf{v}$ along a $\{\mathbf{v}_m\}$ axis are equal to the probability amplitudes of observing ν_{im} and $\nu(t)$. The state $\nu(t)$ (16) at the initial instant is described by the vector $\mathbf{v}(0)=\{\cos\theta_m, \sin\theta_m, 0\}$ at an arbitrary instant t:

$$\mathbf{v}(t)=\{\cos\theta_m, \sin\theta_m\cos(2\pi t/l_m), \ \sin\theta_m\sin(2\pi t/l_m)\}.$$

The evolution of $\nu(t)$ is equivalent to rotation of the vector about the axis $\mathbf{v}_{im}^R$, whereby $\mathbf{v}$ describes a cone with apex angle θ_m. The period of the rotation is equal to the period $T=l_m$ of the oscillations (Fig. 2a).

Electron and muon neutrinos correspond in the basis $\{\mathbf{v}_m\}$ to the vectors

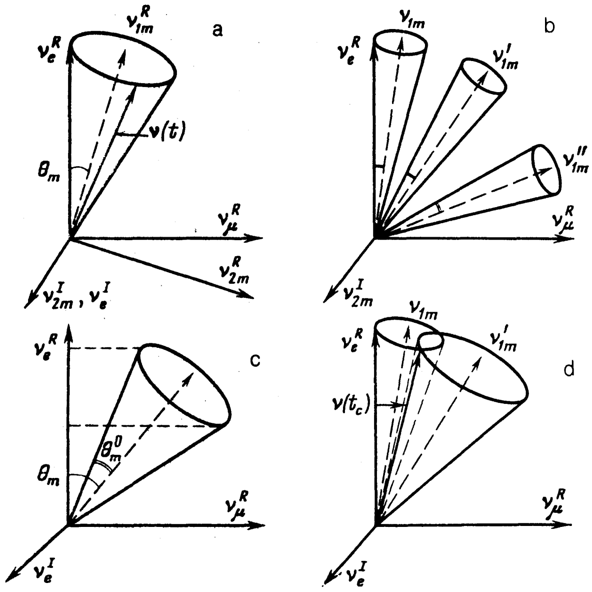

FIG. 2. Graphical representation of oscillations: a) case of vacuum or of a medium with constant density, b) adiabatic regime, c) graphic determination of $\overline{P}$ and A_P in the adiabatic regime, d) jumplike change of the density at the instant t_c.

$$\mathbf{v}_e=\{\cos\theta_m,\ \sin\theta_m,\ i\sin\theta_m\},$$
$$\mathbf{v}_\mu=\{-\sin\theta_m,\ \cos\theta_m,\ i\cos\theta_m\}.$$

The scalar product of $\mathbf{v}_e$ and $\mathbf{v}(t)$ determines the probability amplitude of observing ν_e in the state $\nu(t)$: $\langle\nu_e|\nu(t)\rangle = (\mathbf{v}_e\mathbf{v}(t))$. The change of the components of $\mathbf{v}$ along $\mathbf{v}_e$ and $\mathbf{v}_m$ in the course of rotation of the vector $\mathbf{v}$ about $\mathbf{v}_{im}^R$ duplicates the oscillatory picture.

To determine the depth of the oscillations A_P and the average probability $\overline{P}$ in the case of constant density and of the adiabatic regime (see below), it suffices to consider the real plane of $\{\nu_m\}$ and $\{\nu_\alpha\}$, $\alpha = e, \mu$. The values of A_P and P are

$$A_P=(\mathbf{v}_e{}^R\mathbf{v})^2_{max}-(\mathbf{v}_e{}^R\mathbf{v})^2_{min},$$
$$\overline{P}={}^1/_2[(\mathbf{v}_e{}^R\mathbf{v})^2_{max}+(\mathbf{v}_e{}^R\mathbf{v})^2_{min}], \tag{32}$$

where $(\mathbf{v}_e^R\mathbf{v})_{\max}$ and $(\mathbf{v}_e^R\mathbf{v})_{\min}$ are the maximum and minimum components of $\mathbf{v}$ along $\mathbf{v}_e^R$.

In a medium of variable density, θ_m is variable and accordingly the basis $\{\mathbf{v}_m\}$ rotates relative to the vectors $\mathbf{v}_e$ and $\mathbf{v}_m$. The relative positions of $\{\mathbf{v}_m\}$ and $\{\mathbf{v}_\alpha\}$ are uniquely fixed by the density at the particular instant. The relative positions of $\{\mathbf{v}_m\}$ and $\{\mathbf{v}_\alpha\}$ are uniquely fixed by the density at the particular instant. The motion of the vector $\mathbf{v}(t)$ depends then on the character (rate) of the change of the density with time.

In the adiabatic regime, the absence of $\nu_{1m}\leftrightarrow\nu_{2m}$ transitions and the invariance of the admixtures of ν_{im} in the neutrino state $\nu(t)$ correspond to conservation of the angle of rotation of $\mathbf{v}(t)$ relative to $\mathbf{v}_{im}^R$ (Fig. 2b). The evolution of ν in the adiabatic regime constitutes rotation of the cone axis (i.e., the rotation angle of $\mathbf{v}(t)$ remains unchanged and is equal to the angle between $\mathbf{v}$ and $\mathbf{v}_{im}^R$ at the initial instant. It is easy to obtain from this expressions for A_P and $\overline{P}$. The maximum and minimum components of $\mathbf{v}$ along $\mathbf{v}_e^R$ are equal to

$$(\mathbf{v}_e{}^R\mathbf{v})_{max}=\cos(\theta_m-\theta_m{}^0),\quad (\mathbf{v}_e{}^R\mathbf{v})_{min}=\cos(\theta_m+\theta_m{}^0)$$

(Fig. 2c). Substituting them in (32) we get (24) and (25). Note that here A_P and $\overline{P}$ do not depend on the phase of the rotation.

If the adiabatic condition is not met, the transitions $\nu_{1m}\leftrightarrow\nu_{2m}$ become important. The angle $\nu(t)$ of rotation around ν_{1m} is not conserved in this case. The limiting case is a jumplike change of the density at a certain instant $t_c\,(\rho\to\rho')$. At this instant the position of the vector $\mathbf{v}_{1m}^R$ changes abruptly, $\theta_m(\rho)\to\theta_m(\rho')$, and consequently $\mathbf{v}_{1m}^R\to\mathbf{v}_{1m}^{R'}$. At $t>t_c$, the vector $\mathbf{v}(t)$ will rotate about a new position $\mathbf{v}_{1m}^{R'}$, and the rotation angle will be equal to the angle between $\mathbf{v}$ and $\nu_{1m}^{R'}$ at the instant t_c (Fig. 2d). The values of $\overline{P}$ and A_P at $t\geqslant t_c$ depend on the phase of the system rotation phase at the instant t_c.[7] If the adiabatic condition is met, $\overline{P}$ and A_P are determined not only by $\theta_m(\rho)$ but also by the oscillation phase.

6. DIVERGENCE OF NEUTRINO WAVE PACKETS

The foregoing results correspond to a complete overlap of the wave packets that describe the motion of the eigenstates of the neutrinos in a medium. Just as in a vacuum, these states have different group velocities. The influence of the substance becomes noticeable over large spatial scales and in definite ranges of Δm^2. The wave packet divergence also becomes significant for these scales and ranges of Δm^2.

Let us find the group velocities v_i of the eigenstates ν_{im} in a medium of constant density:

$$v_i=\frac{dE_i}{dk}\approx\frac{d}{dk}(k+M_i{}^d(k))=1+\frac{dM_i{}^d(k)}{dk}.$$

Substituting here M_i^d from (8), we obtain for the arithmetic mean

$$^1/_2(v_1+v_2)=1-\frac{m_1{}^2+m_2{}^2}{2}\frac{1}{2k^2},$$

the same expression as in vacuum, and for the velocity difference we have

$$\Delta v^m=v_1-v_2=-\frac{\Delta m^2}{2k^2}\cos 2\theta\frac{(1-\rho/\rho_R)+\mathrm{tg}^2\,2\theta}{[(1-\rho/\rho_R)^2+\mathrm{tg}^2\,2\theta]^{1/2}}$$
$$=\Delta v^B\cos 2\theta[(-n+\mathrm{tg}\,2\theta)/(n^2+1)^{1/2}]. \tag{33}$$

Here $\Delta v^B = -\Delta m^2/2k^2$ is the difference between the group velocities in a vacuum. It follows from (33) that $|\Delta v^m|$ decreases as ρ approaches ρ_R. At resonance we have $\Delta v^m = \Delta v^B\sin 2\theta\ll\Delta v^B$. The quantity Δv^m vanishes at $\rho=\rho_R'=\rho_R/\cos^2 2\theta$ (ρ_R' is the resonance density for the $\nu_1\leftrightarrow\nu_1$ oscillations in the medium). At $\rho>\rho_R'$ the difference Δv^m reverses sign and becomes in the limit

$$\Delta v=(\Delta m^2/2k^2)\cos 2\theta\approx-\Delta v^B.$$

The cause of this result is that in a dense medium ($\rho\gg\rho_R$) the group velocity of ν_{1m} is determined by the mass m_2, and the group velocity of ν_{2m}, conversely, by m_1. At all values of ρ for the resonance channel ($\rho/\rho_R>0$) the interaction with the substance decreases $|\Delta v^m|:|\Delta v^m|<\Delta v^B|$, and consequently decreases the divergence of the neutrino packets.

We discuss now the effects of packet divergence for oscillations in the adiabatic regime. The states ν_{im} evolve in this regime independently, and we can introduce for them definite group velocities v_i. In contrast to a medium with constant ρ, now v_i and Δv^m are functions of the density. The divergence of the packets over a path L is then

$$D=\int_0^L dr\,\Delta v^m[\rho(r)].$$

For the resonance branch we have $|\Delta v^m|<|\Delta v^B|$, therefore D can be estimated at $|D|\leqslant|\Delta m^2|L/(2k^2)$. The packet divergence can be neglected if $D\ll c\tau$, where $c\tau$ is the length of the wave packet. At $D\gtrsim c\tau$ the divergence effects are substantial. In media with $\rho>\rho_R'$ and $\rho<\rho_R'$ the divergences have opposite sign. Therefore if a neutrino beam propagates in a substance having a monotonically varying density and crosses the resonance layer, the packet divergences ahead and past the resonance layer will cancel each other. A situation is possible wherein ν_{im} packets that have diverged in front of the layer $\rho=\rho_R$ become closer together behind this layer and the overlap is restored.

We consider now the case of fully divergent packets ($D\gg c\tau$). In this case the oscillation depth is $A_P=0$. The probabilities of observing ν_e in the packets ν_{1m} and ν_{2m} are respectively

$$P_1=\cos^2\theta_m{}^0\cos^2\theta_m,\quad P_2=\sin^2\theta_m{}^0\sin^2\theta_m.$$

The sum of these probabilities is

$$P_s=P_1+P_2=\cos^2\theta_m{}^0\cos^2\theta_m+\sin^2\theta_m{}^0\sin^2\theta_m.$$

Obviously, P_s coincides with the average probability $\overline{P}$ for fully overlapping packets. Thus, the effect $P(\nu_e\to\nu_e)$ summed over the packets ν_{im} when these packets had diverged, coincides with the averaged probability for total overlap. In experiments in which the signals are averaged and summed over rather large time intervals, these cases are indistinguishable.

If the divergence is not complete, the summed P_s also coincides with P, and the depth of the oscillations is proportional to the overlap of the packets:

$$A_P=A_P{}^0\frac{1}{\tau}\int_{|D|/c}^{\infty}dt\exp(-t/\tau)=A_P{}^0\exp(-|D|/c\tau).$$

Here A_P^0 is the depth of the oscillations in the case of total overlap.

If the adiabatic condition is not met, $\nu_{1m}\leftrightarrow\nu_{2m}$ transitions become significant, and the pattern of the packet divergence becomes more complicated.

7. REALIZATION OF ADIABATIC REGIME IN SPECIFIC OBJECTS. SUPPRESSION FACTOR

The adiabatic condition is met for a wide range of values of the parameters Δm^2 and $\sin^2 2\theta$ in the sun[4] and in envelopes and cores of collapsing stars.[6,7] The following are typical consequences of the oscillations.

The neutrinos created in dense central regions of the stars cross layers in which ρ ranges from $\rho_{\max}$ to $\rho_{\min}$. The densities $\rho_{\max}$ and $\rho_{\min}$ determine in accordance with (5) the resonance interval, i.e., the interval of $E/\Delta m^2$ for which the resonance condition is met inside the given object. We put $E/\Delta m^2=X$. Then $X_{\min}=m_N\cos 2\theta/2\sqrt{2}G_F\rho_{\max}$ is determined by the maximum (actually, central) density, $X_{\max}=m_N\cos 2\theta/2\sqrt{2}G\rho_{\min}$. Given Δm^2, ($X_{\min}$–$X_{\max}$) determines the interval of resonant energy. This is just the interval in which strong neutrino conversion should occur.

As $\rho_{\min}\to 0$ we have $X_{\max}\to\infty$. In this case the upper limit of the strong-conversion interval is determined either by the dimensions of the object (R) or by the adiabaticity condition. In fact, as X increases the length of the oscillations at resonance increases: $l_m^R=l_\nu/\sin 2\theta=4\pi X/\sin 2\theta$. Starting with a certain $X_a=\min(2\Delta r_R, R)$, where Δr_R is the spatial width of the resonance, the layer of matter becomes too thin for a strong conversion to occur.[4,10] As a rule, the adiabaticity condition imposes the stronger restriction. At $x>x_a$ the effect of the material weakens.

We shall describe the effect of the passage of the neutrino through matter by a suppression factor $P(X,\sin^2 2\theta)$. By definition, P establishes a connection between the neutrino-generation spectrum $F_0(E)$ and the spectrum at the exit

$$F_f(E)=P(X,\sin^2 2\theta)F_0(E)$$

(we assume here averaging over the oscillations). Let us discuss the properties of the suppression factor. We consider first a pointlike neutrino source in a region with

$$P(X,\sin^2 2\theta)=\overline{P}_f(X,\sin^2 2\theta),$$

where $\overline{P}_f$ is the average probability at the exit (see Sec. 4). At $X<X_a$, i.e., in the region of the adiabatic regime, the effect is determined by the initial condition for n_0 (30). The dependence on X is contained in ρ_R. Separating it explicitly, we can write the initial condition in the form

$$n_0=\frac{1}{\mathrm{tg}\,2\theta}\left(\frac{X}{X_0}-1\right),\qquad(34)$$

where $X_0=m_N\cos 2\theta/2\sqrt{2}G\rho_0$—this is the value of $E/\Delta m^2$ at which the resonance condition is met for the initial density ρ_0. Substituting (34) in (30) we get

$$P(X,X_0,\sin^2 2\theta)=\frac{1}{2}\left\{1-\frac{(X/X_0-1)\cos 2\theta}{[(X/X_0-1)^2+\mathrm{tg}^2\,2\theta]^{1/2}}\right\}.\qquad(35)$$

Let us discuss the result (35) (see Fig. 3). The entire X-interval of the strong effect of the substance can be divided into three parts:

A. Region with resonance on. At $X\ll X_0$ Eq. (35) leads to the vacuum result $P^B=1-\sin^2 2\theta/2$, $A_P=\sin^2 2\theta$. For these neutrinos there is no resonance layer in the object, and the effect of the substance is negligibly small. As X approaches X_0, $P(X)$ decreases, and the depth of the oscillations increases. $X=X_0$ is the point at which the resonance is turned on: $P=1/2$, the oscillations in the layer have maximum depth. With further increase of X, the probability P

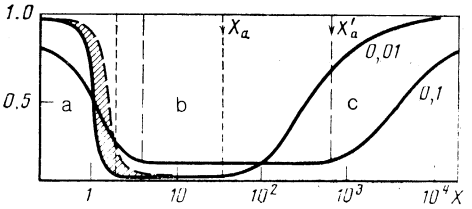

FIG. 3. Suppression factor for flux of neutrinos of initial type as a function of $X=E/\Delta m^2$ (in units of X_c) for different $\sin^2\theta$ (numbers at the curves). Dashed line—suppression factor averaged over the neutrino-source distribution.

continues to fall and flattens out at a level corresponding to maximum suppression: $P \approx \sin^2 \theta$. The width A of the region in which P changes from P' to $\sin^2 \theta$ is proportional to $\sin 2\theta$; the smaller the mixing in the vacuum, the narrower the region in which the resonance is on. The shape of the $P(X)$ curve recalls here the resonance curve.

B. Maximum suppression region. At $X \gg X_0 a$, the nonoscillatory regime conditions are realized: $A_P \approx 0$, the effect at the exit depends little on the initial conditions and is determined in practice only by the mixing angle: $P = \sin^2 \theta$. The value of X_a gives the upper limit of this region. X_a decreases likewise with decrease of θ and the region B becomes narrower.

C. Region of violation of adiabaticity: $X > X_a$. With increase of X, the value of P tends to unity and the form of $P(X)$ depends on the density distribution.

In real objects there exists a spatial distribution $N_\nu(r)$ of the neutrino sources, and the neutrinos are produced in layers with different initial densities $\rho_0(r)$. The suppression factor, with allowance for the source distribution, is equal to

$$\bar{P}(X, \sin^2 2\theta) = \int_0^R dr\, N_\nu(r) P(X, X_0(\rho_0(r)), \theta) \cdot 4\pi r^2,$$

where N_ν is normalized to

$$\int_0^R N_\nu(r) 4\pi r^2\, dr = 1.$$

Averaging over the source distribution leads to the following: 1) The factor P turns out to be larger than the factor $P(X, X_0, \theta)$ corresponding to production at the center with $\rho_0 = \rho_{\max}$. The left edge of the "tub" (Fig. 3) becomes less steep, and the region in which the resonance is on broadens. 2) P becomes dependent on the distribution of the density of the material. 3) The region of maximum suppression becomes narrower.

8. CONCLUSION

The adiabatic-approximation results were obtained by us in terms of the effective mixing and of the evolution of the eigenstates of the neutrinos in the medium. This method seems most convenient for the description of ν oscillations and, in particular, for the transition to the case of oscillations in vacuum. There are also other methods. The evolution in the neutrino system can be tracked by considering directly the variation of the matrix $\hat{M}$ with density.[9] Instead of equations for the eigenstates one writes here an equation for the operator that determines the change of ν_{im}. This method was used in Ref. 9 to obtain an expression for $\bar{P}_f$ at the exit.

The average probability can be obtained from the differential equation for P.[6] Omitting from Eq. (1) of Ref. 6 the terms containing derivatives of order higher than the first, we get

$$M(M^2 + 4\bar{M}^2)\dot{P} - 2\bar{M}^2 \dot{M}(2P - 1) = 0.$$

Integration of this equation with the initial condition $\bar{P} = 1 - \sin^2 2\theta_m^0/2$ leads to expression (24).

Let us summarize our results.[1)]

1) If the density of the matter changes slowly enough, the neutrino oscillations take place in the adiabatic regime.

2) The adiabaticity condition is met for large intervals of Δm^2 and $\sin^2 2\theta$ in the sun and in the cores and envelopes of collapsing stars.

3) A feature of adiabatic neutrino propagation is that the eigenstates of the neutrinos in the medium evolve independently, and the transitions between them are negligibly small. The eigenstate admixtures in a given neutrino state do not change in the course of propagation and are equal to their values at the initial instant of time.

4) Neutrino oscillations in the adiabatic regime can lead to an almost complete conversion of one type of neutrino to the other in a wide energy interval. This interval is determined by the density differential and by the adiabaticity condition. The smaller the vacuum mixing, the stronger the attainable suppression of the flux of neutrinos of the initial type. This is the fundamental difference from the case of oscillations in vacuum.

6) In the case of divergence of the wavepackets ν_{im} in the adiabatic regime, the probability P of observing neutrinos of the initial type, summed over the packet, is equal to the averaged probability $\bar{P}$ for the case when there is no divergence of the packets.

The authors thank V. S. Berezinskiĭ and G. T. Zatsepin for helpful discussions of this work.

[1)]After sending this article to press, we obtained Ref. 12, in which equations for neutrino eigenstates are likewise considered and an averaged probability at the exit is obtained, according with our result in Ref. 7 and Eqs. (30) of the present article.

[1]B. M. Pontecorvo, Zh. Eksp. Teor. Fiz. **33**, 549 (1958) [Sov. Phys. JETP **6**, 429 (1959)].
[2]L. Wolfenstein, Phys. Rev. **D17**, 2369 (1978); *ibid.* **D20**, 2643 (1979).
[3]V. Barger, K. Shisnant, S. Pakvasa, and R. J. N. Phillips, *ibid.* **D22**, 2718 (1980).
[4]S. P. Mikheev and A. Yu. Smirnov, Yad. Fiz. **42**, 1441 (1985) [Sov. J. Nucl. Phys. **42**, 913 (1985)].
[5]H. Bethe, Phys. Rev. Lett. **56**, 1305 (1986).
[6]S. P. Mikheev and A. Yu. Smirnov, Zh. Eksp. Teor. Fiz. **91**, 7 (1986) [Sov. Phys. JETP **64**, 4 (1986)].
[7]S. P. Mikheyev and A. Yu. Smirnov,Proc. VI-th Moriond Workshop on Massive Neutrinos in Particle Physics and Astrophysics, Tignes, France, Jan. 25–Feb. 1, 1986, Ed. Frontieres, p. 355.
[8]S. P. Rosen and J. P. Gelb, *ibid.*, p. 1.
[9]A. Messiah, *ibid.*, p. 373. Preprint Saclay PhT/86-46, 1986.
[10]J. Bouchez, M. Gribier, J. Rich. M. Spiro, *et al.*, Preprint Saclay DPhE/86-10, 1986.
[11]V. K. Ermilov, V. A. Tsarev, and V. A. Chechin, Preprint FIAN-45 (1986).
[12]V. Barger, R. J. N. Phillips, and K. Whisnant, Preprint, Univ. of Wisconsin, Madison, MAD/PH/280, April 1986.

Translated by J. G. Adashko