Resonant-radiation transfer with partial redistribution in frequency

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Resonant-radiation transfer with allowance for natural, Doppler, and collision broadening is considered. An equation for the spectral density of excited atoms is derived in the spatial diffusion approximation. This equation is asymptotically accurate in the case of almost coherent scattering under the conditions $k_D R\gamma/\Delta\omega_D \gg 1$ and $\gamma_c/\gamma \ll 1$. In the opposite limiting case of total redistribution in frequency, it is shown that this approach leads to results accurate to 30%, and can therefore be used with all the parameters of the problem. The de-excitation rate and the density of the excited atoms in volumes of finite size are calculated in this approximation, and a solution of the stationary problem for a uniform semi-infinite layer is obtained. The results are compared with experimental and numerical simulation data.

1. INTRODUCTION

Radiation transfer with complete redistribution in frequency has been intensively investigated, following the classical studies of Biberman and Holstein,[1,2] and is by now well understood. Matters are different when partial redistribution in frequency becomes significant. In particular, great theoretical interest attaches to the case when the wing of the spectral line in which the volume becomes optically transparent is determined by the natural broadening. Since the Dopper broadening usually exceeds noticeably the natural one, the volume must be of sufficient optical thickness for this purpose. At the same time, the pressure must be low enough to prevent collisions from causing total loss of coherence by scattering. The physical conditions under which such a situation can develop are typical of rapid high-temperature processes, when a plasma contains high-multiplicity waves for which the natural broadening can exceed the broadening by pressure, and of astrophysical applications, characterized by low densities and large object sizes.

It was shown in a number of papers, by direct numerical solution of the transport equation[3,4] and by Monte Carlo simulation of the problem,[5,6] that partial redistribution influences strongly the electric properties of the plasma. Astrophysical observations (see the bibliography in Ref. 7) and laboratory measurements of the de-excitation rate of the argon 1048 Å line[8] confirm these conclusions. Many radiation transfer data obtained from numerical and physical experiments are thus available at present. There is, however, no theory capable of yielding reasonable estimates of, e.g., the de-excitation time, the density of the excited atoms, and the spectrum of the output radiation, which would permit comparison of results obtained by various workers for resonant media with differing parameters. The aim of the present paper is to show that resonant-radiation transfer with partial redistribution in frequency has features that permit simple relations to be obtained for the parameters of this problem.

The theory proposed is based on the obvious fact that if the scattering is almost coherent (precisely the situation realized in the natural-broadening wing) the photon leaves the volume before its frequency reaches the optically transparent wing of the spectral line. Thus, if the characteristic frequency of the outgoing photons is designated x_{eff}, the optical thickness

$$\tau_{eff}=k(x_{eff})R\gg 1, \tag{1.1}$$

where R is the characteristic dimension of the volume and $k(x)$ is the absorption factor. Consequently, in contrast to total redistribution in frequency, for which $\tau_{\text{eff}}\sim 1$ (Refs. 1 and 2), the problem has a large parameter that permits diffusion theory to be used.

Incoherence of the scattering, even if small, plays a fundamental role, since it changes the photon frequency, thereby affecting the rate of its diffusion. We consider two mechanisms that alter the frequency of a scattered photon: the Doppler effect and collision broadening. They are essentially different: the Doppler effect in the far wing of the line leads to a gradual change of the photon frequency (diffusion into the wing of the spectral line[9]), whereas collisions cause infrequent jumplike changes of the frequency within the limits of the spectral line.

Equations describing the excited atoms in the volume are derived in Sec. 2 on the basis of the foregoing phyical principles. In Sec. 3 is considered emission of resonant radiation from volumes of finite dimensions, and equations are derived for the de-excitation rate and for the total number of excited atoms in the volume. These equations are valid both in the region of almost-coherent radiation (1.1) and for total distribution in frequency. The results are compared with the calculated experimental data. The stationary problem for a semi-infinite space is considered in Sec. 4 for almost coherent scattering.

2. DERIVATION OF FUNDAMENTAL EQUATIONS

The equations that describe the transfer of resonant radiation with incomplete redistribution in frequency, in terms of the distribution function in frequency and the spectral density, are quite well known.[7] A formal quantum-mechanical derivation of these equations under various conditions can be found in Refs. 10 and 11. Taking this circumstance into account, we introduce the needed relations, using simple qualitative considerations. The dimensionless frequency x is equal to the ratio of the frequency to the Doppler width. The absorption factor is in this case

$$k(x)=\pi^{1/2}k_D\varepsilon(x), \tag{2.1}$$

$$\varepsilon(x)=\frac{a}{\pi^{1/2}}\int_{-\infty}^{\infty}du\frac{e^{-u^2}}{a^2+(x-u)^2},\quad a=\frac{\gamma+\gamma_c}{2\Delta\omega_D},$$

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where k_D is the absorption factor in the Doppler core of the spectral line, γ is the rate of spontaneous emission, and τ_c is the frequency of the broadening collisions. If $a \ll 1$ and x_0 is defined by

$$e^{-x_0^2}=a/\pi^{1/2}x_0^2, \tag{2.2}$$

it can be easily seen that

$$k(x)=k_D e^{-x^2}, \quad x\ll x_0, \tag{2.3}$$

$$k(x)=k_D a/\pi^{1/2}x^2=k_L/x^2, \quad x\gg x_0. \tag{2.4}$$

To describe the scattering of a photon under nonstationary conditions, a time-dependent distribution function $L(x,x',t)$ must be introduced and taken to be the probability that an atom that had absorbed a photon of frequency x' is capable of emitting it after a time t at a frequency x. Obviously,

$$\int_{-\infty}^{\infty} L(x,x',t)\,dx=1.$$

Note that by virtue of the Heisenberg uncertainty principle this quantity is meaningful, for times on the order of the excited-atom lifetime, only for frequencies much higher than the natural broadening, but in the present case this is of no importance, since the transfer is in the far wing of the spectral line. Under these conditions it is reasonable to assume that

$$L(x,x',t)=\varepsilon(x)+[L_{coh}(x,x')-\varepsilon(x)]\exp(-\gamma_c t), \tag{2.5}$$

where L_{coh} is a Doppler-effect-related redistribution function for scattering that is coherent in the rest system of the atom. In (2.5), γ_c is the frequency of the collisions that lead to loss of coherence. It will be shown in Sec. 3 that the Doppler effect is significant only in the case when the collision frequency is low; it can therefore be assumed in the calculation of $L_{\text{coh}}(x,x')$ that the excited atom does not change velocity during its lifetime. The corresponding expression for $L_{\text{coh}}(x,x')$ can be found in Refs. 7 and 12. Averaging $L(x,x',t)$ over the lifetime of the excited atom, we obtain in accordance with Ref. 12

$$L(x,x')=(1-P_c)L_{coh}(x,x')+P_c\varepsilon(x), \tag{2.6}$$

where $P_c = \gamma_c/(\gamma+\gamma_c)1$ is the broadening-collision probability, while $L(x,x')$ has the obvious meaning of the probability that a photon absorbed at the frequency x' is reradiated at the frequency x, a concept widely used under stationary conditions.

To describe the excited atoms, we introduce the spectral density $n(\mathbf{r},x,t)$ such that $\gamma n(\mathbf{r},x,t)d\mathbf{r}dxdt$ is equal to the number of atoms that absorb a photon of frequency x at the point $\mathbf{r}$ and at the instant of time t. Obviously, $n(\mathbf{r},x,t)$ is proportional to the radiation intensity averaged over the angles and multipled by the absorption factor. The density of the excited atoms at the point $\mathbf{r}$ and at the instant t, produced by absorbing a photon of frequency x, is equal to

$$\gamma\int_0^{\infty}\exp(-\gamma t')\,n(\mathbf{r},x,t-t')\,dt'$$

and is practically equal to $n(\mathbf{r},x,t)$ in the case of strong confinement, when the characteristic times are long compared with the spontaneous de-excitation time. In the case of strong confinement we can thus, by calculating $n(\mathbf{r},x,t)$, obtain the total density of the excited atoms:

$$N(\mathbf{r},t)=\int_{-\infty}^{\infty} dx\, n(\mathbf{r},x,t). \tag{2.7}$$

The spectral density $n_s(\mathbf{r},x,t)$ of excited atoms is traditionally regarded as the number of atoms capable of emitting a photon of frequency x at the instant of time t (source function). Clearly,

$$n_s(\mathbf{r},t)=\int_{-\infty}^{\infty}dx'\int_0^{\infty}dt'\exp(-\gamma t')L(x,x',t')\,n(\mathbf{r},x',t-t'). \tag{2.8}$$

In this notation, assuming that isotropic scattering and that atoms are excited only by photon absorption, we can write down the equation

$$n(\mathbf{r},x,t)=\int_V H(\mathbf{r}-\mathbf{r}',x)\,n_s(\mathbf{r}',x,t)\,d\mathbf{r}', \tag{2.9}$$

where

$$H(\mathbf{r}-\mathbf{r}',x)=\frac{k(x)\exp[-k(x)\,|\mathbf{r}-\mathbf{r}'|\,]}{4\pi|\mathbf{r}-\mathbf{r}'|^2}. \tag{2.10}$$

Generally speaking, for a complete description of the excited atom it is necessary to include its velocity among the spectral-density parameters. If, however, unexcited atoms uniformly distributed in velocity absorb photons of frequency x, the distribution function of the excited atoms in velocity can be obtained from the equation

$$f(\mathbf{v})=\frac{a}{\pi^{1/2}\varepsilon(x)}\,\frac{\exp(-v_\parallel^2)}{a^2+(x-v_\parallel)^2}\,f_0(v_\perp),$$

where $v_\parallel$ and $v_\perp$ are the longitudinal and transverse components of the dimensionless velocity of the atom relative to the photon propagation direction, and $f_0(v_\perp)$ is the equilibrium distribution function. With due account of this distribution function in the calculation of $L(x,x',t)$, Eq. (2.9) holds for the spectral density $n(\mathbf{r},x,t)$ integrated over the velocity. Equation (2.9) with allowance for (2.8) is integral in time, frequency, and space. In the case of total redistribution, when $L(x,x',t)=\varepsilon(x)$, it leads readily to the Biberman-Holstein equation for the density $N(\mathbf{r},t)$ of the excited atoms. In general, however, further analysis of this equation is possible only for strong confinement. Recognizing that the characteristic times of the problem are long, we can expand $n(\mathbf{r},x,t-t')$ in powers of $t-t'$ and obtain

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\tau(x,x')\,n(\mathbf{r},x',t)\,dx'=-n(\mathbf{r},x,t)$$
$$+\int_V d\mathbf{r}'\int_{-\infty}^{\infty}dx'\,H(\mathbf{r}-\mathbf{r}',x)\,L(x,x')\,n(\mathbf{r}',x',t), \tag{2.11}$$

where

$$\tau(x,x')=\int_0^{\infty} te^{-\gamma t}L(x,x',t)\,dt.$$

In the derivation of (2.11) it was taken into account that allowance for the difference between the integral operator H and a unit operator would be an exaggeration of the accuracy. Next, in the case of strong confinement, the spectral line can be divided in two parts. The region $x \ll x_{\text{eff}}$ is the core in which almost all the excited atoms are concentrated and from which there is no direct radiation. For this region we have

$$n(\mathbf{r}, x, t) \approx N(\mathbf{r}, t)\varepsilon(x).$$

The second part of the spectral lines contains an extremely small number of excited atoms, but it is they which contribute to the radiation. Naturally, the photons can cross over in the course of reradiation from one region to the other, and the radiation transfer is in fact determined by the rate of "additional pumping" from the core of the spectral line to the wing. Let us consider Eq. (2.11) from this point of view. The derivative with respect to time can be discarded in the wing of the spectra line ($x \sim x_{\text{eff}}$), since the additional pumping from the core certainly exceeds this term in the case of strong confinement, and we can expand in terms of the spatial variables, i.e, change to the diffusion approximation:

$$\frac{1}{3k^2(x)}\Delta n(\mathbf{r}, x) - n(\mathbf{r}, x) + \int_{-\infty}^{\infty} dx' L(x, x')\, n(\mathbf{r}, x') = 0, \tag{2.12}$$

where Δ is the spatial Laplacian. It was recognized in the derivation of (2.12) that in the case of almost coherent scattering it would be an exaggeration of the accuracy to distinguish between the operator L and a unit operator. The solution of (2.12) must be matched to the solution in the region of the core, i.e., the following boundary condition must be met at $x \ll x_{\text{eff}}$:

$$n(\mathbf{r}, x) = N(\mathbf{r}, t)\varepsilon(x). \tag{2.13}$$

In addition, at the boundaries of the volume we have

$$n(\mathbf{r}, x)\big|_{\text{on boundary}} = 0. \tag{2.14}$$

The spectral density in the wing depends on time only via the boundary condition (2.13); the corresponding argument was therefore omitted.

Assuming Eq. (2.12) to be solved, we can obtain from (2.11), but integrating with respect to x a closed equation for $n(\mathbf{r}, t)$:

$$\frac{1}{\gamma}\frac{\partial N(\mathbf{r}, t)}{\partial t} = \Delta \int_{-\infty}^{\infty} dx \frac{n(\mathbf{r}, x)}{3k^2(x)} - \alpha N(\mathbf{r}, t) + F(\mathbf{r}, t). \tag{2.15}$$

This equation includes the quenching of the excited atoms by inelastic collisions and the nonradiative atom-excitation sources, which were left out of the preceding equations for simplicity. The cause of their omission from (2.12) is the same as for the terms with time derivatives. It is assumed that $\alpha \ll P_c$, so that no changes of P_c are needed in (2.6).

The most complicated component of Eq. (2.12) is the term with $L_{\text{coh}}(x, x')$; if, however, the condition

$$x_{eff} \gg x_0 \tag{2.16}$$

is met, this term can be radically simplified. The frequency change following each repetition of the radiation is then relatively small and it can be assumed that

$$\int_{-\infty}^{\infty} L_{coh}(x, x')\, n(x')\, dx'$$

$$\approx n(x) + \frac{\partial}{\partial x}\left\{-\overline{\Delta x}\, n(x) + \frac{\overline{\Delta x^2}}{2}\frac{\partial n(x)}{\partial x}\right\}, \tag{2.17}$$

where $\overline{\Delta x}$ and $\overline{\Delta x^2}$ are the mean and mean squared changes of the frequency in the radiation-repetition process. Since under condition (2.16) the absorption depends little on the atom velocity, we have

$$\overline{\Delta x^2} = 1. \tag{2.18}$$

There is no need for special calculation of $\overline{\Delta x}$, for it is clear that

$$\int_{-\infty}^{\infty} L_{\text{coh}}(x, x')\, \varepsilon(x')\, dx' = \varepsilon(x), \tag{2.19}$$

$$\overline{\Delta x} = -1/x.$$

In this case Eqs. (2.12) and (2.15) take the form

$$\frac{1}{3k_L^2}\Delta f(\mathbf{r}, x) + \frac{1}{2}\frac{\partial^2}{\partial y^2} f(\mathbf{r}, x) + \frac{P_c}{x^4}\{f(\mathbf{r}, x) - N(\mathbf{r}, t)\} = 0, \tag{2.20}$$

$$\frac{1}{\gamma}\frac{\partial N(\mathbf{r}, t)}{\partial t} = \frac{2a}{3\pi k_L^2}\Delta \int_0^{\infty} f(\mathbf{r}, x)\, dy - \alpha N(\mathbf{r}, t) + F(\mathbf{r}, t). \tag{2.21}$$

In the derivation of (2.21), account is taken of the fact that the Doppler redistribution plays a role only at small $P_c \sim \tau_{\text{eff}}^{-2}$ (see Sec. 3), therefore $1 - P_c \approx 1$. New variables are introduced in (2.20) and (2.21), such that

$$n(\mathbf{r}, x) = \varepsilon(x) f(\mathbf{r}, x) \approx (a/\pi x^2) f(\mathbf{r}, x), \quad y = x^3/3.$$

To simplify the equations, x and y are used here jointly. It is necessary to satisfy for (2.20) the zero boundary conditions (2.14) and the condition (2.13), which takes the form

$$f(\mathbf{r}, 0) = N(\mathbf{r}, t). \tag{2.22}$$

A few remarks are in order in connection with the boundary conditions. In the presence of broadening collisions, condition (2.22) is automatically satisfied for Eq. (2.20). Assume that there are no collisions. The second term in (2.20) describes then diffusion of photons into the wing of the spectral line, and the boundary condition (2.22) can be replaced by a source having zero frequency and due to the additional pumping from the spectral-line core. Clearly, the power of this source is equal to the rate of change of the density of the excited atoms in the core, a density not offset by quenching and by nonradiative excitation:

$$Q = -\partial N/\partial t + F - \alpha N. \tag{2.23}$$

This shows that, without solving (2.21) beforehand, the source power can be obtained only in the stationary case in the absence of quenching. This situation was investigated in Ref. 13, and Eq. (2.20) with a source $Q = F$ and with $P_c = 0$ is equivalent to the corresponding equation in Ref. 13. Note that the diffusion approximation, although in another physical situation, is discussed also in Ref. 11.

Zero conditions at the volume boundary lead to correct values of the spectral density in the entire volume, apart from a thin boundary layer with thickness on the order of $k^{-1}(x_{\text{eff}})$. In view of the small thickness of this layer, it can be assumed that its outgoing radiation is scattered coherently. Incoherent scattering has been well studied[7] and it is known that if z is the distance to the boundary, the solution near the boundary can be obtained from a solution that satisfies the zero boundary conditions by replacing z with

$$z + {}^2/_3 k^{-1}(x). \tag{2.24}$$

3. DE-EXCITATION RATE AND NUMBER OF EXCITED ATOMS IN THE VOLUME

If there are no extraneous excitation or quenching sources, it can be assumed, for a long time lapse after the initial excitation, that

$$n(\mathbf{r},\, x)=N_0 n(x)\varphi_0(\mathbf{r})\exp(-\beta\gamma t). \tag{3.1}$$

Here $\varphi_0(\mathbf{r})$ is an eigenfunction of the Laplace operator, and β is the de-excitation rate of interest to us. After determining $n(x)$, the de-excitation rate can be obtained from the equation

$$\beta=\frac{\lambda_0}{3R^2}\int_{-\infty}^{\infty}\frac{n(x)\,dx}{k^2(x)}, \tag{3.2}$$

where λ_0 is the corresponding eigenvalue of the Laplace operator and R is the characteristic dimension of the volume.

Expression (3.2) was obtained on the basis of condition (1.1), which is known to be violated in the case of total redistribution in frequency. It does, however, yield reasonable results even in this case. Putting $L(x,x')=\varepsilon(x)$ in (2.12), we obtain with the aid of (3.2)

$$\beta=\int_{-\infty}^{\infty}\frac{\varepsilon(x)\,dx}{1+3k^2(x)R^2/\lambda_0}. \tag{3.3}$$

The calculations for the Doppler and dispersion profiles in the case of strong confinement show that (3.3) gives the correct dependence on the optical thickness, with a coefficient overestimated by approximately 10%, 30%, and 50% for spherical, cylindrical, and planar geometry, respectively. The cause of this result is that $\tau_{\text{eff}}\sim 2(\lambda_0/3)^{1/2}\gtrsim 2$, for total redistribution, as can be seen from (3.3), and it is this which makes the diffusion-approximation results for the spectal density reasonable. To avoid misunderstandings, we note that from Eqs. (2.12) and (2.15) it is impossible to obtain an equation of the diffusion type for $N(\mathbf{r})$. Substitution of the solution (2.12) in (2.15) makes such an equation integral, taking thereby into account the large-scale correlations between remote points of the volume. We can thus assume that Eqs. (2.12) and (2.15) yield a qualitatively correct result in the entire range of variation of τ_{eff}, and in fact the correct one if (1.1) is satisfied.

We turn to the calculation of β for a pure Dopper redistribution. Putting $P_c=0$ in (2.20) we get

$$f(x)=\exp[-{}^1/_3(x/x_{eff})^3], \tag{3.4}$$

where

$$x_{eff}=(3/2\lambda_0)^{1/6}(k_L R)^{1/3}. \tag{3.5}$$

With the aid of (3.2) we obtain

$$\beta_{dif}=(2\lambda_0/3\pi)^{1/2}(1/k_D R). \tag{3.6}$$

The condition (1.1) with x_{eff} from (3.5) takes the form

$$\tau_{eff}=(k_L R)^{1/3}\gg 1. \tag{3.7}$$

The condition for the validity of approximation (2.16) for the frequency is somewhat more stringent:

$$(k_L R)^{1/3}\gg x_0. \tag{3.8}$$

Expression (3.6) with $k_L R\ll 1$ does not describe a transition to a total distribution in frequency. Furthermore, it can be concluded from the discussion of Eq. (3.3), that the reason for this is that at not very high optical densities the diffusion description cannot be used for the frequency variation due to the Doppler effect. This problem can be solved qualitatively in the following manner. We assume in accordance with Ref. 14 that the scattering at $x<x_0$ is accompanied by total redistribution in frequency. Regarding $k(x)$ to be given for this case by (2.3), we can obtain from (2.12) the spectral density at $x<x_0$. Taking $k(x)$ to be determined that at $x>x_0$ by Eq. (2.4), and the diffusion appoximation in frequency to be invalid, we obtain from (2.20) the value of $n(x)$ in this region. Matching these solutions at the point x_0, we obtain for $n(x)$ an expression that is qualitatively correct in both regions. Integrating in (3.2), we get

$$\beta_D=\frac{2}{\pi}\operatorname{arctg}(\eta)\beta_H+\frac{1}{1+\eta}\beta_{dif}, \tag{3.9}$$

where

$$\eta=\frac{\lambda_0}{3}\left(\frac{x_0{}^2}{k_L R}\right)^2,$$

β_{dif} is determined from (3.6), and β_H from (3.3) with $k(x)$ from (2.3). If β_H in (3.9) is replaced by the exact value given by Holstein[2] for the de-excitation rate, Eq. (3.9) becomes exact in two limiting cases—total redistribution in frequency and almost-coherent radiation—and describes approximately correctly the transition between them.

If the principal role is assumed by collisional redistribution in frequency, we get, omitting from (2.20) the term with the derivative,

$$f(x)=[1+(x/x_{eff})^4]^{-1}, \tag{3.10}$$

$$x_{eff}=(3P_c/\lambda_0)^{1/4}(k_L R)^{1/2}, \tag{3.11}$$

$$\beta_c=(\lambda_0/3)^{1/4}(P_c^{3/4}a/(2k_L R)^{1/2}). \tag{3.12}$$

The condition (1.1) for the validity of the diffusion theory takes the term

$$P_c^{1/2}\ll 1. \tag{3.13}$$

Equation (3.12), however, is a good approximation if $P_c\sim 1$.

If both redistribution mechanisms must be taken into account in (2.20) simultaneously, it can be shown that

$$\beta=\beta_{dif}U(\xi), \tag{3.14}$$

where

$$\xi=P_c\left[\frac{12(k_L R)^2}{\lambda}\right]^{1/3}\sim P_c\tau_{eff}^2.$$

The quantity ξ is the criterion that determines the principal redistribution mechanism. The two mechanisms are comparable at

$$P_c\approx\tau_{eff}^{-2}\ll 1. \tag{3.15}$$

The principal role is played by the Doppler redistribution at $\xi\ll 1$ and by the collisional distribution at $\xi\gg 1$, with $U(\xi)=1$ and $U(\xi)=\pi(\xi/4)^{3/4}$, respectively. The calculated $U(\xi)$ are show in Fig. 1, which shows also that, accurate to 25%,

$$U(\xi)=1+\pi(\xi/4)^{3/4}.$$

It should be noted that an expression of this type was

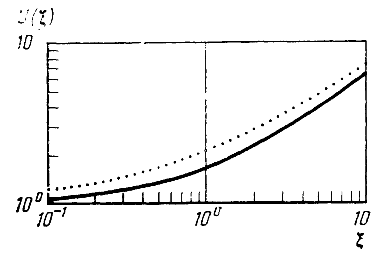

FIG. 1. De-excitation rate (in dimensionless units) with allowance for the Doppler and collisional redistributions in frequency.

first obtained by A. N. Starostin (unpublished) from qualitative estimates of the photon-emission probability.

Let us compare the results with the experimental data on the rate of emission of the 1048 Å line of argon[8] from a cylindrical volume of radius 1.1 cm in the pressure interval $P = 0.0015$–10 Torr. The spontaneous emission rate is $5.5 \cdot 10^8\ \text{sec}^{-1}$.[15] The collision broadening is due to resonant collisions with unexcited atoms. According to Ref. 16, $\gamma_c = 0.046 \gamma N_0 \lambda^3$. Under these conditions, the essential parameters of the problem are: $k_D R = 4 \cdot 10^4 P$, $P_c = 1.74P/(1 + 1.74P)$, $a = 1.2 \cdot 10^{-2}(1 + 1.74P)$, $\tau_{\text{eff}} = 10P^{1/3}$, $\lambda_0 = 5.8$, $x_0 = 2.6$ (P is the pressure in torr). It is seen from Fig. 2 that the de-excitation rate determined from Eqs. (3.9) and (3.14) agrees well with the experimental data in the entire pressure range. The same figure shows for comparison the excitation rates assuming total redistribution.

With an aim at estimating the accuracy of Eq. (3.9), we consider a problem that has been numerically well investigated, that of the number of excited atoms in a planar layer $2L$ thick sustained by a stationary source located at the center of the layer, and take into account only the Doppler redistribution infrequency. The number of exicted atoms can be easily expressed as a series in the eigenfuctions of Eq. (215). Note that in this approximation the eigenfunctions coincide with those of the Laplace operator, and the eigenvalues can be obtained from (3.9) by replacing λ_0 by λ_1. In this case, however, simple estimates show that the first term alone gives the result accurate to $\sim 10\%$. Thus,

$$N = 4/\pi\beta_D, \tag{3.16}$$

where β_D is determined by Eq. (3.9), in which R must be replaced by L, $\beta_H = 0.53/k_D L(\ln k_D L)^{1/2}$ (Ref. 2), and β_{dif} is determined from (3.6) with $\lambda_0 = (\pi/2)^2$. The dependence on the optical thickness, calculated from (3.16), is shown in Fig. 3 and compared there with the results of a numerical calculation.[4] It can be seen from the figure that Eq. (3.9) is fairly accurate at all optical thicknesses, and the result at sufficiently large optical thicknesses is practically equal to that of Ref. 13.

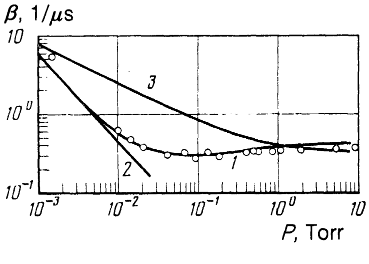

FIG. 2. De-excitation rate of 1048 Å line of argon: Curves—1—author's results, 2—total redistribution with allowance for only the Doppler effect, 3—total redistribution with allowance for only natural and collisional broadening. Points—experimental results.[8]

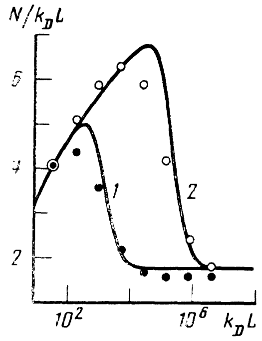

FIG. 3. Integrated density of excited atoms in a planar layer: curve 1—$a = 4.7 \cdot 10^{-3}$, 2—$a = 4.7 \cdot 10^{-5}$; ○, ●—results of numerical calculation.[4]

4. STATIONARY PROBLEM FOR SEMI-INFINITE SPACE

We assume that in a semi-infinite space, the quenching rate α and the rate of nonradiative excitation $D = \alpha$ are independent of the coordinate z measured from the surface into the interior of the volume.

For a Dopper distribution in frquency, the problem has the following frequency and distance scales:

$$x_{eff} = (a/\pi\alpha)^{1/3}, \tag{4.1}$$

$$R_{eff} = (2/3\pi)^{1/2}(1/\alpha k_D). \tag{4.2}$$

The condition (1.1) takes then the form

$$\tau_{eff} = (^2/_3)^{1/2}(a/\pi\alpha)^{1/3} \gg 1. \tag{4.3}$$

The solution of (2.20) and (2.21) shows that

$$f(z, y) = \frac{2}{\pi}\int_0^\infty \frac{\sin(pz)e^{-py}}{p(p+1)}\,dp. \tag{4.4}$$

In particular, the density of the excited atoms is

$$N(z) = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(pz)}{p+1}\,dp. \tag{4.5}$$

The coordinate and frequency in (4.4) and (4.5) are in units of (4.2) and (4.1).

For a large depth $z \gg 1$ we easily obtain from (4.4)

$$f(z, y) = (2/\pi)\,\text{arctg}(z/y). \tag{4.6}$$

It can be seen from this equation that at frequencies $x > z^{1/3}$ the spectral density differs from the equilibrium value at all depths. The optical thickness is then $\tau_{\text{eff}} z^{1/3} \gg 1$, so that the optical characteristics of the medium are not at equilibrium even at large optical depths. An important consequence of this circumstance is that the radiation from the volume is not at equilibrium at any frequency.

Near the surface, $z \lesssim 1/\tau_{\text{eff}}$, we obtain with the aid of the substitution (2.24)

$$f(z, y) = \left(z + \frac{2}{3}\frac{x^2}{\tau_{eff}}\right)\frac{2}{\pi}\int_0^\infty \frac{e^{-py}}{p+1}\,dp. \tag{4.7}$$

Thus, in contrast to deep layers, in which the spectral density decreases monotonically with increase of frequency, the frequency dependence of the density has a maximum near the surface.

We consider now the radiation emitted from the medium. At equilibrium ($\alpha \gg 1$) the intensity of the radiation from the surface is $a/4\pi^2 k_L$. With the aid of (4.7) we find that in this case the radiation intensity referred to the equilibrium value is

$$I(x,\mu)=\frac{2}{\pi}\left(\mu+\frac{2}{3}\right)\frac{x^2}{\tau_{eff}}\int_0^\infty \frac{e^{-pv}}{p+1}dp, \tag{4.8}$$

where μ is the cosine of the angle measured from the normal to the surface. It can be seen from (4.8) that the angular dependence of the radiation is the same as for purely coherent scattering and the radiation intensity as a function of frequency has a maximum and drops to zero at high frequencies.

At high frequencies, in fact, the radiation is determined by the collisional redistribution in frequency. Without dwelling on the details of the calculations, we present the final results: at $x \gg \tau_{\text{eff}} P_c^{1/2}$ we have

$$I(x,\ \mu)=(3P_c)^{1/2}(\mu+{}^2/_3). \tag{4.9}$$

It is clear from (4.8) and (4.9) that the maximum of the radiation intensity is preserved up to pressures at which $P_c \sim \tau_{\text{eff}}^{-2}/$ At still higher pressures the spectrum of the radiation is determined, in accordance with (3.15), by the collisions. The solution of Eqs. (2.20) and (2.21) for this case encounters no fundamental difficulties and is not given here.

These results are in qualitative agreement with the numerical calculation of Ref. 3. A quantitative comparison is impossible, since the (4.3) is not valid under the conditions of Ref. 3.

5. CONCLUSION

The examples considered show that the equations derived in Sec. 2 can be useful for a qualitative analysis of resonant-radiation transfer in finite volume, in the entire range of parameters of the maximum, from optical thicknesses $ak_D R \ll 1$, when it can be reliably assumed that the total distribution in frequency is governed by the Doppler effect, to pressures at which $P_c = 1$ and the total distribution in frequency is collision-governed.

Since the results of Sec. 3 yield a complete set of eigenvalues, they solve in principle any problem involving the densities of excited atoms in volumes of finite size. The eigenvalues are given by Eqs. (3.9) and (3.14), in which λ_0 must be replaced by λ_1. The eigenvalues for the total distributions, which are needed in (3.9), can be found in Refs. 17 and 18. The eigenfunctions agree in the approximation considered with those of the Laplace operator. When condition (1.1) is met, this result is asymptotically accurate. In the case least favorable for the diffusion approximation, that of total distribution in frequency, an analysis of the eigenfunctions obtained in Refs. 17 and 18 in the limit of large optical thicknesses shows that for Doppler and Lorentz profiles they agree, within $\sim 10\%$, with the eigenfunctions of the Laplace operator in the entire volume except in the immediate vicinity of the boundaries.

The spectrum of the emitted radiation can be obtained from the equation

$$I(x)=\frac{\gamma}{3k^2(x)}\int_V d\mathbf{r}\,\Delta n(\mathbf{r},x). \tag{5.1}$$

Estimates show that Eq. (5.1) describes correctly the radiation at the maximum of the line emitted from the volume, in the range from almost-coherent scattering, when it is asymptotically correct, to total redistribution in frequency. It is not applicable, however, in the line wing, and particularly in the self-reversal region, where it can lead to errors amounting to several orders of magnitude.

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