Contribution to the nonlinear theory of thermal instability

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The evolution of large-scale disturbances under conditions of thermal instability is investigated in the framework of one-dimensional gasdynamic equations with allowance for processes of heating and radiative cooling of the gas. The characteristic thermal times in the problem are found in this case to be considerably shorter than the gasdynamic time, and thermal-conduction effects are unimportant. Two physically important classes of motion of matter in such a system are indicated: explosive condensation of the gas (the formation of dense cold regions) and explosive rarefaction of the gas (the formation of rarefied hot regions). Certain analytical solutions describing these features of the motion are obtained and investigated. The asymptotic behavior of the solutions near the singularities is investigated, and the question of the realization of these solutions in a problem with "natural" initial conditions is considered. The possibility of the formation of intermediate structures of the Zel'dovich "pancake" type upon development of a thermal instability in the regime of explosive condensation is indicated.

1. INTRODUCTION

One of the important dynamical processes in the interstellar medium and in stellar atmospheres is thermal instability. The observed "patchiness" of the interstellar medium[1] and the formation of condensations in the solar atmosphere[2] can be attributed to this instability. Apparently, thermal instability plays an important role in processes of star formation.[3] Also well known in plasma physics is the superheating instability—a variant of thermal instability associated with the competition between Joule heating and radiative cooling of a plasma.[4]

A thorough analysis of the linear stage of thermal instability is given in the well-known paper of Field.[5] As regards the nonlinear stage of the instability, it has been investigated in many papers, predominantly by numerical methods, and in each paper specific conditions inherent to the problems formulated were considered. At the same time, there is an important general case in which the equations describing the nonlinear dynamics of the one-dimensional thermal instability can be solved analytically, making it possible to build up a more complete picture of the development of the instability. In Sec. 2 of the article we obtain nonlinear equations describing the long-wavelength thermal instability, and the physically interesting solutions are classified. Sections 3 and 4 are devoted to obtaining and investigating certain analytical solutions of the corresponding problems. In Sec. 5 we briefly discuss the results obtained.

2. THE NONLINEAR EQUATIONS OF THE THERMAL INSTABILITY, AND A METHOD FOR SOLVING THEM

We shall consider a gas with density ρ and temperature T, moving with velocity v under the action of a gradient of the pressure p (we neglect gravity and other forces). The gasdynamics equations have the usual form

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \tag{2}$$

$$T\left(\frac{\partial S}{\partial t}+v\frac{\partial S}{\partial x}\right)+\mathscr{L}(\rho,T)-\frac{1}{\rho}\frac{\partial}{\partial x}\left(\varkappa\frac{\partial T}{\partial x}\right)=0. \tag{3}$$

Here S is the entropy per unit mass of the gas, $\mathscr{L}$ is the difference between the rate of radiative cooling (we assume the gas to be optically transparent) and the rate of heating per unit mass, and $\varkappa$ is the thermal-conductivity coefficient. In the case of the interstellar medium, e.g., the heating is determined by the absorption of subcosmic rays and (or) the ultraviolet and x radiation of the stars.[1] The equation of state specifying the dependences of p and S on ρ and T is assumed to be known.

The system of equations (1)–(3) has simple equilibrium solutions: $T=T^{(0)}=\text{const}, v=0, \mathscr{L}(\rho^{(0)},T^{(0)})=0$. The instability of these equilibrium solutions [or, more generally, of any stationary solutions of the system (1)–(3)] is a thermal instability. As already noted, the linear stage of this instability, characterized by small deviations of the physical quantities from their equilibrium values, was investigated by Field,[5] who linearized the system of equations (1)–(3) together with the equation of state of an ideal gas. Since we are interested in the development of the instability in the framework of the nonlinear system of equations (1)–(3), we shall consider the limit of large-scale disturbances (in other words, we shall make use of the long-wavelength approximation). In this case it is possible, first, to neglect the thermal conduction, and, secondly, to take into account the fact that the time of establishment of local thermal equilibrium in the gas becomes substantially shorter than the characteristic gasdynamic time. Namely, since in the long-wavelength limit the individual terms in the function $\mathscr{L}(\rho,T)$ are considerably greater than the other terms in Eq. (3), we can use instead of Eq. (3) the thermal-equilibrium condition

$$\mathscr{L}(\rho,T)=0. \tag{4}$$

This approach, of course, presupposes that in the range of parameters under consideration the condition $(\partial\mathscr{L}/\partial T)_\rho>0$ is fulfilled, since only in this case is the thermal equilibrium (4) stable with $\rho=\text{const}$. Otherwise, a rapid "jump"from the unstable equilibrium (4) to some stable

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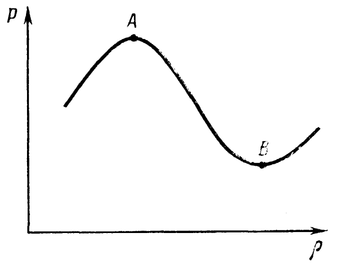

FIG 1.

equilibrium state (with the gas almost stationary) occurs first, and only then will the gas be set in motion.

Solving the algebraic equation (4) for T, we obtain the equilibrium value of the temperature for the given density: $T_* = T_*(\rho)$. To this temperature corresponds a pressure $P(\rho) = p(\rho, T_*(\rho))$ as a function of ρ alone. Thus, in the long-wavelength approximation the thermal instability is described by the system of equations (1), (2), in which one must substitute the function $P(\rho)$ in place of $p(\rho,T)$, and this will be implied below without explicit stipulation. In Fig. 1, as an example, we give the dependence $P(\rho)$ for the interstellar medium.[1]

The dispersion equation for small disturbances proportional to $\exp(-i\omega t + ikx)$ has in the case under consideration the form[1)]

$$\omega^2 = k^2 (dP/d\rho), \tag{5}$$

whence follows the instability condition $dP/d\rho < 0$ (the segment AB in Fig. 1).

Let the gas be carried over sufficiently rapidly into the unstable state (e.g., such a situation occurs in the passage of shock waves in the spiral arms of a galaxy[3)]). Then the system of equations (1), (2) with the known dependence $P(\rho)$ describes one-dimensional unstable motion of the gas and can be solved analytically. With the aim of obtaining visualizable results, we approximate the unstable region of $P(\rho)$ as follows:

$$P(\rho) = A + B\rho^{1-\alpha}, \tag{6}$$

where $A > 0$, $B > 0$, and $\alpha > 1$ are constants. The presence of three fitting parameters A, B, and α makes it possible to obtain a sufficiently accurate approximation in the most widely different cases. We introduce the dimensionless variables $\tilde{x} = x/L$, $\tilde{t} = ct/L$, $\tilde{v} = v/c$, and $\tilde{\rho} = \rho/\rho_0$, where L is the characteristic spatial scale of the initial disturbance, ρ_0 is the characteristic value of the gas density (e.g., $\rho_0 = \rho^{(0)}$), and $c^2 = (\alpha - 1)B\rho_0^{-\alpha}$. The quantity c (the characteristic gasdynamic velocity) is usually equal in order of magnitude to the velocity of sound in the gas under consideration. Then, omitting the tildes, we rewrite Eq. (2) in the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \rho^{-(1+\alpha)}\frac{\partial \rho}{\partial x} = 0. \tag{7}$$

Equation (1) in dimensionless variables keeps the same form. The system of equations (1), (7) with different values of the parameter α has arisen in a whole series of papers (going back to Chaplygin[6]) devoted to the study of different hydrodynamic instabilities with a linear dispersion equation analogous to (5). Thus, e.g., the case $\alpha = 2$, corresponding to a "Chaplygin gas,"[6] arises in the problem of the breaking of a neutral current layer in a plasma[7] and in problems concerning the long-wavelength Buneman instability[8-10] and parametric instability[11] of a plasma. In this case, a highly effective method of solution is to change to Lagrangian variables, in which Eqs. (1) and (7) become linear, making it possible to obtain an explicit Lagrangian description of the motion of the gas.[7] The case $\alpha = 3$, describing the breakway of constrictions in plasma pinches with current, was investigated in Ref. 12. A further series of cases is mentioned in the recent paper Ref. 13.

We shall consider the case of arbitrary α. One of the most effective methods of solving problems of the type under consideration is a hodograph transformation, which consists in going over from the functions $\rho(x,t)$ and $v(x,t)$ to the functions $x(\rho,v)$ and $t(\rho,v)$, where ρ and v are now considered as the independent variables. The equations for $x(\rho,v)$ and $t(\rho,v)$ then take the form

$$\frac{\partial}{\partial \rho}(x - vt) = \rho^{-(\alpha+2)}\frac{\partial}{\partial v}t\rho, \quad \frac{\partial}{\partial v}(x - vt) = -\frac{\partial}{\partial \rho}t\rho, \tag{8}$$

whence

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^{\alpha+2}}\frac{\partial^2}{\partial v^2}\right)t\rho = 0, \tag{9}$$

$$x - vt = \int \left(\frac{1}{\rho^{\alpha+2}}\frac{\partial (t\rho)}{\partial v}d\rho - \frac{\partial (t\rho)}{\partial \rho}dv\right). \tag{10}$$

The integrand of (10) in the (ρ,v) plane is, according to (8), a total derivative.

Because of its linearity, Eq. (9) admits, in principle, the possibility of construction of a general solution. For definiteness, we shall confine ourselves to seeking those solutions which correspond to gas-density profiles that are symmetric about a certain plane $x = 0$. Then it is easy to see that the velocity $v(x)$ will be an odd function of x. Knowing the solution of the linear equation (9), we can determine the function $x(\rho,v)$ from the relation

$$x = vt - \int_0^v \frac{\partial (t\rho)}{\partial \rho}dv, \tag{11}$$

which follows from the general formula (10). Therefore, we shall concentrate attention on the solution of Eq. (9). We make the change of variables $t(\rho,v) = \rho^{1/2}\Phi(\rho^{-\alpha/2}, \alpha v/2)$. Then, for the function $\Psi = \Phi(r,z)\exp(i\varphi/\alpha)$, by means of (9) we obtain the Laplace equation in the cylindrical coordinates r, z, φ. A somewhat different way of reducing the system (1), (7) to the three-dimensional Laplace equation was proposed recently in Ref. 13. The known methods of solution of the Laplace equation make it possible to obtain a large set of physically interesting solutions and to investigate different regimes of flow of the unstable gas [if the approximation (6) used is extended to all values of ρ, from zero to infinity]. The most interesting solutions describe the explosive rarefaction of matter with simultaneous heating ($\rho \to 0, T \to \infty$), and the explosive condensation (collapse) of matter with simultaneous cooling ($\rho \to \infty, T \to 0$). Whereas the second case corresponds to the breaking of compression waves (as in "stable" gas dynamics), the first case can be interpreted as the breaking of rarefaction waves.[11] We shall consider these cases separately.

3. EXPLOSIVE RAREFACTION OF THE GAS IN THE PROCESS OF THE THERMAL INSTABILITY

In this section we consider solutions describing explosive gas rarefaction ($\rho\to 0$) accompanied by strong heating of the gas ($T\to\infty$). We are interested, first, in the behavior of the gas flow near the singularity $\rho\to 0$ that (we shall assume) arises in the plane $x=0$ at $t=0$ (this is ensured by a particular choice of the coordinate origin), and, secondly, in the question of the realization of this flow under ordinary (natural) initial conditions.

Changing to spherical coordinates in the Laplace equation for the function Ψ, we obtain a family of solutions describing the explosive rarefaction of the gas:

$$t=-C\rho^{(l\alpha-1)/2}U_l(\alpha\rho^{\alpha/2}v/2), \tag{12}$$

where

$$U_l(w)=\frac{(1+w^2)^{-l/2}\left[\mathrm{P}_{-l}^{-1/\alpha}\left(\frac{w}{(1+w^2)^{1/2}}\right)+\mathrm{P}_{-l}^{-1/\alpha}\left(-\frac{w}{(1+w^2)^{1/2}}\right)\right]}{2\mathrm{P}_{-l}^{-1/\alpha}(0)},$$

$P_\nu^\mu(x)$ is an associated Legendre function of the first kind on the cut $-1<x<1$ (see e.g., Ref. 14), and C is an arbitrary positive constant. Below we assume that $C=1$, corresponding to a certain concrete choice of L.

The solutions written out correspond to self-similar solutions of the system of equations (1), (7), of the form

$$\rho=(-t)^{2(1-\gamma)/\alpha}R\left(\frac{x}{(-t)^\gamma}\right),\quad v=(-t)^{\gamma-1}V\left(\frac{x}{(-t)^\gamma}\right), \tag{13}$$

where $\gamma=1-\alpha/(l\alpha-1)$. From the requirement that $\rho=0$ at $t=0$ there follows the inequality $\gamma<1$. The solutions (12) agree with the natural boundary conditions only in the case $\gamma\geqslant 0$, when the function $U_l(w)$ has zeros. Thus, in (12) and (13), we have $0\leqslant\gamma<1$ or $1+1/\alpha\leqslant l<+\infty$.

From the formula (12) we can obtain the following asymptotic forms: For $|x|\ll|t|^\gamma$,

$$\rho=(-t)^{2(1-\gamma)/\alpha},\quad v=-\frac{2(1-\gamma)}{\alpha}\frac{x}{t}, \tag{14}$$

while for $|x|\gg|t|^\gamma$ (in particular, at the time $t=0$ of the singularity),

$$\rho\infty|x|^{2(1-\gamma)/\gamma\alpha},\quad v\infty x|x|^{-1/\gamma}. \tag{15}$$

The formulas (15) describe the formation of characteristic density and velocity profiles with breaks.

The expressions (15) are valid for $0<\gamma<1$. For $\gamma=0$ (which corresponds to the condition $l=1+1/\alpha$), the density vanishes on a whole segment of the x axis simultaneously (the length of this segment is determined by the shape of the initial disturbance), and the velocity of the gas becomes infinitely large.

If the solution of a particular problem with concrete initial conditions is represented in the form of a sum of multipoles (12) with different l, then near the singularity $\rho\to 0$ the main role, as a rule, will be played by the multipole with the smallest l (and, correspondingly, with the smallest γ). In this sense the solution with $\gamma=0$ is physically distinct and deserves great attention. Therefore, we shall give here relations for the functions $R(x)$ and $V(x)$; these relations are obtained by setting $l=1+1/\alpha$ in (12), making use of (11), and comparing the results with (13):

$$1+(\alpha/2)^2R^\alpha V^2=R^{\alpha^2/(\alpha+2)}, \tag{16}$$

$$\int_1^R\frac{dR}{R}\frac{4R^{-2\alpha/(\alpha+2)}/(\alpha+2)+(\alpha-2)R^{-\alpha}}{(R^{-2\alpha/(\alpha+2)}-R^{-\alpha})^{1/2}}=2x. \tag{17}$$

The formulas (16) and (17) can also be obtained by integrating the self-similar equations to which the system (1), (7) is reduced when the relations (13) are substituted into it. The behavior of the functions $R(x)$ and $V(x)$ is depicted in Fig. 2. In the particular case $\alpha=2$, from (16) and (17) we obtain $R=\cos^{-2}x$ and $V=\frac{1}{2}\sin 2x$, in agreement with Ref. 9.

It is possible that the arguments expounded here concerning the physically distinct nature of the solution with $\gamma=0$ will become more convincing if we give the exact solution of the problem with definite, sufficiently natural boundary conditions—namely, if we consider the solution which, at $t\to-\infty$, describes the equilibrium state of the gas. To this end, as in the recent paper Ref. 13, we shall make use of the Laplace equation for the function Ψ in toroidal coordinates η, θ, φ (Ref. 14). Then the solution can be written in the form

$$\Psi=(\mathrm{ch}\,\eta-\cos\theta)^{1/2}\sum_n C_n\mathrm{P}_{n-1/2}^{-1/\alpha}(\mathrm{ch}\,\eta)\exp(in\theta+i\varphi/\alpha), \tag{18}$$

where $P_\mu^\nu(x)$ is an associated Legendre function of the first kind (see Ref. 14). Confining ourselves to the harmonic with $n=0$ and setting $C_0=1$ (which corresponds to a specific choice of L), we obtain for t the expression

$$t=-\rho^{(\alpha-2)/4}(\mathrm{sh}\,\eta)^{1/2}\mathrm{P}_{-1/2}^{-1/\alpha}(\mathrm{ch}\,\eta), \tag{19}$$

where

$$\eta=\frac{1}{2}\ln\frac{(\rho^{-\alpha/2}+1)^2+(\alpha v/2)^2}{(\rho^{-\alpha/2}-1)^2+(\alpha v/2)^2}. \tag{20}$$

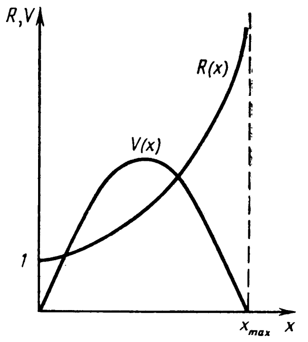

FIG. 2.

Investigation shows that as the density and velocity tend to their equilibrium values ($\rho \to 1, v \to 0$), the value of t in (19) tends to $-\infty$ in accordance with the law

$$t = K \ln[(\rho-1)^2 + v^2],$$
$$K = \frac{\Gamma[(\alpha+2)/\alpha]}{2^{(4-\alpha)/2\alpha}\Gamma[(\alpha+1)/\alpha]\Gamma^2[(\alpha+2)/2\alpha]}. \quad (21)$$

[This corresponds to the limit $\eta \to \infty$ in (19).] Solving (21) for ρ and v with the aid of (11), we obtain, for $-t \gg 2K$,

$$\rho = 1 - \cos(x/2K)\exp(t/2K),$$
$$v = \sin(x/2K)\exp(t/2K). \quad (22)$$

In the formulas (22) it is assumed that the plane $x = 0$ corresponds to one of the density minima.

The limit $\eta \to 0$ in formula (19) is also of interest. In this case we obtain

$$t = -\frac{1}{2^{1/\alpha}\Gamma[(\alpha+1)/2]}\rho^{(2-\alpha)/4}\eta^{(\alpha+2)/2\alpha},$$

and therefore, for $\rho \to 0$ and arbitrary $v\rho^{\alpha/2}$, we have

$$t = -Q\rho^{\alpha/2}(1 + (\alpha/2)^2\rho^\alpha v^2)^{-(\alpha+2)/2\alpha}, \quad (23)$$

where $Q = 2^{(2-\alpha)/2\alpha}/\Gamma[(\alpha+1)/\alpha]$. This asymptotic form is the exact solution of the problem, and coincides with (12) if in (12) we set $l = 1 + 1/\alpha$ and select the constant C in the necessary way. Thus, the solution (19) satisfying the natural initial conditions approaches at $t \to 0$ the self-similar solution (13), (16), (17) with $\gamma = 0$. Here it is necessary to stay at a sufficient distance from the planes $x_j = \pi K(2j+1)$ (with j an integer), in which the density tends to infinity. The corresponding criterion has the form $|x - x_j| \gg t^{2/(\alpha+2)}$; it becomes softer and softer as the singularity is approached.

Returning to formula (18), we note that for $n > 0$ the unstable disturbance are localized in space.[13]

4. EXPLOSIVE CONDENSATION (COLLAPSE) IN THE PROCESS OF THE THERMAL INSTABILITY

We turn now to an account of the results of an investigation of the character of the gas flow in regions of appearance of cold condensations ($\rho \to \infty, T \to 0$). One of the solutions describing the condensations has already been obtained, in fact, in Sec. 3 [formula (19)]. However, in this solution the entire gas condenses in a finite time on a discrete number of planes, positioned at equal distances (equal to $2\pi K$) from each other. The gas density becomes infinite in all Lagrangian elements of the gas simultaneously. This behavior of the condensations is exceptional and is not typical for general initial conditions. Therefore, since we are interested in precisely the flows near the singularity $\rho \to \infty$, we shall consider the following continuous set of solutions of Eqs. (1), (7):

$$x = -\frac{a_1 v}{\rho} + \frac{2}{3}a_2 v^3 - \frac{2a_2}{\alpha-1}\frac{v}{\rho^\alpha},$$
$$t = -\frac{a_1}{\rho} + a_2 v^2 - \frac{2a_2}{\alpha^2-1}\frac{1}{\rho^\alpha}, \quad (24)$$

where $a_1, a_2 > 0$ are constants. Since $\alpha > 1$, we can neglect the last terms in (24) as $\rho \to \infty$. Then, changing the scales in such a way that $a_1 = a_2 = 1$, we obtain, as $\rho \to \infty$,

$$t = v^2 - 1/\rho, \quad x = {}^2/_3 v^3 - v/\rho. \quad (25)$$

This flow [as can be seen from formulas (9) and (10)] implies inertial motion of the gas, when the role of the pressure forces is negligibly small. It can be shown that the solution (25) corresponds to the only structurally stable singularity of the flow of a cold gas of noninteracting particles.[15] The term "structural stability" here, as usual, implies preservation of the character of the motion upon a small change of the initial conditions.

In the cases $a_1 = 0$ and $a_2 = 0$ the formulas (25) also describe the solutions (24) with $\rho \to \infty$, but these solutions are scarcely typical.

The fact that when condensation appears the gas flow proceeds in the same way as in a gas of noninteracting particles is physically fairly obvious, since for the chosen approximation of $P(\rho)$ in the form (6) we have $P(\rho) \to$ const as $\rho \to \infty$ (and, correspondingly, $\nabla p \to 0$). The latter circumstance also obtains in the three-dimensional case, and this makes it possible to detect a similarity between the "thermal collapse" considered here and the development of condensations in the process of long-wavelength gravitational instability.[16] In this case the theory describes the formation of planar structures—Zel'dovich "pancakes."[16] In the collisional gas under consideration the "pancakes" should contain shock waves, as in the case of the gravitational instability. It is not ruled out that the onset of flattened cold condensations during the development of the thermal instability is not directly connected with the long-wavelength approximation but is preserved in the more general case.

5. DISCUSSION OF THE RESULTS

In this paper we have discovered two physically important types of motion of a gas experiencing thermal instability: explosive condensation (the formation of cold dense regions) and explosive rarefaction (the formation of rarefied hot regions). These processes have been investigated analytically for the case when the approximation (6) for the effective pressure $P(\rho)$ is extended to the entire ρ axis. The approximation (6) is not, of course, the only one possible. For example, the curve $P(\rho)$ (see Fig. 1) for not very large ρ can also be approximated by a polynomial in powers of ρ; some of the coefficients of this polynomial should then be negative. Problems of this type are treated analogously. In the particular case $P(\rho) = D - E\rho^2$, where $D, E > 0$ and ρ is not very large, Eqs. (1) and (2) reduce to the familiar equations describing the self-focusing and self-modulation of waves (see Ref. 17 and the literature cited therein). The solutions of these equations are well known; they can describe the phenomena of explosive condensation and rarefaction at not very high densities.

In the real situation of a nonmonotonic dependence $P(\rho)$ (see Fig. 1) a complete analytical investigation of the thermal instability is difficult, since we encounter here the complicated problem of the transition from unstable to stable flow, and vice versa. A rough estimate of the final state of the gas in this case can be obtained from the condition that the final and initial pressures of the gas be equal,[1] and this makes it possible to affirm the presence of strong compression (or rarefaction) of the gas in the real final state.

The features of the long-wavelength thermal instability that have been elucidated here should evidently also be manifested in more-complicated situations, e.g., when one takes into account gravitational forces or a magnetic field. In this connection we recall the numerical experiment of Ref. 18, in which, in the framework of the gasdynamic equations with allowance for thermal processes and gravitation, the large-scale flow of interstellar gas in galactic spiral waves was modelled, and, in conditions of thermal instability, interesting dynamical phenomena, to a certain extent analogous to those considered by us, were observed.

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[1)] Naturally, formula (5) also follows from the corresponding expression of Ref. 4 in the long-wavelength limit.

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