Stability of stationary periodic structures for weakly supercritical convection and in related problems

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We investigate stationary spatially periodic and weakly supercritical structures produced in a convective layer and in related problems (instability of the surface of a dielectric liquid in an external electric field, dissipative structure in reaction-diffusive systems, etc.). The regions of instability to small perturbations are found for each of three possible types of such structures, viz., rolls forming a one-parameter family of solutions, and rhombs and hexagons forming three-parameter families. It is shown that besides the previously known quadratic cellular structures, there can exist structures made up of rhombic cells with vertex angle differing substantially from $\pi/2$. The existence of a critical vertex angle of such rhombic cells is discussed.

1. INTRODUCTION

To study the motion of a liquid in weakly supercritical convection, we start with the evolution equation.[1]

$$\frac{\partial U_{\mathbf{k}}}{\partial t}=\gamma_k U_{\mathbf{k}}-2\int \alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}U_{\mathbf{k}_1}U_{\mathbf{k}_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\,d\mathbf{k}_1\,d\mathbf{k}_2 -\frac{4}{3}\int \mu_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}U_{\mathbf{k}_1}U_{\mathbf{k}_2}U_{\mathbf{k}_3}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3, \qquad (1)$$

where U_k is the Fourier transform of the real scalar function $u(r,t)$ that describes the distribution of the velocity field and of the liquid temperature in the horizontal plane. Here $\mathbf{r}$ is a two-dimensional radius vector and $U_{-\mathbf{k}} = U^*_{\mathbf{k}}$; the coefficients α,γ, and μ are real (the coefficients $\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ П $\mu_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ are frequently called matrix elements).

The right-hand side of (1) is in fact a Landau expansion in powers of U. A liquid at rest corresponds in this case to the trivial solution $U_{\mathbf{k}} = 0$.

An important characteristic of the convection problem is the presence or absence of symmetry of the physical conditions of the problem relative to a horizontal plane passing through the middle of the convective layer. The presence of such a symmetry corresponds formally to invariance of (1) to reversal of the sign of U. The presence, however, of even powers of U in (1) violates this invariance. It is important in this case that the symmetry breaking is due as a rule to relatively weak effects, such as the temperature dependence of the viscosity, the thermocapillary effect, and others; this leads to numerical smallness of the coefficients of the even powers of U, particularly of the characteristic values of $\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ compared with $\mu_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$; this explains why a term proportional to U^3 is taken into account in (1) besides the term proportional to U^2 (Refs. 1 and 2).

The stability growth rate γ_k, which determines the linear stage of the development of convective motion, usually reaches a maximum at a certain finite value $k = k_0$, which we set equal to unity. In a weakly supercritical situation, γ_k in the vicinity of this point can be represented in the form[1,2]

$$\gamma_k \approx \varepsilon-(k^2-1)^2, \qquad (2)$$

where $\varepsilon \ll 1$ is a parameter that determines the supercriticality.

According to (2), modes with wave vectors $\mathbf{k}$ located outside the narrow ring $(k^2-1)^2 \lesssim \varepsilon$ attenuate rapidly. In investigations of stationary solutions of (1), the matrix elements $\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $\mu_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ which describe the nonlinear interaction of the modes, therefore can be expanded in a series in the vicinity of the points $|\mathbf{k}_j| = 1$. In view of the condition $\sum_j \mathbf{k}_j = \mathbf{k}$ the first term of this expansion for α is a constant, while for μ it is a function of a single argument θ, which is the vertex angle of the rhomb made up of the vectors $\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3$.[1] We note further that the function $\mu(\theta)$ must satisfy the obvious conditions $\mu(\theta)>0$ and $\mu(\pi-\theta)=\mu(\theta)$. We shall find it also convenient to put $\mu(\theta) = 1$ at $\theta = 0$, something always attainable by a scale transformation. In this case the requirements that α be small compared with μ means that $0 \leqslant \alpha \ll 1$ (if $\alpha < 0$ the problem reduces to the preceding reversal of the sign of U).

In individual cases it is possible to obtain for the function $u(t,\mathbf{r})$ a local equation in coordinate space. An example is the problem of convection between boundaries with poor thermal conductivity, with allowance for the capillary effect, where the corresponding equations is[3]

$$\frac{\partial u}{\partial t}+u+(2+\varepsilon)\Delta u+\Delta^2 u-4\alpha\nabla(u\nabla u)-\nabla[\nabla u(\nabla u)^2]. \qquad (3)$$

It is easily seen that (3) is reduced to (1) by Fourier transformation. In that case

$$\mu(\theta)={}^1/_3(1+2\cos^2\theta). \qquad (4)$$

The matrix elements for some other formulation of the problem were calculated in Ref. 1. In each specific case such a calculation is a problem in itself. Our present results will therefore be formulated without specifying the $\mu(\theta)$ dependence, to illustrate the general laws that govern the phenomenon.

We shall assume that the convective layer is infinite in the horizontal plane, so that the linearized equation (1) has a continuous eigenvalue spectrum. We are interested in the

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present paper in stationary solutions (1), which correspond in coordinate space to structures that form periodic lattices (Bénard cells).

The most complete analysis of the stability of such structures to small perturbations is contained in Ref. 1. This analysis, however, pertains only to structures with $|\mathbf{k}| = 1$. Yet besides these solutions there exist also continuously adjacent families of solutions for which $|\mathbf{k}|$ can take on arbitrary values from the region $(k^2-1)^2 \lesssim |\varepsilon|$, see Eq. (2). Some of these solutions can also be stable to small perturbations. Attention to these circumstances was drawn by Refs. 4 and 5, where the light was cast on this question for the case when the only stable structures are quasi-one-dimensional lattices of (1) in the form of rolls ($\alpha = 0$).

The case $\alpha \neq 0$ differs qualitatively from that considered in Refs. 4 and 5 in that at $\alpha \neq 0$ there can exist, besides stable rolls, also stable stationary solutions of (1), having hexagonal symmetry.[1,7,8] Rigid transitions of the type of first-order phase transitions can take place between these rolls and the hexagons.

The present paper is devoted to a generalization of the approach developed in Refs. 1,4, and 5 to include the case $|\mathbf{k}| \neq 1, 0 < |\varepsilon| \ll 1, 0 \leqslant \alpha \ll 1$; this generalization yields stability criteria for the stationary periodic solutions of (1), similar to the Eckhaus criterion[9] which is valid in the one-dimensional situation.

We proceed to investigate the stationary solutions of Eq. (1). The wave vectors of the investigated stationary structure will be denoted by $\mathbf{q}$, with the symbol $\mathbf{k}$ retained for the wave vectors of the perturbations.

2. STABLE ROLLS

At $\varepsilon \geqslant 0$ Eq. (1) has a stationary solution that describes quasi-one-dimensional rolls and corresponds, in coordinate representation, to

$$u(\mathbf{r}) = A_q \cos qx + O(\alpha\gamma) + O(\gamma^{3/2}), \tag{5}$$

$$A_q = \gamma_q^{1/2}, \tag{6}$$

where q lies in the interval

$$-\varepsilon^{1/2}/2 \leqslant q-1 \leqslant \varepsilon^{1/2}/2, \tag{7}$$

within which $\gamma_q \geqslant 0$, see (2), and the x axis is directed along $\mathbf{q}$.

Investigation of the stability of the solution (5) to small perturbations leads to a standard eigenvalue problem. An approximate solution of this problem, with the same accuracy as (5), has shown that the solution (5) is stable to small perturbations if the three following three inequalities hold[1,4,5]:

$$0 \leqslant q-1 \leqslant (\varepsilon/12)^{1/2}, \tag{8}$$

$$2\mu(\theta) A_q^2 \geqslant \varepsilon, \tag{9}$$

$$A_q \geqslant \frac{\alpha + (\alpha^2 + 2\varepsilon\mu)^{1/2}}{2\mu}, \tag{10}$$

where the condition (9) should be met for all values of θ (here and elsewhere μ without an argument stands for $\mu(\pi/3)$).

The reason for three different criteria is that in the approximation considered there exist three different types of "dangerous" perturbations, which will be called below, following Ref. 1, internal, external, and resonant. For internal perturbations, the criterion of stability to which is given by (8), the proper mode is a superposition of two plane waves with wave vectors $\mathbf{k} = \mathbf{q} \pm \mathbf{p}$, where $\mathbf{p}$ is an arbitrary small vector ($p^2 \lesssim \varepsilon$). For external perturbations, the stability to which is ensured by satisfaction of condition (9), the proper mode is $\sim \exp(\Gamma t + i\mathbf{kr})$ where $(k^2-1)^2 \lesssim \varepsilon$ and the angle between the vectors $\mathbf{k}$ and $\mathbf{q}$ is not close to $\pi/3$ and $2\pi/3$. Finally, the proper mode of the resonant perturbations is a superposition of plane waves with wave vectors $\mathbf{k}_{1,2}$ that form a resonant triad with the vector $\mathbf{q}$, i.e., they satisfy the conditions $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{q} = 0$ and $(k_{1,2}^2 - 1)^2 \lesssim \varepsilon$.

It can be seen from (8)–(10) that the most stable structures are those with $q = 1$, having the maximum (for the given ε) value of A_q, see (2) and (6). For such structures, (8) is satisfied identically, (9) goes over into the condition

$$\mu(\theta) \geqslant 1/2, \tag{11}$$

and (10) to the condition

$$\varepsilon \geqslant \varepsilon_R \equiv \left(\frac{\alpha}{u - 1/2}\right)^2. \tag{12}$$

3. RHOMBIC STRUCTURES

The general form of the stationary solution of Eq. (1), describing in coordinate representation a lattice of rhombic cells, is

$$u = A_1 \cos(\mathbf{q}_1\mathbf{r}) + A_2 \cos(\mathbf{q}_2\mathbf{r}) + B_1 \sin(\mathbf{q}_1\mathbf{r}) + B_2 \sin(\mathbf{q}_2\mathbf{r}) + \ldots, \tag{13}$$

where $(q_{1,2}^2 - 1)^2 \lesssim \varepsilon$, the ellipsis stands for discarded terms of higher order of smallness, cf. (5), and in the case $\alpha \neq 0$ the angle θ_0 between the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ is not close to $\pi/3$ (the case $\theta_0 \approx \pi/3$ will be discussed in the next section). We note next that both coefficients B_1 and B_2 can be made equal to zero by translation along $\mathbf{q}_1$ and $\mathbf{q}_2$. Putting the Fourier transform (13) in (1) we find that in this case

$$A_{1,2}^2 = (\gamma_{1,2} - 2\mu_0\gamma_{2,1})/(1 - 4\mu_0^2), \tag{14}$$

where we have introduced the notation $\gamma_{1,2} \equiv \gamma_{q\,1,2}, \mu_0 \equiv \mu(\theta_0)$.

Investigation of the stability of the solution (13), (14) to nonresonant perturbations leads to the criterion

$$2(\mu_1 A_1^2 + \mu_2 A_2^2) \geqslant \varepsilon, \tag{15}$$

cf. (9). Here $\mu_{1,2} \equiv \mu(\theta_{1,2})$, where $\theta_{1,2}$ are the angles between the perturbation wave vector $\mathbf{k}$ and the vectors $\mathbf{q}_{1,2}$ $(\theta_1 + \theta_2 = \theta_0)$.

The proper mode of the resonant perturbation is of the same form as in the case of rolls. The difference is that now the vectors $\mathbf{k}_{1,2}$ can form resonant triads with either $\mathbf{q}_1$ or $\mathbf{q}_2$. Each such triad leads to a separate stability criterion, so that stability of the investigated solution to resonant perturbation calls for simultaneous satisfaction of both criteria. This leads to the condition

$$2\mu A_1^2 + (\mu_{11} + \mu_{12}) A_2^2 - \varepsilon \geqslant \{4\alpha^2 A_1^2 + A_2^4 (\mu_{11} - \mu_{12})^2\}^{1/2} \tag{16}$$

and to a second condition obtained from (16) by interchanging the subscripts 1 and 2. Here $\mu_{ij} \equiv \mu(\theta_{ij})$, where θ_{ij} is the angle between the vectors $\mathbf{q}_j$ and $\mathbf{k}_j$ $(\theta_{11}+\theta_{12}=\theta_{21}+\theta_{22}=\pi/3)$.

The most cumbersome is an investigation of the stability of (13) and (14) to internal perturbations whose wave vectors are close to $\mathbf{q}_{1,2}$. In this case it is convenient to write the perturbed solution in the form

$$\tilde{u}=\sum_{j=1}^{2}(A_j+a_j)\cos(\mathbf{q}_j\mathbf{r})+b_j\sin(\mathbf{q}_j\mathbf{r}), \quad (17)$$

$$a_j,\ b_j\sim\exp(\Gamma t+i\mathbf{pr}), \quad (18)$$

where $\mathbf{p}$ is a small wave vector $(p^2\lesssim\varepsilon)$.

The dispersion equation obtained for Γ after substituting the Fourier transform of (17) in (1) is deferred to Appendix 1. For its investigation it is expedient to represent Γ as a series in powers of p^2 (Ref. 10). At $p^2=0$ it has two zero roots $\Gamma_1=\Gamma_2=0$ and two other roots given by

$$\Gamma_{3,4}^2+(A_1^2+A_2^2)\Gamma_{3,4}+4(1-4\mu_0^2)A_1^2A_2^2=0. \quad (19)$$

It can be seen from (19), the criterion of stability to in-phase perturbations $(p^2=0)$ takes the simple form

$$\mu_0\leqslant 1/2. \quad (20)$$

Comparing (12) with (20), we conclude that rolls and rhombs have mutually exclusive stability criteria, i.e., coexistence of rolls and of rhombic cells is impossible for a given $\mu(\theta)$ dependence. For the particular case of square cells with $q_1=q_2=1$ and rolls with $q=1$ this result was obtained previously in Refs. 11 and 6.

Allowance for the first nonvanishing corrections in p^2 leads to stability criteria that are in general quite elaborate. We therefore present them only for a few cases of greatest interest. Thus, at $q_1=q_2\equiv q$, i.e., at $\gamma_1=\gamma_2=\gamma_q$, the stability criterion reduces to the requirement

$$\gamma_q\geqslant 2(1+2\mu_0)\varepsilon/(2\mu_0+3). \quad (21)$$

In the case of skewed rhombs, however, at $q_1=1, q_2\neq 1$, i.e., at $\gamma_1=\varepsilon, \gamma_2\equiv\gamma_q<\varepsilon$, the criterion of stability to internal perturbations takes, with allowance for the condition (20), the form

$$\gamma_q\geqslant {}^2/_3\varepsilon(1+\mu_0). \quad (22)$$

The condition (22) determines the limiting skew, i.e., the minimum γ_2 at which the solution (13), (14) still remains stable to internal perturbations.

A few remarks concerning the stability of structures with $q_1=q_2=1$. In this case the conditions (21) and (22) degenerate to (20), and the condition (15) becomes independent of ε. If the actual form of $\mu(\theta)$ is such that these conditions are met, they lead in the general case to constraints on the possible values of θ_0, of the type ϑ_j, Θ_j, where ϑ_j and Θ_j are independent of ε and the difference $\Theta_j-\vartheta_j$ is of the order of unity. In particular, (20) forbids the existence of structures with a value of θ_0 belonging to a certain finite vicinity of the point $\theta_0=0$ (recall that $\mu(0)=1$).

As for the criterion (16), in the considered case $q_1=q_2=1$ it degenerates into the condition which at $\alpha\neq 0$ certainly cannot be met for $\varepsilon\ll\alpha^2$ and, with (15) taken into account, is certainly met at $\varepsilon\gg\alpha^2$. It is clear therefore that there must exist a certain characteristic value $\varepsilon_Q\sim\alpha^2$ such that at $\varepsilon=\varepsilon_Q$ Eq. (16) begins to be satisfied first for a definite value $\theta=\theta_Q$, which can be naturally called the critical vertex angle (several values of θ_Q can occur simultaneously in certain degenerate situations). At $\varepsilon>\varepsilon_Q; \varepsilon-\varepsilon_Q\ll\varepsilon_Q$ there is produced in the vicinity of θ_Q a region of permisible values of θ; the width of this region increases with ε.

Let us illustrate the foregoing using the matrix element (4) as the example. It is easily seen that in this case (15) reduces to (20), which leads to the limitation $\pi/3\leqslant\theta_0\leqslant 2\pi/3$. The most dangerous external perturbations are here those with $\mathbf{k}$ perpendicular to the bisector of the angle θ_0. On the other hand, the condition (16) for stability to resonant perturbations takes the form

$$\theta_0\geqslant\frac{1}{2}\arccos\left[\frac{\alpha^2}{\varepsilon}-\left(\frac{\alpha^4}{\varepsilon^2}+7\frac{\alpha^2}{\varepsilon}+\frac{1}{4}\right)^{1/2}\right]. \quad (23)$$

In this case $\theta_Q=\pi/2$, $\varepsilon_Q=20\alpha^2/3$. If $\varepsilon>\varepsilon_Q$, Eq. (23) can be written in the form $\pi/2-\delta\theta_Q\leqslant\theta_0\leqslant\pi/2$, where $\delta\theta_Q\sim(\varepsilon-\varepsilon_Q)^{1/2}$ at $\varepsilon-\varepsilon_Q\ll\varepsilon_Q$.

Thus, in the case of (4) the critical vertex angle of rhombic structures with $q_1=q_2=1$ is found to be $\pi/2$. At an arbitrary $\mu(\theta)$ dependence, expanding the matrix elements in (16) in powers of $\delta\theta\equiv\pi/2-\theta_0$ and retaining terms $\sim(\delta\theta)^2$, we obtain the criterion for having $\theta_Q=\pi/2$. This criterion is quite elaborate, and we write here only the sufficient condition that follows from it at $q_1=q_2=1$:

$$\mu''(\pi/2)>0, \quad \mu''(\pi/2)>\mu''(\pi/6). \quad (24)$$

In this situation,[2)]

$$\varepsilon_Q=\frac{\alpha^2[1+2\mu(\pi/2)]}{[\mu(\pi/3)+\mu(\pi/6)-\mu(\pi/2)-{}^1/_2]^2}. \quad (25)$$

This expression for ε_Q was in fact obtained earlier in Ref. 11.

We conclude this section by emphasizing that the condition $\varepsilon_Q=\pi/2$ is not obligatory: in some cases cells of oblique-angle rhombs can be stable and those with right angles unstable.

4. HEXAGONAL STRUCTURES

Besides the solutions investigated above, there is one more type of stationary spatioperiodic solutions:

$$u(\mathbf{r})=\sum_{j=1}^{3}[A_j\cos(\mathbf{q}_j\mathbf{r})+B_j\sin(\mathbf{q}_j\mathbf{r})]+\ldots, \quad (26)$$

where the vectors $\mathbf{q}_j$ satisfy the conditions $(q_{1,2,3}^2-1)^2\lesssim\max\{|\varepsilon|;\alpha^2\}; \mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3=0$.

We confine ourselves next to the study of regular hexagons, for which $A_1=A_2=A_3\equiv A$; $B_1=B_2=B_3\equiv 0$; $q_1=q_2=q_3\equiv q$ (generally speaking, $q\neq 1$). In this case, substituting the Fourier transform of (26) in (1), we obtain for A an equation whose solution is

$$A=-\frac{\alpha\pm[\alpha^2+\gamma(1+4\mu)]^{1/2}}{1+4\mu}. \quad (27)$$

Expression (27) agrees with a known result,[1,7,8] according to which the hexagons are produced at a negative ε equal to

$$\varepsilon_{min}=-\alpha^2/(1+4\mu), \quad (28)$$

in abrupt fashion, i.e., having immediately a finite amplitude

$$|A_{min}|=\alpha/(1+4\mu). \tag{29}$$

The solution (27) exists for q in the interval $(q^2-1)^2 \leqslant \varepsilon - \varepsilon_{\min}$, see (2) so that the only hexagonal-symmetry solution that exists at $\varepsilon = \varepsilon_{\min}$ is the one with $q = 1$.

An investigation, perfectly analogous to that carried out in Refs. 1 and 8, of the stability of solutions (26) and (27) to external nonresonant perturbation, leads for the case $q = 1$ to the stability criterion

$$2A^2(\mu_1+\mu_2+\mu_3)\geqslant\varepsilon, \tag{30}$$

where $\mu_{1,2,3} \equiv \mu(\theta_{1,2,3})$, and $\theta_{1,2,3}$ are the angles between the vector $\mathbf{k}$ and the vectors $\mathbf{q}_{1,2,3}$, cf. (9) and (15).

We note further that for hexagonal structures internal perturbations are simultaneously also resonant.[1] The perturbed solution can be written in this case in a form similar to (17), (18):

$$\tilde{u} = \sum_{j=1}^{3}[A+a_j\exp(\Gamma t+i\mathbf{pr})]\cos(\mathbf{k}_j\mathbf{r}) + \sum_{j=1}^{3} b_j\exp(\Gamma t+i\mathbf{pr})\sin(\mathbf{k}_j\mathbf{r}), \tag{31}$$

where $p^2 \lesssim \max\{|\varepsilon|;\alpha^2\}$.

Substitution of (31) in (1) leads, after linearization with respect to small a_j, and b_j, to the cumbersome dispersion equation (A.2) written out explicitly in Appendix 2. Without dwelling on the elaborate intermediate calculations, we present the result of its investigation. First, an investigation of the stability to in-phase perturbations ($p^2 = 0$) shows that the solution with the smaller $|A|$ [lower sign in (27)] is always unstable. As for the solution with the larger $|A|$ [upper sign in (27)] it is stable for $\mu > \frac{1}{2}$ in the region

$$\gamma_q \leqslant \varepsilon_{max} \equiv \frac{2(\mu+1)\alpha^2}{(\mu-{}^1/_2)^2} \equiv 2(\mu+1)\varepsilon_R, \tag{32}$$

where ε_R is defined in (12). In the case $\mu \leqslant \frac{1}{2}$, however, it is stable to in-phase perturbations in the entire region of its existence: $\gamma_q \geqslant \varepsilon_{\min}$, see (27) and (28).

An investigations of perturbations with $0 < p^2 \ll \max\{|\varepsilon|,\alpha^2\}$ leads to a stability criterion that is expediently written in the form

$$0 \leqslant \frac{\varepsilon+2\alpha|A|-(1+4\mu)|A|^2}{\alpha-(\mu-{}^1/_2)|A|} \leqslant \begin{cases} |A|\dfrac{(1+4\mu)|A|-\alpha}{|A|+\alpha}; & |A|<\dfrac{\alpha}{2\mu}, \\ |A|; & |A|\geqslant\alpha/2\mu \end{cases}. \tag{33}$$

That $\alpha - (\mu - \frac{1}{2})|A|$ is positive follows from the criterion of the stability to in-phase perturbations, so that the left-hand inequality in (33) follows from (27) and determines the boundary of the region where solutions (26) exist. The stability conditions themselves, however, determine only the right-hand sides of (33).

The criterion (33) becomes considerably simpler near the points $\varepsilon = \varepsilon_{\min,\max}$ and also at $|A| = \alpha/2\mu$. Thus, for example, at $\varepsilon - \varepsilon_{\min} \ll \varepsilon_{\min}$ the conditions (33) reduce to the inequalities

$$\frac{4}{3\alpha}(\varepsilon-\varepsilon_{min})\leqslant|A|-\frac{\alpha}{1+4\mu}\leqslant\left(\frac{\varepsilon-\varepsilon_{min}}{1+4\mu}\right)^{1/2}. \tag{34}$$

5. DISCUSSION OF RESULTS. PHASE DIAGRAMS

Thus, stable rolls of type (5) form a continuous one-parameter family of solutions of Eq. (1), while stable rhombic (13) and hexagonal (26) structures form continuous three-parameter solutions of (1). These include as a particular case the select structures with $|\mathbf{q}_j| = 1$ investigated in Refs. 1, 4, and 6–8. The independent parameter that determines the particular type of solution in each family can be taken to be the value of the wave number q for rolls, $q_{1,2}$ and the vertex angle for vertices, and the values $q_{1,2,3}$ for hexagons.

Figure 1, which can naturally be called a phase diagram, shows the regions of existence of stable stationary rolls, regular rhombic structures ($q_1 = q_2 = q$) with fixed vertex angle ($\theta_0 = \text{const}$), and regular hexagonal structures ($q_1 = q_2 = q_3 \equiv q$). The independent coordinates on this diagram are ε and $|A|$, where A is the amplitude of the corresponding structure. The value of $|A|$ is uniquely related to γ_q, see (6), (14), and (27). As to the wave number q, it has according to (2) at $\gamma_q < \varepsilon$, two values, $q_1 < 1$ and $q_2 > 1$, for each value of γ_q. The regions corresponding to stable hexagons and rhombs on the phase diagram (see the figure) are therefore doubly degenerate: to each point of these regions correspond two different stable structures having identical

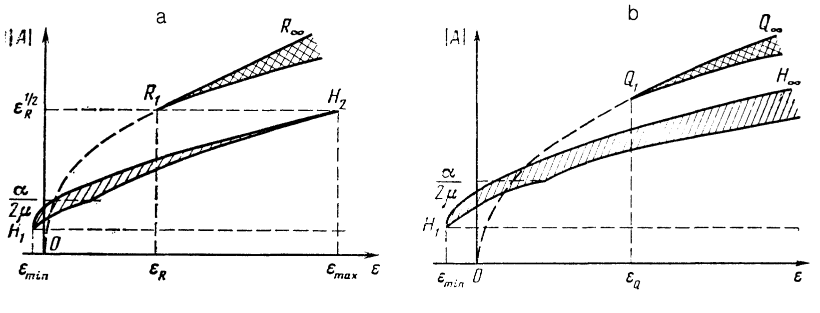

FIG. 1. Phase diagrams: The regions occupied by stable hexagons are singly hatched; those occupied by rolls (a) and by rhombs (b) are cross hatched.

values of γ_q. Exceptions are points belonging to the upper boundaries of the regions, for which $q_1 = q_2 = 1$. The regions corresponding to stable rolls are not degenerate, since rolls with $q < 1$ are unstable [see (8)].

The regions with stable rolls ($\mu(\theta) > \frac{1}{2}$, Fig. 1a) and with stable rhombic structures ($\mu(\theta) < \frac{1}{2}$, Fig. 1b) are never closed.[3)] As for the region with stable hexagons, it is not closed at $\mu(\pi/3) \leqslant \frac{1}{2}$ and is closed, according to (32), at $\mu(\pi/3) > \frac{1}{2}$ (the last result was obtained independently, but not published, by A. A. Nepomnyashchiĭ).

The kinks on the lower boundaries of the existence regions correspond to transitions to different branches of the stability criteria. The situation shown in the figure corresponds to the minimum number of kinks, when the conditions (10) and (16) for the stability of rolls and rhombic cells to resonant perturbations are more stringent for all ε than the conditions (8), (21) and (9), (15) of stability to internal and external perturbations. The generalizations of the figure to the case of another relation between the different stability criteria are obvious.

A situation is also possible with $\mu(\pi/3) > \frac{1}{2}$ i.e., the region with stable hexagons is closed, but the $\mu(\theta)$ dependence is such that $\mu(\theta) < \frac{1}{2}$ in some region of θ that not located in a small vicinity of the point $\theta = \pi/3$. In this case the stable structure competing with hexagons will be not rolls but rhombic cells.

We point out also that situations are possible wherein none of the considered simple stationary solutions of Eqs. (1) is stable. By way of example we consider the case $\alpha = 0$ and $\mu(\theta)$ having two sharp minima at $\theta = \pi/2$ and $\theta = \pi/4$, so that $\mu(\pi/2) < \frac{1}{2}$ and $\mu(\pi/4) < \frac{1}{2}$. It is assumed here that $\mu(\theta) > \frac{1}{2}$ everywhere except close to these two points.

Close to these points, however, $\mu(\theta)$ is approximated, with sufficient accuracy, by the expressions

$$\mu(\theta) \approx \mu(\theta_j) + {}^1/_2\mu''(\theta_j)(\theta-\theta_j)^2, \quad j=1;\,2, \quad \theta_1=\pi/4, \quad \theta_2=\pi/2. \tag{35}$$

In this case the hexagons are certainly unstable, since $\alpha = 0$, and the rolls (5), (6) are unstable to external perturbations whose wave vectors make an angle $\pi/2$ or $\pi/4$ with the vector $\mathbf{q}$, see (9). Next, the rhombic structures (13) and (14), whose vertex angles θ_0 do not satisfy the condition $\mu(\theta_0) < \frac{1}{2}$, are unstable to in-phase internal perturbations, see (20). It remains to investigate rhombic structures with θ_0 close to $\pi/2$ and $\pi/4$, whose stability is determined by the criterion (15). Using (35), we easily verify that under the condition

$$2\mu''(\pi/2) > \mu''(\pi/4), \quad 2\mu(\pi/4) < \mu(\pi/2) + {}^1/_2$$

square cells ($\theta_0 \approx \pi/2$) are unstable to external perturbations whose wave vector is directed along the diagonal of the square made up of the vectors $\mathbf{q}_{1,2}$. Instability of rhombic cells with vertex angle close to $\pi/4$, however, is ensured by the condition $\mu(\pi/2) < \frac{1}{2}$. In this case stable stationary solutions of (1) may turn out to be quasi-periodic solution with modulated amplitude, of the type considered in Ref. 5, solutions with selected centers,[12], and other more complicated low-symmetry solutions.[12] The stability criteria for such solutions are usually more stringent than for the high-symmetry solutions considered above, and their details are outside the scope of the present paper.

We point out in conclusion that Eq. (1) considered here is quite general and describes not only a convective layer of liquid, but also other distributed dynamic systems with aperiodic instability for finite k and with nonlinear stabilization, see, e.g., Refs. 13 and 14. In addition, in a description of an almost conservative (Hamiltonian) system with low dissipation, such as a liquid dielectric in an external electric field,[11] the charged surface of liquid helium,[15] and others, the problem can also be reduced in many cases to Eq. (1), in which $\partial U/\partial t$ is replaced by $\partial^2 U/\partial t^2 + \nu \partial U/\partial t$, where ν is a small damping coefficient ($\nu > 0$). Obviously, this replacement affects neither the form of the stationary solutions of such an equation, nor the analysis of their stability to small perturbations.

We note also that besides the investigated stability of solutions (5), (13), and (26) to small perturbations, considerable interest attaches to their stability to perturbations of finite amplitude and to the associated problem of determining the regions of attraction of the initial perturbations to various asymptotic states of the considered dynamic system.

A detailed treatment of these questions will be the subject of a separate paper, but we state here the following without proof:

1. In the region $0 < \varepsilon - \varepsilon_{\min} \ll |\varepsilon_{\min}|$ the minimum amplitude $\delta A_{\min}$ of a perturbation that disrupts a hexagonal structure is estimated at $\delta A_{\min} \sim (\varepsilon - \varepsilon_{\min})^{1/2}$.

2. A similar estimate with the substitution ($\varepsilon_{\min} \to \varepsilon$) is valid in the region $0 < \varepsilon - \varepsilon_R \ll \varepsilon_R$ for rolls.

3. In the region $0 < \varepsilon_{\max} - \varepsilon \ll \varepsilon_{\max}$ (at $\mu > \frac{1}{2}$), on the other hand, the hexagonal structure can be disrupted by a perturbation with amplitude $\delta A_{\min} \sim (\varepsilon_{\max} - \varepsilon)$.

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APPENDIX 1.

The dispersion equation that determines the stability of the rhombic solutions (13) and (14) to internal perturbations (17) and (18) is

$$\begin{vmatrix} \Gamma + 4\xi_1^2 + 2A_1^2 & 4\mu(\theta_0)A_1A_2 & 4i(q_1^2-1)\xi_1 & 0 \\ 4\mu(\theta_0)A_1A_2 & \Gamma + 4\xi_2^2 + 2A_2^2 & 0 & 4i(q_2^2-1)\xi_2 \\ -4i(q_1^2-1)\xi_1 & 0 & \Gamma + 4\xi_1^2 & 0 \\ 0 & -4i(q_2^2-1)\xi_2 & 0 & \Gamma + 4\xi_2^2 \end{vmatrix} = 0, \tag{A.1}$$

where $\xi_{1,2} = \mathbf{p}\mathbf{q}_{1,2}$.

APPENDIX 2.

The dispersion equation that determines the stability of hexagonal solutions (26) and (27) to small internal perturbations (31) can be written in the form

$$\begin{vmatrix} \Omega_1 & C & C & \Xi_1 & 0 & 0 \\ C & \Omega_2 & C & 0 & \Xi_2 & 0 \\ C & C & \Omega_3 & 0 & 0 & \Xi_3 \\ -\Xi_1 & 0 & 0 & \omega_1 & -2\alpha A & -2\alpha A \\ 0 & -\Xi_2 & 0 & -2\alpha A & \omega_2 & -2\alpha A \\ 0 & 0 & -\Xi_3 & -2\alpha A & -2\alpha A & \omega_3 \end{vmatrix} = 0, \quad (A.2)$$

where the symbols are ($j = 1, 2, 3$)

$$\xi_j = \mathbf{p}\mathbf{q}_j; \quad \Xi_j = 4i(q^2-1)\xi_j; \quad C = 4\mu A^2 + 2\alpha A;$$
$$\omega_j = \Gamma - 2\alpha A + 4\xi_j^2; \quad \Omega_j = \omega_j + 2A^2$$

(by virtue of the condition $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0$, only two of the three parameters ξ_{123} are independent, since $\xi_1 + \xi_2 + \xi_3 = 0$).

1)In convection problems, the state of the liquid is usually described by a definite dimensionless parameter R (the Rayleigh number), which is proportional to the temperature difference between the lower and upper surfaces of the liquid layer. The quantity ε is proportional to $(R - R_c)/R_c$, where R_c is the value of R at which an immobile liquid becomes unstable to the onset of convection.

2)We emphasize that we are dealing now with a small vicinity of the point $\theta_0 = \pi/2$. For rather complicated $\mu(\theta)$ dependences, there can exist, besides this region, other regions of stable values of θ_0, at a finite distance from the point $\theta_0 = \pi/2$. These regions have their own values of ε_Q, generally different from the ε_Q, defined by (25).

3)Of course, the point here is only that these regions are not closed at $\varepsilon \ll 1$. At sizable supercriticality, higher bifurcations occur in the problem and lead to instability of the simple stationary structures considered here.[2,12]

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