Density-density correlator in a model of a disordered metal on a Bethe lattice

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The density-density correlator in a model of a disordered granulated metal on a Bethe lattice is calculated by the method of supersymmetry. In the limit of large distances and low frequencies general expressions are obtained for this correlator both in the insulating and in the metallic region. In the insulating region an explicit expression for the localization length is obtained. Near the transition point this length is inversely proportional to the distance from the transition point. In the metallic region the diffusion coefficient D is calculated. As the transition point is approached D decreases sharply in accordance with an exponential law, making it possible to speak of the existence of a "quasijump." A comparison is made with other models. The role of the noncompactness of the symmetry group of the supersymmetric σ-model used is discussed.

1. INTRODUCTION

In recent years a large number of papers have been devoted to the study of the Anderson metal-insulator transition. The systematic solution of this problem for realistic physical models presents considerable difficulties. One of the few models that can be investigated analytically is the model of a disordered metal on a Bethe lattice (Cayley tree). However, even on such a simplified lattice, the solution of the problem is not simple. The Anderson transition on a Bethe lattice has been studied by many authors. The Anderson model on a Bethe lattice was studied in Refs. 1–3. In Ref. 4, Shapiro proposed a certain model of one-dimensional scatterers, which was studied by means of the resistance-composition hypothesis proposed in Ref. 5 for the investigation of the conductivity of wires. A model of a granulated metal on a Bethe lattice was studied in Refs. 6–8. In all these papers certain averaged quantities were calculated, knowledge of which made it possible to make definite statements about the properties of the models that had been studied. The investigation carried out in these papers has made it possible, e.g., to establish firmly the existence of the transition.

Complete information on the kinetics can be obtained by calculating the density-density correlator. This correlator at coinciding points was calculated in Refs. 6 and 7. However, to calculate such quantities as the diffusion coefficient or permittivity, it is necessary to know the density-density correlator at large distances. In a recent paper[8] a certain correlator at noncoinciding points was investigated. The correlator calculated in Ref. 8 differs from the density-density correlator, but coincides with the latter in the region of localization in the low-frequency limit. This did not permit the author of Ref. 8 to calculate the diffusion coefficient in the metallic region, although the localization length in the insulating region was obtained.

Below we calculate directly the density-density correlator at noncoinciding points. The form obtained for this correlator makes it possible to determine the behavior of the diffusion coefficient in the metallic region and of the dielectric permittivity and localization length in the insulating region. It is shown that as the transition point is approached the diffusion coefficient falls off very rapidly (in accordance with an exponential law). This rapid decrease should give a "quasijump" on the curve of the dependence of the diffusion coefficient on the disorder, although in the strict sense there is no jump in the diffusion coefficient. In the insulating region the localization length near the transition point is inversely proportional to the distance from this point. It is shown that the unusual critical behavior is a consequence of the noncompactness of the symmetry group of the supersymmetric σ-model. A comparison is made with the results of Refs. 2–4.

2. THE BASIC EQUATIONS

The kineticts of a system of metallic granules is described by the supersymmetric σ-model on a lattice. The effective Lagrangian F in such a model is written in the form[6]

$$F[Q]=-\sum_{i,j}J_{ij}\,\mathrm{STr}\,Q_iQ_j-\frac{i}{4}(\omega+i\delta)\pi\nu\sum_i\mathrm{STr}\,\Lambda Q_iV_i. \quad (1)$$

The first term in (1) describes the interaction of the granules, and the second term is the effective Lagrangian of the isolated granules. The letters ω and ν denote the frequency and the density of states in the granules, V denotes the volume of the granules, and the symbol STr denotes the supertrace. The supermatrices Q and Λ in (1) have dimensions 8×8 and are equal to

$$Q=UQ_0\bar{U},\quad Q_0=\begin{pmatrix}\cos\hat{\theta} & i\sin\hat{\theta}\\ -i\sin\hat{\theta} & -\cos\hat{\theta}\end{pmatrix},$$
$$U=\begin{pmatrix}u & 0\\ 0 & v\end{pmatrix},\quad \Lambda=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \quad (2)$$

The form of the matrices u, v, and θ depends on the presence of time-reversal symmetry and central symmetry. A bar above a supermatrix denotes charge conjugation. The corresponding formulas can be found in the review Ref. 9. We note only that $u\bar{u}=v\bar{v}=1$.

The calculation of the density-density correlator $K(r_1,r_2)$ in the model described reduces to the calculation of the following correlator:

$$K(r_1,r_2)=-2\pi^2\nu^2\int(Q_{13}{}^{12})_{r_1}(Q_{31}{}^{21})_{r_2}\exp(-F[Q])\prod dQ_i, \quad (3)$$

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where $F[Q]$ is determined by the expression (1). The superscripts in (3) label the blocks distinguished explicitly in (2), and the subscripts label the elements in these blocks.

Below we shall calculate the density-density correlator (3) for a speical type of lattice—the Bethe lattice (Cayley tree). As in Ref. 6, we consider the case of an arbitrary branching ratio m. We assume that the only interactions are between nearest neighbors, for which all J_{ij} are the same and equal to J.

The structure of the Cayley tree makes it possible to reduce the calculation of the integral (3) for arbitrary r_1 and r_2 to the calculation of a definite integral of the solution of certain integral equations. To obtain these equations, as in Ref. 6 we introduce the function

$$\Psi(Q_0)=\int \exp(-F'[Q])\prod_j{}' dQ_j, \tag{4}$$

where

$$F'[Q]=-2J\sum_{i,j}{}' \operatorname{STr} Q_iQ_j-\frac{i}{4}(\omega+i\delta)\pi\nu V\sum_i \operatorname{STr}\Lambda Q_i.$$

The symbols Σ' and Π' denote sums and products over all sites of one of the $m+1$ branches entering the site with index 0. Here, in Π_j' and Σ_j' the index j cannot take the value 0, while in Σ_{ij}' it can. It is assumed that the jth site is closer than the ith site to the base of the branch. The function $\Psi(Q_0)$ depends only on Q_0. Considering the function $\Psi(Q_1)$ at a neighboring site, it is not difficult to set up the equation

$$\Psi(Q)=\int \exp\left[\frac{\alpha}{4}\operatorname{STr} QQ'+\frac{\beta}{4}\operatorname{STr}\Lambda Q'\right]\Psi^m(Q')\,dQ',$$
$$\alpha=8J,\quad \beta=-i(\omega+i\delta)\pi\nu V. \tag{5}$$

In (5) the integration is over supermatrices of the form (2).

Knowledge of the function $\Psi(Q)$ makes it possible to calculate the correlator $K(r_1,r_2)$ at coinciding points, and this was done in Refs. 6 and 7. However, the information that can be obtained from the form of this function is not complete. To calculate such quantities as the diffusion coefficient, localization length, etc., it is necessary to calculate the correlator $K(r_1,r_2)$, the asymptotic form of this correlator at large distances being the most interesting.

Using the definition (4) of the function $\Psi(Q)$, we write the correlator $K(r_1,r_2)$ in the form

$$K(r_1,r_2)=-2\pi^2\nu^2\int (Q_{13}{}^{12})_{r_1}(Q_{31}{}^{21})_{r_2}\Psi(Q_{r_1})\Psi(Q_{r_2})$$
$$\times\exp\left(\frac{\alpha}{4}\tilde{\sum_{i,j}}\operatorname{STr} Q_iQ_j\right)\tilde{\prod_i} Z(Q_i)\,dQ_i, \tag{6}$$
$$Z(Q)=\Psi^{m-1}(Q)\exp\left(\frac{\beta}{4}\operatorname{STr}\Lambda Q\right). \tag{6a}$$

In the integral (6) in the product $\tilde{\Pi}$ and the sum $\tilde{\Sigma}$ the labels are those of the sites lying on the path between the sites r_1 and r_2 (including the points r_1 and r_2). In the sum Σ_{ij} the indices i and j denote, as before, nearest neighbors, the jth site being closer to r_2, and the ith site closer to r_1.

To calculate $K(r_1,r_2)$ we shall make use of the scheme proposed in Refs. 9 and 10 for the calculation of this same correlator in wires. We introduce the function

$$P(r,Q)=N(r)\int \exp\left(\frac{\alpha}{4}\tilde{\sum_{i,j}}\operatorname{STr} Q_iQ_j\right)$$
$$\times\Psi(Q_{r_2})Q_{r_2}{}^{21}\tilde{\prod_i}{}' Z(Q_i)\,dQ_i, \tag{7}$$

where in the product $\tilde{\Pi}_i$ the sites are those on the path between r_1 and r_2, excluding the site r_1, r is the distance between the sites r_1 and r_2, and $N(r)=(m+1)m^{r-1}$ is the number of sites situated at a distance r from any fixed site.

The function $P(r,Q)$ (7) is defined for all $r\geqslant 1$. We shall complete the definition of this function by setting

$$P(0,Q)=Q^{21}\Psi(Q);\quad P(r,Q)=0,\quad r<0. \tag{8}$$

Using the definitions (7) and (8), it is not difficult to obtain the following equation for $P(r,Q)$:

$$P(r,Q)-[m+\delta(r-1)]\int \exp\left(\frac{\alpha}{4}\operatorname{STr} QQ'\right)$$
$$\times Z(Q')P(r-1,Q')\,dQ'$$
$$=\delta(r)\Psi(Q)Q^{21}. \tag{9}$$

Correspondingly, for the density-density correlator we obtain

$$K(r)\equiv N(r)K(r_1,r_2)=-2\pi^2\nu^2\int Q_{13}{}^{12}P_{31}(r,Q)Z(Q)\Psi(Q)\,dQ. \tag{10}$$

In principle, the formulas (5), (6a), (9), and (10) solve the posed problem of the calculation of the density-density correlator. The correlator $K(r)$ differs from $K(r_1,r_2)$ by the factor $N(r)$. To be precise, $K(r)$ characterizes the rapidity of the decay of the correlations, since $K(r_1,r_2)$ always decays exponentially because of the exponential growth of the number of sites situated at a distance r from a specified site.

Of course, all the formulas written out above are applicable for systems with any type of symmetry, irrespective of whether or not magnetic and spin-orbit interactions are present. The subsequent calculations must be carried out for each model separately. In this paper we shall consider only the case of a system with broken time-reversal symmetry (a unitary ensemble). In this case (model II in the classification proposed in Ref. 7), the matrix $\hat{\theta}$ determining Q has the form

$$\hat{\theta}=\begin{pmatrix}\theta & 0\\ 0 & i\theta_1\end{pmatrix},\quad 0<\theta<\pi,\quad 0<\theta_1<\infty.$$

The equations (5), (9), and (10) can be simplified considerably if we make use of the invariance of the original Lagrangian (1), (2) to the replacement $U\to U_0U$, where U_0 is any matrix of the form (2) satisfying the condition $U_0\overline{U}_0=1$.

Because of this invariance the solution Ψ of Eq. (5) can depend only on the variables θ and θ_1. Integrating over the other variables in (5), we bring this equation to the form[6]

$$\frac{\Psi(\lambda)-1}{\lambda_1-\lambda}=\int\Gamma(\mathbf{nn}',\mathbf{n}_1\mathbf{n}_1')\frac{\Psi(\lambda')Z(\lambda')-1}{\lambda_1'-\lambda'}\frac{d\mathbf{n}'\,d\mathbf{n}_1'}{(2\pi)^2}, \tag{11}$$

where

$$\Gamma(\mathbf{nn}',\mathbf{n}_1\mathbf{n}_1')=\frac{\alpha^2}{2}\left[\exp(-\alpha\mathbf{n}_1\mathbf{n}_1')\frac{d}{d\alpha}\exp(\alpha\mathbf{nn}')\right.$$

$$-\exp(\alpha \mathbf{n}\mathbf{n}')\frac{d}{d\alpha}\exp(-\alpha \mathbf{n}_1\mathbf{n}_1')\Big],$$

$$\lambda=(\lambda,\lambda_1),\quad \lambda=\cos\theta,\quad \lambda_1=\operatorname{ch}\theta_1,$$

$\mathbf{n}$ and $\mathbf{n}'$ are vectors on a sphere, and $\mathbf{n}_1$ and $\mathbf{n}_1'$ are vectors on a hyperboloid. Correspondingly, $\mathbf{n}\mathbf{n}'$ ($\equiv \mathbf{n}\cdot\mathbf{n}'$) is an ordinary scalar product, and $\mathbf{n}_1\mathbf{n}_1' = n_{1z}n_{1z}' - n_{1x}n_{1x}' - n_{1y}n_{1y}'$ is a scalar product on a hyperboloid. The components of the vectors $\mathbf{n}$ and $\mathbf{n}_1$ are equal to

$$\begin{aligned}\mathbf{n}&=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta),\\ \mathbf{n}_1&=(\operatorname{sh}\theta_1\cos\varphi_1,\ \operatorname{sh}\theta_1\sin\varphi_1,\ \operatorname{ch}\theta_1).\end{aligned}\qquad(12)$$

The same expressions define the vectors $\mathbf{n}'$ aand $\mathbf{n}_1'$. if we make the replacement $\theta\to\theta'$, $\theta_1\to\theta_1'$, $\varphi\to\varphi'$, and $\varphi_1\to\varphi_1'$.

The function $Z(\lambda)$ is defined by the expression (6a) and can be rewritten in the form

$$Z(\lambda)=\Psi^{m-1}(\lambda)\exp[\beta(\lambda-\lambda_1)].\qquad(13)$$

In Eq. (11) we can immediately integrate over φ' and φ_1'. This makes it possible to obtain an equation in which the integration is performed only over the two variables λ' nd λ_1':

$$\Psi(\lambda)=\int_{-1}^{1}\int_{1}^{\infty}L_0(\lambda,\lambda')\frac{\lambda_1-\lambda}{\lambda_1'-\lambda'}Z(\lambda')\Psi(\lambda')\,d\lambda'\,d\lambda_1' +\exp[\alpha(\lambda-\lambda_1)],$$

where

$$L_0(\lambda,\lambda')=\tfrac{1}{2}\alpha^2\exp[\alpha(x-x_1)][I_0(\alpha y_1)I_0(\alpha y)(x_1+x) -I_1(\alpha y_1)I_0(\alpha y)y_1+I_1(\alpha y)I_0(\alpha y_1)y],\qquad(14)$$

in which I_0 and I_1 are Bessel functions of imaginary argument, and

$$x_1=\lambda_1\lambda_1',\quad x=\lambda\lambda',\quad y_1=[(\lambda_1^2-1)(\lambda_1'^2-1)]^{1/2},$$

$$y=[(1-\lambda^2)(1-\lambda'^2)]^{1/2}.\qquad(14a)$$

We note an important property of the kernel $L_0(\lambda,\lambda')$ and of the function $\Psi(\lambda)$:

$$\int_{-1}^{1}\int_{1}^{\infty}L_0(\lambda,\lambda')\frac{\lambda_1-\lambda}{\lambda_1'-\lambda'}d\lambda'\,d\lambda_1'=1-\exp[\alpha(\lambda-\lambda_1)];$$

$$\Psi(\lambda)=1\quad\text{for}\quad\lambda=(1,1).\qquad(15)$$

The invariance of the Lagrangian (1) under the replacement $U\to U_0U$ makes it possible to simplify Eq. (9) as well. As in the problem of wires,[9,10] we seek a solution P in the form

$$P=-ivR\bar{u},\quad R=\begin{pmatrix}f(\lambda,\lambda_1) & 0\\ 0 & if_1(\lambda,\lambda_1)\end{pmatrix}.\qquad(16)$$

Substituting (16) into (9) and integrating over all the variables except λ' and λ_1', we obtain

$$\mathbf{f}(r,\lambda)-[m+\delta(r-1)]\times\int_{-1}^{1}\int_{1}^{\infty}L(\lambda,\lambda')\mathbf{f}(r-1,\lambda')Z(\lambda')\frac{\lambda_1-\lambda}{\lambda_1'-\lambda'}d\lambda'\,d\lambda_1' =\delta(r)\Psi(\lambda)\begin{pmatrix}(1-\lambda^2)^{1/2}\\ (\lambda_1^2-1)^{1/2}\end{pmatrix},\qquad(17)$$

where $\mathbf{f}=\begin{pmatrix}f\\ f_1\end{pmatrix}$ is a two-component vector. The matrix $L(\lambda,\lambda')$ has the form

$$L(\lambda,\lambda')=\begin{pmatrix}L_{11}(\lambda,\lambda') & L_{12}(\lambda,\lambda')\\ L_{21}(\lambda,\lambda') & L_{22}(\lambda,\lambda')\end{pmatrix}\qquad(18)$$

and acts on the vector $\mathbf{f}$ in the usual way:

$$\begin{aligned}L_{11}(\lambda,\lambda')&=\tfrac{1}{2}\alpha^2\exp[\alpha(x-x_1)]\{I_1(\alpha y)I_0(\alpha y_1)\\ &\times(x+x_1)+I_0(\alpha y)I_0(\alpha y_1)y-I_1(\alpha y)I_1(\alpha y_1)y_1\\ &+[\alpha(\lambda_1-\lambda)(\lambda_1'-\lambda')]^{-1}[I_0(\alpha y)I_0(\alpha y_1)y-I_1(\alpha y)I_1(\alpha y_1)y_1]\},\\ L_{22}(\lambda,\lambda')&=\tfrac{1}{2}\alpha^2\exp[\alpha(x-x_1)]I_1(\alpha y_1)I_0(\alpha y)(x+x_1)\\ &-I_0(\alpha y_1)I_0(\alpha y)y_1+I_1(\alpha y_1)\{I_1(\alpha y)y+[\alpha(\lambda_1-\lambda)(\lambda_1'-\lambda')]^{-1}\\ &\times[I_0(\alpha y)I_0(\alpha y_1)y_1-I_1(\alpha y_1)I_1(\alpha y)y]\},\\ L_{12}(\lambda,\lambda')&=\tfrac{1}{2}\alpha\exp[\alpha(x-x_1)][(\lambda_1-\lambda)(\lambda_1'-\lambda')]^{-1}\\ &\times[I_0(\alpha y)I_0(\alpha y_1)(1-\lambda^2)^{1/2}(\lambda_1'^2-1)^{1/2}+I_1(\alpha y)I_1(\alpha y_1)\\ &\times(1-\lambda'^2)^{1/2}(\lambda_1^2-1)^{1/2}],\quad L_{21}(\lambda,\lambda')=L_{12}(\lambda',\lambda).\end{aligned}$$

The variables x, x_1, y, and y_1 are defined by the expressions (14a).

Substituting (16) and (17) into (10), we bring the expression for the correlator $K(r)$ to the form

$$K(r)=2\pi^2v^2\int_{-1}^{1}\int_{1}^{\infty}\frac{(1-\lambda^2)^{1/2}f(r,\lambda)+(\lambda_1^2-1)^{1/2}f_1(r,\lambda)}{(\lambda_1-\lambda)^2}\times Z(\lambda)\Psi(\lambda)\,d\lambda\,d\lambda_1.\qquad(19)$$

The equations (14) and (16)–(18) and the integral (19) solve completely the posed problem of the calculation of the density-density correlator $K(r)$. We note that up to now we have not made any approximations. Of course, an exact analytical calculation of the correlator $K(r)$ is impossible. However, we can investigate the most interesting asymptotic forms—in particular, the asymptotic form at large distances r. Introducing the new function

$$\mathbf{p}=[Z(\lambda)]^{1/2}(\lambda_1-\lambda)^{-1}\mathbf{f}(\lambda),$$

we bring Eq. (17) to the form

$$\mathbf{p}(r,\lambda)-[m+\delta(r-1)]\hat{M}\mathbf{p}(r-1,\lambda) =\delta(r)\frac{\Psi(\lambda)[Z(\lambda)]^{1/2}}{\lambda_1-\lambda}\begin{pmatrix}(1-\lambda^2)^{1/2}\\ (\lambda_1^2-1)^{1/2}\end{pmatrix}.\qquad(20)$$

In (20) the action of the operator $\hat{M}$ on an arbitrary function $\varphi(\lambda)$ is specified by the integral

$$\hat{M}\varphi(\lambda)\equiv\int_{-1}^{1}\int_{1}^{\infty}[Z(\lambda)]^{1/2}L(\lambda,\lambda')[Z(\lambda')]^{1/2}\varphi(\lambda')\,d\lambda'\,d\lambda_1'.\qquad(20a)$$

The expressions (18) and (20a) show that the operator $\hat{M}$ is real and symmetric. The eigenfunctions $\varphi_E(\lambda)$, satisfying the equation

$$\hat{M}\varphi_E(\lambda)=E\varphi_E(\lambda),\qquad(21)$$

are orthogonal to each other and form a complete set. The eigenvalues E are real.

By expanding the function $\mathbf{p}(\lambda)$ in (20) in eigenfunctions of the operator $\hat{M}$, we can find the solution of Eq. (20). Substituting this solution into (19), we bring the expression for the correlator $K(r)$ to the form

$$K(r)=\frac{2\pi^2\nu^2(m+1)}{m}\sum_E B_E(mE)^r,$$

$$B_E=\Big[\int_{-1}^{1}\int_{1}^{\infty}\frac{(1-\lambda^2)^{1/2}\varphi_{E1}(\lambda)+(\lambda_1^2-1)^{1/2}\varphi_{E2}(\lambda)}{\lambda_1-\lambda}$$

$$\times[Z(\lambda)]^{1/2}\Psi(\lambda)\,d\lambda\,d\lambda_1\Big]^2, \qquad (22)$$

where the summation is performed over all the eigenvalues of the operator $\hat{M}$; φ_1 and φ_2 are the components of the vector function φ. For $r=0$ the summation can be performed immediately, if we make use of the completeness of the set of eigenfunctions. In this case we obtain

$$K(0)=\frac{2\pi^2\nu^2(m+1)}{m}\int_{-1}^{1}\int_{1}^{\infty}\frac{\lambda_1+\lambda}{\lambda_1-\lambda}Z(\lambda)\Psi^2(\lambda)\,d\lambda\,d\lambda_1. \qquad (23)$$

Of course, this formula can also be written directly using the integral (3). It was the quantity $K(0)$ that was studied in Ref. 6.

The subsequent calculation in (22) can be carried out in the low-frequency limit for the metallic and insulating regions separately. The critical value α_c separating these regions is determined by the equation[6]

$$m\Gamma_{00}(\alpha_c)=1, \qquad (24)$$

$$\Gamma_{n\rho}(\alpha)=\alpha\left[K_{i\rho}(\alpha)\frac{d}{d\alpha}I_{n+1/2}(\alpha)-I_{n+1/2}(\alpha)\frac{d}{d\alpha}K_{i\rho}(\alpha)\right],$$

where $K_\nu(\alpha)$ and $I_\nu(\alpha)$ are Bessel functions of imaginary argument. For $\alpha>\alpha_c$ the conductivity is nonzero, while for $\alpha<\alpha_c$ the system does not conduct.

3. THE INSULATING REGION

In the region of localized states in the limit of low frequencies β all the equations are simplified considerably. This is connected with the fact that the main contribution is made by values $\lambda_1\sim 1/\beta\gg 1$ (Refs. 6, 7, 9, 10). Passing to the limit $\lambda_1\gg 1$ and making the change of variables $2\beta\lambda_1=\exp t$, we bring Eq. (14) to the form

$$\Psi(t)=\int_{-\infty}^{\infty}\bar{L}(t-\tau)\exp\frac{t-\tau}{2}Z(\tau)\Psi(\tau)\,d\tau,$$
$$Z(t)=\exp(-{}^1/_2e^t)\,\Psi^{m-1}(t), \qquad (25)$$
$$\bar{L}(t)=\frac{\alpha^{1/2}}{(2\pi)^{1/2}}\exp(-\alpha\operatorname{ch}t)\left[\frac{\operatorname{sh}\alpha}{\alpha}\operatorname{ch}t+\frac{\operatorname{ch}\alpha}{\alpha}-\frac{\operatorname{sh}\alpha}{2\alpha^2}\right].$$

In the derivation of (25) it was assumed that for small β the solution of Eq. (14) is a function only of $\beta\lambda_1$, and this made it possible to perform the integration over λ. From the equality (15) it follows that

$$\int_{-\infty}^{\infty}\bar{L}(t)e^{-t/2}\,dt=1. \qquad (26)$$

The equality (26) can also be obtained by direct integration of $\bar{L}(t)$ (25).

Passing to the limit of small β and large λ_1, in place of (20) we obtain

$$\bar{p}(r,t)-[m+\delta(r-1)]\int_{-\infty}^{\infty}\bar{M}(t,\tau)\bar{p}(r-1,\tau)\,d\tau$$
$$=\Psi(t)[Z(t)]^{1/2}e^{-t/2}, \qquad (27)$$

where

$$\bar{M}(t,\tau)=\bar{L}(t-\tau)[Z(t)Z(\tau)]^{1/2},$$

and $\bar{p}$ is the second component of the vector $\mathbf{p}\beta^{1/2}\exp(-t/2)$. The first component is small in the limit $\beta\to 0$, $\beta\lambda_1=\text{const}$. This can be seen directly from the right-hand side of (20). The first formula in (22) for the correlator $K(r)$ preserves its form. For the coefficients B_E in (22) we obtain

$$B_E=\frac{1}{\beta}\Big[\int_{-\infty}^{\infty}\Psi(t)[Z(t)]^{1/2}e^{t/2}\varphi_E(t)\,dt\Big]^2. \qquad (28)$$

Here the functions φ_E are eigenfunctions of the symmetric operator $\bar{M}$:

$$\int_{-\infty}^{\infty}\bar{M}(t,\tau)\varphi_E(\tau)\,d\tau=E\varphi_E(t) \qquad (29)$$

and satisfy the orthogonality and completeness properties. Using these properties, we can obtain $K(0)$ in the localized region:

$$K(0)=\frac{2\pi^2\nu^2(m+1)}{m\beta}\int_{-\infty}^{\infty}Z(t)e^t\Psi^2(t)\,dt. \qquad (30)$$

This formula can also be easily obtained directly from (23). An investigation of the dependence of $K(0)$ on α was carried out in Refs. 6 and 7. It was shown that $\beta K(0)$ decreases monotonically with increase of α, reaching a certain critical value at $\alpha=\alpha_c$. More interesting is the behavior or $K(r)$ at large r, since this correlator describes the decay of the wavefunctions.

To elucidate the asymptotic form of the correlator $K(r)$ at large r we shall consider the properties of the eigenfunctions $\varphi_E(t)$ of the operator $\bar{M}$. First of all, we shall determine the asymptotic form as $t\to-\infty$. In this limit the function $\Psi(t)$ satisfying Eq. (25) tends to unity.[6,7] Therefore, for the determination of the asymptotic form as $t\to-\infty$, in place of (29) we have

$$\int_{-\infty}^{\infty}\bar{L}(t-\tau)\varphi_E(\tau)\,d\tau=E\varphi_E(t). \qquad (31)$$

From Eq. (31) we obtain

$$\varphi_E(t)=(2/\pi)^{1/2}\sin[-\rho t+\delta(\rho)],\quad t\to-\infty, \qquad (32)$$

where the parameter ρ is a certain function of E, and $\delta(\rho)$ is a phase that depends on ρ. As $t\to\infty$ the function $\varphi_E(t)$ should decay to zero for all $E>0$, because of the presence in $\bar{M}(t,\tau)$ of the factor $[Z(t)Z(\tau)]^{1/2}$ which decays in this region. Therefore, the eigenvalue spectrum of the operator $\bar{M}(t,\tau)$ is continuous. The general factor multiplying the sine ensures that the functions $\varphi_E(t)$ are orthonormal:

$$\int_{-\infty}^{\infty}\varphi_E(t)\varphi_{E'}(t)\,dt=\delta(\rho-\rho').$$

Substituting (32) into (31), we can determine the dependence $E(\rho)$:

$$E(\rho)=\Gamma_{0\rho}(\alpha), \tag{33}$$

where $\Gamma_{0\rho}$ is determined by the expression (24).

We now elucidate the behavior of the function φ_E as $\rho\to 0$. In this limit, for $t\gg 1$ the function approaches the asymptotic form

$$\varphi_E(t)\approx c(\rho)u(t), \quad t\gg 1. \tag{34}$$

The function $u(t)$ in (34) no longer depends on ρ. The coefficient $c(\rho)$ is determined by matching with the solution as $t\to -\infty$. Equating the asymptotic forms (32) and (34), and also their derivatives, at $t\sim 1$, we obtain $c(\rho)\sim\rho$, $\delta(\rho)\sim\rho t_0$, $t_0\sim 1$.

This reasoning makes it possible to write the function $\varphi_E(t)$ for $\rho\to 0$ and all t in the form

$$\varphi_E(t)=\rho v(t). \tag{35}$$

We note that $v(t)$ tends to a finite limit $v_0(t)$ as $\alpha\to\alpha_c$. This follows from the fact that the function $\Psi(t)$ (Refs. 6, 7) and, consequently, the kernel $\bar{M}(t,\tau)$ (27) tend to a finite limit $\Psi_0(t)$.

Owing to the continuity of the spectrum of the kernel $\bar{M}(t,\tau)$, the sum over the energies E in (22) must be replaced by an integral over ρ. At large r the main contribution is made by the region of small ρ. Substituting (35) into (28), and then (28) into (22), and expanding $E(\rho)$ for small ρ, we obtain

$$K(r)=\frac{2\pi^2\nu^2(m+1)}{m\beta}a^2e^{-r/4l}\int_0^\infty \rho^2 e^{-rb\rho^2}d\rho, \tag{36}$$

where

$$a=\int_{-\infty}^{\infty} v(t)\Psi(t)[Z(t)]^{1/4}e^{t/2}\,dt,$$

$$l^{-1}=-4\ln[mE(0)]=-4\ln[m\Gamma_{00}(\alpha)],$$

$$b=\frac{1}{2}\left|\frac{\partial^2 E(0)}{\partial\rho^2}\right|=-\frac{1}{2}\left[\frac{\partial^2\Gamma_{0\rho}(\alpha)}{\partial\rho^2}\right]_{\rho=0}.$$

A convenient characteristic of the wave functions is the long-time limit of the density-density correlation function in the time representation. Denoting this limit by $p_\infty(r)$, calculating the integral over ρ in (36), and going over to the time representation $\tilde{K}(r,t)$ with the aid of (5), we obtain

$$\tilde{K}(r,t\to\infty)\equiv p_\infty(r)=\frac{\pi^{1/2}\nu}{2V}\frac{m+1}{m}a^2(br)^{-1/4}e^{-r/4l}. \tag{37}$$

The formula (37) is applicable for $r\gg 1$, and the coefficients a and b remain finite as the transition point α_c is approached. The only quantity that has singular behavior near the transition point is the localization length l. Comparing (36) with (24), we obtain the asymptotic behavior of l as $\alpha\to\alpha_c$:

$$l=\text{const}/(\alpha_c-\alpha). \tag{38}$$

We note that the quantity $m\Gamma_{00}(\alpha)$ decreases monotonically with decrease of α. It follows from this that the localization length l decreases monotonically. The expression (37) has the same form as the corresponding expression for one-dimensional systems.[9–11] Formula (37) is also applicable for $m=1$, which corresponds to a one-dimensional chain of granules. Of course, in the one-dimensional case there are no singularities for finite α, since the function $\Gamma_{00}(\alpha)$ decreases monotonically from 1 to 0 as α varies from ∞ to 0. For $m=1$ and $\alpha\gg 1$ the localization length l is proportional to α. In this limit the model on a lattice can be replaced by a continuum model, for which the formulas obtained in Refs. 9 and 10 are applicable. The diffusion coefficient D to which the localization length in the continuum model is proportional (we have in mind the classical diffusion coefficient) coincides, to within a numerical factor, with the parameter α.

Knowledge of the function $p_\infty(r)$ makes it possible to calculate the response to an external electric field $\mathscr{E}$ of the following symmetry: We select a particular site and let the potential at each point situated at a distance r from this site be equal to $\mathscr{E}r$. We shall calculate the polarizability $\varkappa$ in this field. It is not difficult to see that it is equal to

$$\varkappa=e^2\sum_r r^2 p_\infty(r), \tag{39}$$

where e is the electron charge. Near the transition point, when l is large, the main contribution to the sum (39) is made by large r. In this limit we can use the expression (37). Calculating the sum in (39), we obtain

$$\varkappa=\text{const}\cdot l^{7/4}\sim(\alpha_c-\alpha)^{-7/4}. \tag{40}$$

Naturally, for large values of $\varkappa$ the dielectric permittivity is proportional to $\varkappa$.

In Ref. 6 the correlator $K(0)$ was investigated. The quantity $p_\infty^{-1}(0)$ was called an integral permittivity, and it was shown that this quantity tends to a finite value as the transition point is approached. Although the expressions given by formulas (39) and (40) correspond to the real dielectric permittivity, the quantity $p_\infty(0)$ also has an important physical meaning. This quantity is proportional to the average probability that a quantum particle that was at some particular point at time $t=0$ will be found at the same point after an infinite time. The fact that this probability does not vanish as the transition point is approached seems to be not at all trivial. It was this fact which stimulated the assertion in Ref. 6 that the physical dielectric constant is finite at the transition point.

In real physical systems, evidently, the response to a spatially uniform field should grow as the transition point is approached. At the same time, it can be postulated that the response to some particular field localized in a small region of space can remain finite.

The expression (37) and (38) were obtained near the transition point in the presently considered model on a Cayley tree by Zirnbauer[8] in a calculation of a certain correlator $K^{(1)}$ differing from the density-density correlator. The fact that in the insulating region these two correlators become equal as the frequency $\omega\to 0$ enabled him to obtain the formulas (37), (38) for the density-density correlator as well.

In principle, the equations (14), (18), (21) and (22) derived here make it possible to calculate the density-density correlator for all frequencies; it is evident, however, that this can only be done numerically.

4. THE METALLIC REGION

Before calculating the density-density correlator, we shall consider the classical diffusion of a particle on a Bethe lattice. We shall find the probability $W_N(r)$ that a particle executing a random walk on a Bethe lattice will be found after N steps at a distance r from the point from which it started. This probability coincides, to within a normalization factor, with the density-density correlator. Let the probabilities of the moves from some site to the neighboring sites be equal. Then each such probability is equal to $1/(m+1)$. It is not difficult to write a recursion equation for $W_N(r)$:

$$W_{N+1}(r)=W_N(r-1)\frac{m+\delta(r-1)}{m+1}+\frac{W_N(r+1)}{m+1},$$

$$W_0(0)=1, \quad W_N(-r)=0. \tag{41}$$

Equation (41) is linear, and can be solved by means of Fourier transformations.

Let the number of steps per unit time be equal to n. In real physical processes the number n is determined by internal characteristics of the system. In the limit of low physical frequencies ω the solution of Eq. (41) gives

$$W(\omega,r)=\frac{1}{(m-1)D}\left[1-\frac{i(\omega+i\delta)}{(m-1)D}\right]^{-r}$$

$$\approx\frac{1}{(m-1)D}\exp\left[\frac{i(\omega+i\delta)r}{(m-1)D}\right]. \tag{42}$$

Here $D=n/(m+1)$.

If we go over from the frequency representation $W(\omega,r)$ to the time representation, we immediately obtain the probability of finding the particle at a distance r at time t. The coefficient D in (42) is proportional to the probability of a move from a particular site to a neighboring site per unit time. It can be identified with the diffusion coefficient. Consequently, if for some model on a Bethe lattice one can obtain the law (42), one can make definite statements about the diffusion coefficient. Below it will be shown that in the model (1)–(3) under consideration the density-density correlator $K(\omega,r)$ in the metallic region is indeed described, to within a normalization factor, by formula (42), and this makes it possible to calculate the diffusion coefficient D.

In the limit $\alpha\gg 1$ the correlator $K(r)$ (3), (10) with the Lagrangian (1) was calculated in Ref. 6. At low frequencies this correlator has the form

$$K(r)=\frac{(2\pi\nu)^2}{cm}e^{-\beta r/cm}. \tag{43}$$

The expression (43) differs from (42) only in the normalization. Recalling the connection between β and the physical frequency ω (5), we obtain for the diffusion coefficient

$$D=\alpha/\pi\nu V. \tag{44}$$

We now show that for all $\alpha>\alpha_c$ the correlator $K(r)$ has the form (43), if the frequency is small and r is greater than a certain characteristic length that depends, for $\beta\to 0$, only on α. For this we shall make use of the eigenfunction expansion (22). First of all we note that the spectrum of the eigenfunctions φ_E of the operator $\bar{M}$ [see (20a) and (21)] in the metallic region $\alpha>\alpha_c$ is discrete. This is connected with the fact that the function $\Psi(\lambda)$ and, consequently, $Z(\lambda)$ fall off to zero with increase of λ_1 (Refs. 6, 8). Therefore, the kernel in the integrand of (20a) differs substantially from zero only in a finite range of variation of its arguments, and this region also remains finite as $\beta\to 0$. Correspondingly, the spectrum also remains discrete in this limit. The character of the decrease of the function $\Psi(\lambda)$ will be discussed in more detail later. The discreteness of the spectrum makes it possible to calculate the sum (22) at large distances $r\gg\xi(\alpha)$, where $\xi(\alpha)$ is a certain characteristic length. In this limit, only one term, with the larget value $E_0(\beta)$, remains. The length $\xi(\alpha)$ is inversely proportional to the difference of the logarithms of this eigenvalue and of the next eigenvalue, and remains finite as $\beta\to 0$.

For $\beta=0$ the largest eigenvalue $E_0(0)$ and the corresponding eigenfunction $\varphi_0(\lambda)$ of the operator $\bar{M}$ can be found exactly:

$$E_0(0)=\frac{1}{m}, \quad \varphi_0(\lambda)=[\Phi_\beta(\lambda)]_{\beta=0},$$

$$\Phi_\beta(\lambda)=\frac{c_1(\beta)[Z(\lambda)]^{1/2}}{\lambda_1-\lambda}\begin{pmatrix}-\dfrac{\partial\Psi}{\partial\lambda}(1-\lambda^2)^{1/2}\\ \dfrac{\partial\Psi}{\partial\lambda_1}(\lambda_1^2-1)^{1/2}\end{pmatrix}, \tag{45}$$

where $c_1(\beta)$ is a normalization factor, equal to

$$c_1(\beta)=\left[\int_{-1}^{1}\int_{1}^{\infty}\left[\left(\frac{\partial\Psi}{\partial\lambda}\right)^2(1-\lambda^2)+\left(\frac{\partial\Psi}{\partial\lambda_1}\right)^2(\lambda_1^2-1)\right]\Psi^{m-1}\frac{d\lambda\,d\lambda_1}{(\lambda_1-\lambda)^2}\right]^{-1/2}. \tag{46}$$

We note that $\Phi_\beta(\lambda)$ for $\beta\neq 0$ is not an eigenfunction of the operaor $\hat{M}$. In (46) we have used the equality (13).

In order to see that $E_0(0)$ and $\varphi_0(\lambda)$ are indeed an eigenvalue and eigenfunction of the operator $\hat{M}$, we substitute (45) into (21), (20a), and (18). In the left-hand side of (21) we integrate by parts. After this we can convince ourselves that for $\beta=0$ the first of the equations obtained coincides with the derivative of the two sides of Eq. (14) with respect to λ, multiplied by $-(1-\lambda^2)^{1/2}$, while the second coincides with the derivative with respect to λ_1, multiplied by $(\lambda_1^2-1)^{1/2}$. This proves that the expressions (45) are an eigenvalue and eigenfunction of the operator $\hat{M}$ (20a) for $\beta=0$.

If β is nonzero but small, the eigenfunctions and eigenvalues are close to the corresponding quantities for $\beta=0$. In the expression (22) for B_E we can simply set $\beta=0$. For the coefficient B_0 corresponding to the eigenvalue (45) we obtain

$$B_0=c_1^2(0)\left[\int_{-1}^{1}\int_{1}^{\infty}\left[(\lambda_1^2-1)\frac{\partial\Psi}{\partial\lambda_1}-(1-\lambda^2)\frac{\partial\Psi}{\partial\lambda}\right]\Psi^m\frac{d\lambda\,d\lambda_1}{(\lambda_1-\lambda)^2}\right]^2. \tag{47}$$

The subsequent calculation of the coefficient B_0 can be carried out by integrating by parts in (47). Here we must exercise some care, since for $\lambda=\lambda_1=1$ the integrand is not defined. In order to circumvent this difficulty, we can calculate the integral over the region $-1<\lambda<1-\delta$, $1+\delta_1<\lambda_1$,

and then in the final answer take the limit $\delta \to 0$, $\delta_1 \to 0$. After straightforward calculations, we obtain

$$B_0 = 4c_1^2(0)/(m+1)^2. \tag{48}$$

In the derivation of (48) we used the equality (15) for the function $\Psi(\lambda)$.

For the calculation of $E_0(\beta)$ for small but nonzero values of β we multiply both sides of Eqs. (21) and (20a) by the vector $\Phi_\beta(\lambda)$ (45) and integrate both sides over λ and λ_1. In the left-hand side, as in the case with $\beta = 0$, we again integrate by parts and use the derivatives of both sides of Eq. (14), but now with $\beta \neq 0$. For small β the eigenfunction φ_E of the operator $\hat{M}$ (21), (20a) coincides with φ_0 (45). Performing the necessary transformations, we obtain in the limit $\beta \to 0$

$$E_0(\beta) = \frac{1}{m}\left\{1 + \frac{\beta}{m} c_1^2(0) \int_{-1}^{1}\int_{1}^{\infty} \frac{\Psi^m(\lambda)}{(\lambda_1-\lambda)^2}\left[(\lambda_1^2-1)\frac{\partial \Psi}{\partial \lambda_1} - (1-\lambda^2)\frac{\partial \Psi}{\partial \lambda}\right] d\lambda\, d\lambda_1\right\}$$

$$= \frac{1}{m}\left[1 - \frac{2\beta c_1^2(0)}{m(m+1)}\right]. \tag{49}$$

In (49) and everywhere below we understand by $\Psi(\lambda)$ the solution of Eq. (14) with $\beta = 0$. Substituting (5), (46), (48), and (49) into (22), we obtain

$$K(r) = \frac{4\pi\nu}{V(m-1)D} \exp\left[\frac{i(\omega+i\delta)r}{(m-1)D}\right], \tag{50}$$

where the diffusion coefficient is equal to

$$D = \frac{m}{2\pi\nu V}\frac{m+1}{m-1}\int_{-1}^{1}\int_{1}^{\infty} \Psi^{m-1}\left[(\lambda_1^2-1)\left(\frac{\partial \Psi}{\partial \lambda_1}\right)^2 + (1-\lambda^2)\left(\frac{\partial \Psi}{\partial \lambda}\right)^2\right] \frac{d\lambda\, d\lambda_1}{(\lambda_1-\lambda)^2}. \tag{51}$$

The expression (50) corresponds to formula (42) for classical random walks on a Bethe lattice, and, as already discussed, the diffusion coefficient D is proportional to the probability of a move from one side to a neighboring site per unit time. In the derivation of the expressions (50), (51) we have used only the smallness of the quantities β and r^{-1}. The frequency β should be smaller than $c_1^{-2}(0)$ (46), in order to ensure that the correction is small in comparison with the perturbed eigenvalue (45). The relative magnitudes of β and r^{-1} can be arbitrary.

Solving Eq. (14) for $\beta = 0$ and using (51), we can construct, at least numerically, the entire curve of the dependence $D(\alpha)$ for $\alpha > \alpha_c$. We can calculate $D(\alpha)$ analytically in the limits $\alpha - \alpha_c \ll \alpha_c$ and $\alpha \gg 1$. For $\alpha \gg 1$ values of λ and λ_1 close to unity are important. In this limit the function Ψ is equal to[6]

$$\Psi = \exp[-c(\lambda_1-\lambda)], \tag{52}$$

where

$$c = \alpha(m-1)/m. \tag{52a}$$

Substituting this solution into (50), (51), we arrive at the expressions (43), (44) already obtained in this limit.

We turn to the investigation of the behavior of the diffusion coefficient B in the neighborhood of the transition point. A scheme for the analytical investigation of the function $\Psi(\lambda)$ was proposed in Ref. 6. Following this scheme, we rewrite Eq. (11) in the form

$$(\hat{\Gamma}^{-1}-m)U(\lambda) = -F[U],$$

$$\hat{\Gamma}\Phi(\lambda) = \int \Gamma(\mathbf{nn}', \mathbf{n}_1\mathbf{n}_1')\Phi(\lambda)\frac{dn'\, dn_1'}{(2\pi)^2}, \tag{53}$$

where

$$U = (1-\Psi)/(\lambda_1-\lambda),$$

$$F[U] = \{[1-U(\lambda_1-\lambda)]^m + mU(\lambda_1-\lambda) - 1\}/(\lambda_1-\lambda).$$

We shall study the solutions that decay as $\lambda_1 \to \infty$. The solution $\Psi = 1$ corresponds only to the insulating region $\alpha < \alpha_c$. As $\lambda, \lambda_1 \to 1$, U approaches a constant. For functions Ψ of the form described, $F[U] \geqslant 0$.

First of all we can write the asymptotic form of the solution Ψ for $\lambda_1 \to \infty$. For such λ_1 the function Ψ does not depend on λ, and in place of Eq. (11) we can write Eq. (25), having set $Z = \Psi^{m-1}$ in the latter, since we are considering the limit $\beta = 0$. In Ref. 6 the asymptotic form of the solution Ψ was obtained:

$$\Psi \propto \exp[-b_1(\theta_1^{1/2}+\gamma)\exp(b_2\theta_1^{1/2})], \tag{54}$$

where b_1 and b_2 are numerical coefficients and $\beta_1 = \ln(2\lambda_1)$. The parameter γ in (54) is arbitrary; this corresponds to the invariance of the kernel in Eq. (25) under a scale change $\lambda_1 \to A\lambda_1$. If Ψ_s is a particular solution, e.g., that corresponding to the asymptotic form (54) with $\gamma = 0$, then for the functions Ψ and U for $\lambda_1 \gg 1$ we have

$$\Psi(\lambda_1) = \Psi_s(A\lambda_1), \quad U(\lambda_1) = AU_s(A\lambda_1), \tag{55}$$

where A is a number determined by matching with the region $\lambda_1 \sim 1$. For the calculation of A we multiply both sides of (53) by $P_{-1/2+i\rho}(\lambda_1)P_n(\lambda)$ and integrate over λ_1 and λ. The product $P_{-1/2+i\rho}(\lambda_1)P_n(\lambda)$ of Legendre functions is an eigenfunction of the operator Γ. Therefore, we obtain

$$[\Gamma_{n\rho}^{-1}(\alpha) - m]U_{n\rho} = -F_{n\rho}, \tag{56}$$

$$U_{n\rho} = \int_{-1}^{1}\int_{1}^{\infty} UP_{-1/2+i\rho}(\lambda_1)P_n(\lambda)\, d\lambda\, d\lambda_1,$$

$$F_{n\rho} = \int_{-1}^{1}\int_{1}^{\infty} F[U]P_{-1/2+i\rho}(\lambda_1)P_n(\lambda)\, d\lambda\, d\lambda_1,$$

where $\Gamma_{n\rho}$ is given by the expression (24).

If we assume that $A \to 0$ as $\alpha \to \alpha_c$, then the important region in the integrals in (56) is the region $\lambda_1 \gg 1$. In this limit the solution does not depend on λ and it is necessary to set $n = 0$. We also set $\rho = 0$ and estimate the integrals obtained. Using the asymptotic form $P_{-1/2}(z) \sim \ln z/z^{1/2}$ for $z \gg 1$, substituting (55) into (56), and making the replacement $\lambda_1 A = z$, we bring (56) to the form

$$[m - \Gamma_{00}^{-1}(\alpha)]\int_A^\infty \frac{U_s(z)}{z^{1/2}} \ln\frac{z}{A}\, dz = \int_A^\infty \frac{F[U_s]}{z^{1/2}} \ln\frac{z}{A}\, dz. \tag{57}$$

Strictly speaking, the form of the integrands is applicable only for $z \gg A$ ($\lambda_1 \gg 1$). However, the region $\lambda_1 \sim 1$ makes a small contribution. The function $U_s(z)$, like $\Psi_s(z)$, varies over scales $z \sim 1$.

At first sight, it appears that integrals in both sides of the equation are of order $\ln(1/A)$. But the integral in the left-hand side is multiplied by the quantity $m - \Gamma_{00}^{-1}(\alpha)$, which tends to zero as $\alpha \to \alpha_c$. This leads to a contradiction, on the basis of which it was stated in Ref. 6 that A remains finite at the transition point. Then the diffusion coefficient need not tend to zero either.

However, as shown by Zirnbauer,[8] this crude estimate of the integral in the left-hand side of Eq. (57) is not exact, since, although the counting of the powers of the parameter A is correct, extra powers of $\ln(1/A)$ arise. To convince ourselves of this, we shall consider the asymptotic forms of the function $U_s(z)$. For $z \gg 1$ the function $U_s \sim A/z$, as can be seen directly from the relationship (53) between U and Ψ and from the decay of Ψ at infinity. For small $A \ll z \ll 1$ the quantity U is small and the asymptotic form of U_s can be found by neglecting the right-hand side $F[U]$ in (53), which for small U is proportional to U^2. In this limit we obtain

$$U_s(z) = a(\varepsilon)\varepsilon^{-1} z^{-1/2} \sin[-\varepsilon \ln z + \delta(\varepsilon)], \tag{58}$$

where $a(\varepsilon)$ and $\delta(\varepsilon)$ are unknown parameters to be determined from the condition for matching at $z \gg 1$ with the asymptotic form (54), in which we must set $\gamma = 0$ and $\theta_1 = \ln(2z)$, and ε is determined from the equation

$$\Gamma_{0s}^{-1}(\alpha) - m = 0. \tag{58a}$$

For $\alpha - \alpha_c \ll \alpha_c$ we have $\varepsilon \sim (\alpha - \alpha_c)^{1/2}$.

For $\varepsilon \to 0$ the asymptotic form (54) has no singularities. Joining of the asymptotic forms (54) and (58) for $z \to 1$ in the limit $\varepsilon \to 0$ gives the limiting expressions $a(\varepsilon) \to a$ and $\delta(\varepsilon)/\varepsilon \to \delta_0$ for $\varepsilon \to 0$.

For $z \ll 1$ the functional $F[U] \sim U_s^2(z) z$. Recalling the asymptotic form for $z \gg 1$, we convince ourselves that in the right-hand side of (57) the regions $z \ll 1$ and $z \gg 1$ make a small contribution. The main contribution is made by the region $z \sim 1$, and this contribution is of order

$$\int_A^\infty \frac{F[U_s]}{z^{1/2}} \ln\frac{z}{A}\, dz \sim \ln\frac{1}{A}.$$

This estimate was obtained in Ref. 6. The region $z \sim 1$ also gives the same in the integral in the left-hand side of (57). However, the region $A \ll z \ll 1$ gives an extra logarithmic factor. Assuming that $\varepsilon^{-1} \sim \ln(1/A)$ (this will be seen from the final result), we have

$$\int_A^1 \frac{U_s(z)}{z^{1/2}} \ln\frac{z}{A}\, dz \sim \int_A^1 \ln\left(\frac{z}{A}\right) \frac{\sin[\varepsilon(-\ln z + \delta_0)]}{\varepsilon} \frac{dz}{z}$$
$$\sim \int_A^1 \ln\frac{z}{A} \ln\left(\frac{1}{z}\right) \frac{dz}{z} \sim \ln^3\frac{1}{A}.$$

The logarithmic contribution from the region $z \ll 1$ was omitted in Ref. 6.

Substituting the estimates obtained into (57), we obtain

$$A = p \exp[-q(\alpha - \alpha_c)^{-1/2}], \tag{59}$$

where p and q are numerical coefficients that depend on m. The formulas (55) and (59) show that the function $\Psi(\lambda_1)$ falls off slowly in the region $\lambda_1 \lesssim 1/A$, and then, according to the asymptotic form (44), the decay becomes sharp. As the transition point is approached the boundary of the region of slow decay tends to infinity. The expression (59) was obtained by Zirnbauer,[8] who arrived at it by combining the results of a numerical and an analytical investigation.

It is possible to propose a simpler way of obtaining (59), which makes it possible, in particular, to find the coefficient q in explicit form. For this we write the function U in the region $\lambda_1 \ll 1/A$. Neglecting the right-hand side in (53), we have

$$U(\lambda, \lambda_1) = a_0 P_{-1/2+i\varepsilon}(\lambda_1).$$

Using the asymptotic form for $\lambda_1 \gg 1$,

$$P_{-1/2+i\varepsilon}(\lambda_1) \approx (2/\pi\varepsilon)\lambda_1^{-1/2} \sin[\varepsilon \ln(2\lambda_1)],$$

and matching with (58), we obtain with the aid of (55) for $\varepsilon \to 0$

$$2a_0/\pi = A^{1/2} a, \quad \varepsilon \ln(1/A) = \pi. \tag{60}$$

Solving Eq. (58a) in the limit of small $\alpha - \alpha_c$, we obtain

$$q = \pi\left[b(\alpha_c) \Big/ \frac{\partial \Gamma_{00}(\alpha_c)}{\partial \alpha} \right]^{1/2}, \tag{60a}$$

where $b(\alpha)$ is defined in (36). Of course, this analysis does not enable us to determine p.

The results obtained for the function Ψ make it possible, using (51), to calculate the diffusion coefficient D. The main contribution to the integral (51) is made by the region $1 \ll \lambda_1 \ll 1/A$. Using (53), (55), and (58), we have for Ψ in this region

$$\Psi(\lambda_1) = 1 - a(A\lambda_1)^{1/2} \ln(2\lambda_1).$$

Substituting this expression into (51), we obtain

$$D = [m(m+1)a^2/12\pi\nu V(m-1)] A \ln^3(1/A).$$

Using the expression (60) for A, we finally obtain

$$D = \frac{pa^2 m(m+1)}{12\pi\nu V(m-1)} \frac{\exp[-q(\alpha-\alpha_c)^{-1/2}]}{(\alpha-\alpha_c)^{3/2}}. \tag{61}$$

The formula (61) completely solves the problem of the calculation of the diffusion coefficient in the critical region. Although a minimum metallic conductivity, the existence of which was predicted erroneously in Ref. 6, is absent, the very sharp decrease of (61) as the transition point is approached makes it possible to speak of the existence of a quasijump near the critical point.

5. DISCUSSION OF THE RESULTS. THE ROLE OF THE NONCOMPACTNESS

1. The density-density correlator calculated in the preceding sections makes it possible to give a complete description of the kinetics of the system under consideration. In the insulating region of $\alpha - \alpha_c$ this correlator decays exponentially at large distances [see (37)], and for $\alpha_c - \alpha \ll \alpha_c$ the localization length l (38) is inversely proportional to the distance $\alpha_c - \alpha$ from the transition point. This behavior of the localization length near the transition point coincides

with the characteristic-length behavior predicted in Ref. 3. The form of the correlator (37) is exactly the same as in the one-dimensional models of Refs. 9–11. The dependence of l on α is specified by the exact expressions (36), (24), which are applicable for all $\alpha < \alpha_c$. It is interesting to note that the authors of Ref. 3 considered an entirely different model of a disordered metal on a Bethe lattice, in which, in particular, time-reversal symmetry was not broken. This agreement of the results is characteristic for the one-dimensional case. In Ref. 11 disordered chains were studied, while in Ref. 9 and 10 wires were investigated. The coincidence of the results in all these cases implies that the model (1) of a granulated metal correctly describes the kinetics. However, this is also clear without calculations. The model (1) differs from, say, the Anderson model in that the density of states in (1) is fixed. But fluctuations of this quantity should not have a qualitative effect on the phenomenon of localization.

The dependence (38) of l on α is very similar to the corresponding dependences of the characteristic lengths in the theory of second-order phase transitions (it has a power-law form). However, this length, diverging at the transition point, occurs only in the exponent in the correlator (37), despite the presence of a power of a distance in the pre-exponential factor. This leads to the result that the application of the traditional scaling relations for the calculation of physical quantities does not give correct results. For example, the polarizability $\varkappa$ (40) is proportional to $l^{3/2}$, and not to l^2 as would follow from ordinary scaling. The probability that a quantum particle remains at its original site after an infinite time does not tend to zero as the transition point is approached, and has a discontinuity at the transition point. This probability is determined by the density-density correlator at coinciding points, which was calculated in Ref. 6. On the basis of this property, it was postulated in Ref. 6 that there exists a maximum dielectric permittivity. It is evident that the correctness of this assertion depends on what form the applied external field takes. If we examine the response to a spatially uniform electric field, this response should diverge, as can be conjectured by looking at the expression (40). However, we cannot exclude the possibility that the response to a field that is nonzero only in some particular small region of space will remain finite, since it is determined by the correlator at coinciding points.

The exponential behavior of the diffusion coefficient (61) seems very interesting. The assertion in Ref. 6 that there exists a minimum metallic conductivity in the model under consideration was the result of an insufficiently accurate investigation of the integral equation. The correct formula (61) shows that the transition is continuous. Nevertheless, the decrease of the diffusion coefficient near the transition point is sharp, making is possible to speak of the existence of a quasijump near the transition point. If we assume that the exponential decrease also applies for real systems, this could explain the very sharp conductivity decrease that has been observed in many experiments and has led people to suspect the existence of a minimum metallic conductivity.

The conductivity in the Anderson model on a Cayley tree was studied numerically in Ref. 2. The authors of Ref. 2 obtained a very sharp decrease of the conductivity near the transition point and concluded that a minimum metallic conductivity exists. The sharp decrease is in agreement with (61). However, for an exact comparison of the results of Ref. 2 with (61) we need a larger number of calculated points near the transition.

In no way can the formula (61) be obtained by means of the scaling hypothesis of Ref. 12. Of course, it may be doubted whether models on a Bethe lattice bear any relation to real systems. However, such doubts seem strange, since the theory proposed in Ref. 12 is very similar to the usual theories of second-order phase transitions, for which the Bethe lattice has simply yielded mean-field theory. Moreover, an attempt has already been made[4] to obtain the conductivity on a Bethe lattice using ideas expressed in Refs. 12 and 5. Application of all these ideas in a certain rather general model on a Bethe lattice has led to the conclusion that the conductivity decreases linearly as the transition point is approached,[4] which is natural from the point of view of scaling but contradicts the exact result (61). In a recent paper[13] scaling relations were used to investigate localization in a space of high dimensionality d. It was concluded that the dielectric permittivity for $d \to \infty$ (corresponding to the Bethe lattice) should remain finite as the transition point is approached. As already discussed above, the result can depend on how the applied field varies in space. However, the author of Ref. 13 calculated the dielectric permittivity at zero momentum, which, it would seem, corresponds to $\varkappa$ (39). But the quantity $\varkappa$ diverges as the transition point is approached [see (40)], contradicting Ref. 13. One may doubt the possibility of describing a localized region by means of only one length.

2. The entire investigation carried out above is based in an essential way on the noncompactness of the symmetry group of the order paramter Q. In paticular, in the derivation of (37) the important contribution is made by the region $\lambda_1 = \operatorname{ch}\theta_1 \sim 1/\omega$, and in the derivation of (61) it is made by the region $\lambda_1 \propto \exp[q(\alpha - \alpha_c)^{-1/2}]$. In the usual theories of phase transitions, however, the group of the order parameter has always been compact. We shall show that in the present case too the replacement of the noncompact group by any compact group quickly leads to scaling. We stress that the compact symmetry group proposed below has no physical meaning and is used only to illustrate the role of the noncompactness.

We shall assume that the new model is described, as before, by the formulas (1) and (2), but make the following change for $\hat{\theta}$ in (2). We suppose, as before, that $\hat{\theta}$ is a diagonal matrix, in which the component $\hat{\theta}_{11}$ corresponds to the compact sector. We change only the component $\hat{\theta}_{22}$. We write the results of this change in the form of the following change of Q_{022}:

$$Q_{022} = \begin{pmatrix} \operatorname{ch}\theta_1 & -\operatorname{sh}\theta_1 \\ \operatorname{sh}\theta_1 & -\operatorname{ch}\theta_1 \end{pmatrix} \to \begin{pmatrix} f(z)z & -x \\ x & -f(z)z \end{pmatrix}, \quad x>0, \qquad (62)$$

where the variables z and x are related by the equation

$$f^2(z)z^2 - x^2 = 1. \qquad (62a)$$

If $f(z) = 1$, we return to the noncompact model already considered, since then $x = \operatorname{sinh}\theta_1$. To construct the compact model we choose $f(z)$ in such a way that Eq. (62a) describes a closed curve. As $f(z)$ we can take, e.g., the function

$$f(z) = \frac{1}{z}\left\{1 + (z^2-1)\left[1 - 2\exp\left(-\frac{R}{z^2-1}\right)\right]\right\}^{1/2}, \qquad (63)$$

where R is a number.

Having changed the original model in this way, we can, as before, obtain Eq. (11). Now, however, $\mathbf{n}_1$ is a vector not on a hyperboloid but on the closed surface obtained from the curve (62a) by rotation about the z axis. Correspondingly, by the scalar product of the vectors $\mathbf{n}$ and $\mathbf{n}_1'$ we understand the expression

$$\mathbf{n}_1\mathbf{n}_1'=f(z)f(z')zz'-\mathbf{x}\mathbf{x}', \quad \mathbf{n}_1=(z,\ \mathbf{x}). \tag{64}$$

The vectors $\mathbf{n}_1$ and $\mathbf{n}_1'$ in (11) are now unit vectors in the sense of the scalar product (64), and $\lambda_1=f(z)z$. In the region $\alpha\gg 1$, as before, the main contribution is given by values of λ and λ_1 close to unity, and this gives the solution (52). Moreover, the entire perturbation theory in the limit $\alpha\gg 1$ in the presently considered compact model with $f(z)$ (63) coincides with the perturbation theory for the correct noncompact model that was studied in the preceding sections, since for values of z close to unity $f(z)$ differs from unity by an exponentially small amount. Despite the coincidence of the perturbation-theory results, the critical behavior of these two models is entirely different.

To elucidate the critical behavior it is necessary to find the eigenfunctions of the kernel Γ. In the previous case the eigenfunction was the product $P_n(\lambda)P_{-1/2+i\rho}(\lambda_1)$. Because of the noncompactness the set of eigenvalues was continuous. However, in the general case too, it is not difficult to construct the eigenfunctions of the changed operator $\tilde{\Gamma}$. Naturally, the variables are again separable and the function is written in the form of a product $P_n(\lambda)\varphi_l(z)$. The function $\varphi_l(z)$ is a generalized spherical harmonic, corresponding to transformations g that carry the set of unit [in the sense of the scalar product (64)] vectors into themselves, so that $\mathbf{n}_1'=g\mathbf{n}_1$. The procedure for constructing such spherical harmonics by means of representations $T^l(g)$ of the group of g is described in the book by Vilenkin.[14] Any function on a compact set of g can be expanded in a Fourier series in these spherical harmonics (on a noncompact set, a Fourier integral is necessary).

We seek solutions of Eq. (11) in the form (52), assuming that c is a function of λ and z. We expand c in a Fourier series:

$$c(\lambda,z)=\sum_{n,l}c_{nl}P_n(\lambda)\varphi_l(z), \tag{65}$$

and substitute (52) and (65) into (11). After this we transform (11) to the form (56), in which we must replace $P_{-1/2+i\rho}$ by φ_l. In the critical region, as usual, only the zeroth harmonic is important. The corresponding coefficient c_{00} (65) is small. Expanding (65) in c_{00} and discarding the other harmonics, we obtain

$$a_2c_{00}^2+(\alpha_c-\alpha)a_1c_{00}=a_0\beta, \tag{66}$$

where a_0, a_1, and a_2 are numerical factors of order unity.

Equation (66) coincides in form with the equation for the mean field in standard theories of phase transitions (in spin models, the term c_{00}^2 is absent and it is necessary to write c_{00}^3). This means that all the nonstandard results obtained in the preceding sections are a consequence only of the noncompactness of the group of the order parameter.

If in addition we assume that c in (52) for $\lambda, z\to 1$, and, consequently, c_{00} are proportional to the diffusion coefficient D, which in any case is true in the region $\alpha\gg 1$, then Eq. (66) can be interpreted as an equation for the diffusion coefficient. For $\alpha>\alpha_c$ and $\beta\to 0$ we have $c_{00}\sim\alpha-\alpha_c$. Precisely this result was obtained in Ref. 4. For $\alpha<\alpha_c$ the quantity $c_{00}\sim\beta/(\alpha_c-\alpha)$. Comparing with (50), for the localization length we obtain $l\sim(\alpha_c-\alpha)^{-1}$, which unexpectedly coincides with (38).

Equation (66) has also been obtained[15] with the aid of a self-consistent approximation. The coincidence of the equations obtained in Refs. 4 and 15 with Eq. (66), which was derived with the aid of specially made incorrect distortions of the correct noncompact σ-model, compels us to think that the approximations used in Refs. 4 and 15 are incorrect. To check this assertion it would be interesting to use the method of Ref. 15 to investigate the transition on a Bethe lattice. We note that almost any change of the correct model leads to (66). For example, it is possible in the original model to calculate the integrals over λ_1 not from 1 to ∞ but from 1 to some $\lambda_{\max}$. Then we again obtain Eq. (66). Evidently, replacement of the noncompact model by any compact model corresponds to replacement of certain strongly fluctuating quantities by their average values. Near the transition, when the averages themselves (e.g., the conductivity) become small, neglect of their fluctuations can lead to incorrect results.

3. The form of the diffusion coefficient (61) also contradicts the predictions of the renormalization-group method in a space of $2+\varepsilon$ dimensions,[16,17] if this method is applied to the supersymmetric model (1). While in no way doubting the correctness of Refs. 16 and 17, we must note that the analysis was performed only for compact models. All the proofs of Refs. 16 and 17 were based on the invariance of the integration measure. For example, in the matrix case it is necessary to integrate over all matrices Q satisfying the conditions $Q^2=1$ and $\mathrm{Tr}\,Q=0$. Such invariance guarantees that, after the renormalizations, only terms of the type $\mathrm{Tr}(\nabla Q)^2$ appear in the Lagrangian. In noncompact models, specifying the matrices with the aid of only these restrictions leads to the necessity of integration over both sheets of the hyperboloid, and this gives divergent integrals. A way of integration over supermatrices Q is described in Ref. 9, but this method cannot be called invariant, since it assume that the matrix $\Lambda=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ is distinct. Therefore, there is in advance no guarantee that terms other than $\mathrm{Tr}(\nabla Q)^2$ are forbidden.

An attempt to discover the existence of particular new terms in the Lagrangian was undertaken in Ref. 18. The existence of such terms could be manifested in nonuniversality of the coefficients of the logarithms. In Ref. 18 it was stated that this nonuniversality is already apparent in perturbation theory. However, the calculational result presented in Ref. 18 is erroneous, as a consequence of the loss of the contribution of one of the diagrams. Correct allowance for all the diagrams leads to a universal coefficient in the order under consideration. It is evident that if the conjectured new terms in the Lagrangian do exist, they are exponentially small and a calculation in the framework of perturbation theory will not make it possible to find them.

At the same time, it is important to go beyond the framework of perturbation theory, since the critical properties are determined not by perturbation theory but by the global symmetry of the order parameter. For example, the compact model defined by means of (63) has the same per-

turbation theory as the correct noncompact model, although the critical behavior of these models is entirely different.

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