The anomalous superfluid current in ^{3}He-A and the index theorem

A. B. Balatskiĭ and V. A. Konyshev

L. D. Landau Institute for Theoretical Physics of the USSR Academy of Sciences, Moscow
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It is shown that when the orbital angular momentum vectors of the Cooper pairs possess texture, the quasiparticle spectrum in ^{3}He-A exhibits an anomaly which is analogous to the chiral anomaly in (2 + 1)-dimensional quantum electrodynamics in the presence of an external field. The role of the latter for ^{3}He-A is played by the texture. Both in quantum electrodynamics and in ^{3}He-A this anomaly is a consequence of the Attyah-Singer index theorem for an operator; in the case of ^{3}He-A this is the Bogolyubov operator for the quasiparticles of the texture. The superfluid current at $T = 0$ is composed of condensate current ρv_s (ρ is the fluid density, v_s is the superfluid velocity) and the current of the fermions occupying the anomalous branch of the spectrum. In an arbitrary texture the intersection of the anomalous branch with the Fermi surface gives rise to a finite density of states and a nonvanishing density ρ_n of the normal component.

INTRODUCTION

Superfluid ^{3}He-A has a series of properties which distinguish this fluid, as a matter of principle, from other superfluid and superconducting systems, properties which are related to the order parameter of the A-phase: the pairing in ^{3}He-A occurs in the state with $L = 1, L_l = 1, S = 1, S_d = 0$, where **l** and **d** are the quantization axes for the orbital and spin angular momenta. The expression for the order parameter has the form

$$\Delta_{\varkappa\nu} = i(\sigma_\mu \sigma_y) d_\mu \Delta_0 (\mathbf{e}_1 + i\mathbf{e}_2)\mathbf{k}/k_F, \tag{1}$$

where σ_i are the Pauli matrices; Δ_0 is the width of the gap, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{l} = \mathbf{e}_1 \times \mathbf{e}_2$ are the unit vectors of the orbital coordinate system and k_F is the Fermi momentum.[1]

A consequence of the vanishing of the order parameter for $\mathbf{k} \| \mathbf{l}$ is the disappearance of the gap from the spectrum of Bogolyubov excitations, which is essential for the distinctive properties of ^{3}He-A for $T \ll T_c$. Thus there arises the problem of finding the quasiparticle spectrum for low temperatures. In addition to the quasiclassical description of the quasiparticles proposed by Volovik and Mineev[2] a series of papers has recently appeared which consider the quantum problem of determining the quasiparticle spectrum in the presence of a texture of the vector **l**.[3–6] In all these papers it was noted that the quasiparticle spectrum in a sufficiently general texture has an asymmetric branch which violates the symmetry with respect to a change of sign of the quasiparticle energies. In Refs. 4–6 it was shown that in the case of a texture with curl $\mathbf{l} \| \mathbf{l}$ the Bogolyubov equations are, in a certain sense, analogous to the Dirac equation in (2 + 1) dimensions for an electron in an external magnetic field, the role of which is played by $\mathbf{B} = \text{curl}\,\mathbf{l}$. As is well known, in this case the spectrum of the electrons or quasiparticles becomes anomalous,[7,8] i.e., there appears a branch in the spectrum which violates the symmetry with respect to the substitution $\mathbf{B} \mapsto -\mathbf{B}$; this leads to the appearance of a vacuum current directed along the magnetic field, i.e., a chiral anomaly.

In the presence of a texture of the vector **l** in ^{3}He-A at $T = 0$ there appears a superfluid current which until now used to be derived by means of a gradient expansion[1]

$$\mathbf{j}_s = \rho \mathbf{v}_s + {}^1/_2\, \text{curl}(\rho \mathbf{l}/2) - {}^1/_2 C_0 \mathbf{l}(\mathbf{l}\cdot\text{curl}\,\mathbf{l}), \tag{2}$$

where $\mathbf{v}_s$ is the superfluid velocity, $\rho = k_F^3/3\pi^2$ is the fluid density, $C_0 \approx \rho$ in the weak coupling approximation. In the sequel we use units with $m = \hbar = 1$.

In this paper we shall prove that the spectrum of the Bogolyubov equation for the quasiparticles has an anomalous branch for arbitrary textures of the vector **l**, even when there is no analogy with the Dirac equation. The vacuum current formed by the uncompensated ground-state fermion momenta will be transported at $T = 0$ by the fermions which occupy the anomalous branch of the spectrum (*v. infra*), and has the expression

$$\mathbf{j}_{\text{an}} = \tfrac{1}{2}\, \text{curl}\, \rho \mathbf{l}/2 - {}^1/_2 C_0 \mathbf{l}\, (\mathbf{l}\cdot\text{curl}\,\mathbf{l}). \tag{3}$$

The total momentum of the fermions which occupy the anomalous branch of the spectrum is completely compensated and does not contribute to the vacuum current (3). In addition, the anomalous branch of the quasiparticle spectrum has no gap, so that the density of the normal component at $T = 0$ is nonvanishing. We show that the existence of an anomalous branch of the spectrum follows from the Atiyah-Singer index theorem for an elliptic operator[9]—in this case the Bogolyubov operator in the texture of the vector **l**. For this purpose we make use of the Atiyah-Patodi-Singer index

$$\eta = \sum_{E>0} 1 - \sum_{E<0} 1,$$

which is a topological characteristic of the spectrum (see Appendix A). Thus, the general phenomenon of spectral asymmetry for the quasiparticle spectrum pointed out in Refs. 3–6, is a consequence of the topology of the order parameter (1), since the Bogolyubov equation is constructed in terms of the order parameter.

The plan of the paper is the following. In the first section we consider the equation for quasiparticles in the case of the texture curl $\mathbf{l} \| \mathbf{l}$. We show that although there is no complete analogy with (2 + 1)-dimensional quantum electrodynamics, there exists an anomalous level which leads to an uncompensated vacuum current (3). In Section 2 an exact solution is found for the problem of determining the spectrum in the case of a texture with curl $\mathbf{l} \perp \mathbf{l}$. The third section discusses the case when the texture of the vector **l** is absent,

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and there exists only a gradient of the chemical potential. However, in this case too there exists an anomalous level leading to an uncompensated vacuum current. The fourth section considers the case of arbitrary textures, and the anomalous level is determined approximately.

Everywhere below it is assumed that $T=0$ and the spin part of the order parameter is assumed constant: $d_\mu = \delta_{\mu y} d$.

1. THE CASE OF A TEXTURE WITH CURL l||l

In this case we write the Bogolyubov equation (for details see Refs. 4–6) in the form

$$i\frac{\partial}{\partial t}\chi=\begin{pmatrix} \hat{\varepsilon} & -i(\Delta_i\partial_i+\partial_i\Delta_i)/2 \\ -i(\Delta_i^*\partial_i+\partial_i\Delta_i^*)/2 & -\hat{\varepsilon}\end{pmatrix}\chi=E\chi, \qquad (1.1)$$

where $\chi = \binom{u}{v}$ is the Bogolyubov spinor, $\hat{\varepsilon} = -(\partial_i^2 + k_F^2)/2$ is the energy operator, and Δ_i is the orbital part of the order parameter (1):

$$\Delta_i=\Delta_0[\mathbf{e}_{1i}(\mathbf{r})+i\mathbf{e}_{2i}(\mathbf{r})]/k_F. \qquad (1.2)$$

In the case under consideration we choose $\mathbf{e}_1$ and $\mathbf{e}_2$ in the form

$$\mathbf{e}_1=\hat{x},\quad \mathbf{e}_2=\hat{y}-\hat{z}Hx,\quad \mathrm{curl}\,\mathbf{l}=H\hat{z}+O(r^2). \qquad (1.3)$$

Since the size of the wave function is considerably smaller than the size of the texture, we restrict our expansion to the linear term. The operator (1.1) can be rewritten in the form

$$\hat{H}=\tau_3(\tilde{\varepsilon}+\hat{p}^2/2)+\alpha\tau_1\hat{p}_x-\alpha\tau_2(\hat{p}_y-\hat{p}_zHx), \qquad (1.4)$$

where τ_i are the Pauli matrices, $\alpha = \Delta_0/k_F$, and $\tilde{\varepsilon} = (p_y^2+p_z^2)/2-\mu$.

Substituting

$$\chi=\chi(x)\exp(ik_zz+ik_yy),\qquad B=k_zH \qquad (1.5)$$

into (1.1) and transforming to the momentum representation $\hat{p}_x\mapsto p$, $x\mapsto i\partial/\partial p$, we obtain (cf. also Ref. 6)

$$\hat{H}=\tau_3(\tilde{\varepsilon}+p^2/2)+\alpha\tau_1p+\alpha\tau_2(Bi\partial/\partial p-p_y). \qquad (1.6)$$

Removing the p_y-dependence in Eq. (1.6) by means of the transformation

$$\chi\to\chi\exp(-ip_yp/B), \qquad (1.6a)$$

we obtain the one-dimensional equation

$$[\tau_3(\tilde{\varepsilon}+p^2/2)+\alpha\tau_1p+\alpha\tau_2(Bi\partial/\partial p)]\chi=E\chi. \qquad (1.7)$$

We consider the potential

$$U=[(\tilde{\varepsilon}+p^2/2)^2+\alpha^2p^2]^{1/2}, \qquad (1.8)$$

which in the case $H=0$ determines the eigenvalues of Eq. (1.7): $E(H=0) = \pm U$. We assume that for the low-lying levels E the quantum eigenvalue problem (1.7) yields values close to the minima of the potential U, since small H correspond to a large value of the mass of the particle which moves in the potential U^2 (see also Section 3).

We determine the minima of the potential U

$$1)\ p_0=0,\quad U^2(p_0)=\tilde{\varepsilon}^2, \qquad (1.9a)$$

$$2)\ (\tilde{\varepsilon}+p_0^2/2)+\alpha^2=0,\quad p_0=\pm[-2(\tilde{\varepsilon}+\alpha^2)]^{1/2},$$

$$U^2(p_0)=-2\alpha^2\tilde{\varepsilon}. \qquad (1.9b)$$

We note that the minimum (1.9a) corresponds to the case considered in Ref. 6 and leads to the term with C_0 in the expression for the vacuum current. However, in reality, as can be seen from Eq. (1.9b), for $\tilde{\varepsilon}<0$ the actual minimum is attained for the values $p_0\approx\pm(-2\tilde{\varepsilon})^{1/2}$. Expanding the quantities in (1.7) around p_0 to first order, and carrying out the rotation

$$\chi\to\chi\exp(i\tau_2u/2),\ \hat{H}=\exp(-i\tau_2u/2)\hat{H}\exp(i\tau_2u/2), \qquad (1.10a)$$

$$\cos u=\alpha(\alpha^2+p_0^2)^{-1/2},\quad \sin u=p_0(\alpha^2+p_0^2)^{-1/2}, \qquad (1.10b)$$

we obtain

$$\hat{H}=-\alpha|p_0|\tau_3+\tau_1(p_0^2+\alpha^2)^{1/2}m+\alpha\tau_2Bi\partial/\partial m, \qquad (1.11)$$

where $m=p-p_0$. In Eq. (1.11) we have neglected quantities of order α^2. As shall be shown below, it suffices for us to know the ground state of the Hamiltonian (1.11):

$$E_0=-\alpha|p_0|\theta(-B)+\alpha|p_0|\theta(B), \qquad (1.12)$$

$$\tilde{\chi}=\theta(-B)\begin{pmatrix} f_0(m)\\ 0\end{pmatrix}+\theta(B)\begin{pmatrix}0\\ f_0(m)\end{pmatrix}, \qquad (1.13)$$

where f_0 is the normalized harmonic oscillator eigenfunction with frequncy $|B|$ in the ground state (see Ref. 6).

Thus, even in the case when the equation (1.7) differs from the Dirac equation, the spectrum of the operator has an asymmetric branch, either for $E<0$, $B<0$, for for $E>0$, $B>0$. We show that such a spectral asymmetry is a consequence of the topology of the spectrum of the original operator (1.7). For this purpose we make use of the Atiyah-Patodi-Singer η-invariant for the original Hamiltonian [see Appendix A; the computation of the index $\eta(H)$ is given in Appendix B]:

$$\eta[H(\tilde{\varepsilon})]=\frac{2}{\pi^{1/2}}\int_0^{+\infty}dy\,\mathrm{Tr}[H\exp(-y^2H^2)]=2\theta(-\tilde{\varepsilon})\,\mathrm{sgn}\,B, \qquad (1.14)$$

from which it follows that the number of levels with $E>0$ of the Bogolyubov Hamiltonian differs from that with $E<0$. If at the same time $\tilde{\varepsilon}>0$, then the spectrum of the operator (1.7) is symmetrical. If one considers the spectral flow of the Hamiltonian $\hat{H}$ as a function of $t\equiv\tilde{\varepsilon}$, $-\infty<\tilde{\varepsilon}<+\infty$, then, as can be seen from Appendix A, the number of levels intersecting the value $E=0$ at some point $\tilde{\varepsilon}=\varepsilon^*$ is equal to

$$\frac{1}{2}\{\eta[H(\tilde{\varepsilon}\to+\infty)]-\eta[H(\tilde{\varepsilon}\to-\infty)]\}$$

$$-\frac{1}{2}\int d\tilde{\varepsilon}\,\frac{d\eta}{d\tilde{\varepsilon}}=-\mathrm{sgn}\,B. \qquad (1.15)$$

Making use of results from Appendix A one can show that $d\eta/d\tilde{\varepsilon}=0$.

Equation (1.5) confirms the results of Ref. 6, where it was shown, starting from different considerations, that the spectrum of the operator (1.7) has one branch which crosses the level $E=0$ at the point $\tilde{\varepsilon}=\varepsilon^*$ (Fig. 1).

Thus, in going from $\tilde{\varepsilon}<0$ to $\tilde{\varepsilon}>0$ the spectrum of the Bogolyubov operator gets restructured. The direction of the intersection (the change of the sign of E from minus to plus) depends on the sign of B. This implies that the asymmetry of the spectrum (1.12) of the linearized operator which we have obtained is a consequence of the topology of the spectrum of the original Hamiltonian (1.7). The coefficient 2 in

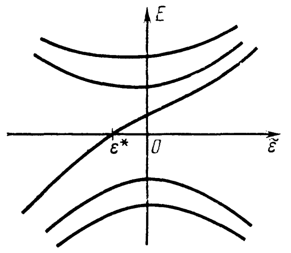

FIG. 1. The spectrum of the operator (1.7) as a function of $\tilde{\varepsilon}$ for $B<0$. One of the branches of the spectrum intersects the axis $E=0$ for $\tilde{\varepsilon}=\varepsilon^*$.

Eq. (1.14) is a consequence of the twofold degeneracy $p_0 \mapsto -p_0$ of the spectrum (1.12). In addition, it was shown in Ref. 6 that if one neglects $\alpha\tau_1 p$ in the Hamiltonian (1.6), the spectrum is symmetric under $E \mapsto -E$, and if one takes into account this term one is led to the result that one branch of the spectrum intersects the axis $E=0$, and for sufficiently large values of $\tilde{\varepsilon} \to +\infty$ the spectrum is deformed slightly, but exhibits no asymmetry in the number of levels, therefore $\eta(H)=0$ for $\tilde{\varepsilon}>0$. On the other hand, for $\varepsilon^* < \tilde{\varepsilon} < 0$ there is no intersection, and an asymmetry arises only for $\tilde{\varepsilon} \to -\infty$, a fact which is reflected in the value of $\eta(H)$ in Eq. (1.14).

After these remarks we calculate the current. We introduce the standard expansion

$$\psi = \sum_s u_s a_s + v_s^* a_s^+, \qquad (1.16)$$

where s is a complete set of quantum numbers; a_s and a_s^+ are the annihilation and creation operators for quasiparticles in the state s; u_s, v_s are the components of the Bogolyubov spinor [Eq. (1.11)]. Then the current operator can be written in the form

$$\mathbf{j} = (\psi^+ \hat{\mathbf{p}}\psi - \hat{\mathbf{p}}\psi^+\psi) = \sum_s (a_s^+ a_s u_s^* \hat{\mathbf{p}} u_s - a_s a_s^+ v_s^* \hat{\mathbf{p}} v_s)$$

$$= \sum_{s,E_s<0} u_s^* \hat{\mathbf{p}} u_s - \sum_{s,E_s>0} v_s^* \hat{\mathbf{p}} v_s. \qquad (1.17)$$

In Eq. (1.17) we have taken into account the two possible values of the spin projection.

In the case under consideration the current flows along the z axis:

$$j_z = \sum_{s,E_s<0} k_z |u_s|^2 - \sum_{s,E_s>0} k_z |v_s|^2, \qquad (1.18)$$

and for the level with $n=0$ we obtain

$$j_z = \int \frac{dk_y\, dk_z}{(2\pi)^2} k_z \theta(-E_0) |u_0(x=0)|^2$$

$$- \int \frac{dk_y\, dk_z}{(2\pi)^2} k_z \theta(E_0) |v_0(x=0)|^2. \qquad (1.19)$$

Substituting

$$u_0(x=0) = \int \frac{dm}{2\pi} u_0(m), \qquad v_0(x=0) = \int \frac{dm}{2\pi} v_0(m), \qquad (1.20a)$$

and taking into account that $\cos u \approx 0$, $|\sin u| \approx 1$ for $p_0 \gg a$, we obtain from (1.10)

$$\begin{pmatrix} u_0(m) \\ v_0(m) \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\theta(-B) \begin{pmatrix} f_0(m) \\ f_0(m) \end{pmatrix} + \theta(B) \begin{pmatrix} f_0(m) \\ -f_0(m) \end{pmatrix} \right]$$

$$\times \exp(-ip_y p/B). \qquad (1.20b)$$

Substituting (1.20b) into (1.20a) we have

$$|u_0(x=0)|^2 = {}^1/_2 |f_0(p_y/B)|^2, \qquad |v_0(x=0)|^2 = {}^1/_2 |f_0(p_y/B)|^2. \qquad (1.20c)$$

For $H>0$, substituting (1.20c) into Eq. (1.19), we obtain, after the substitution $k_z = r\cos\theta$, $k_y = r\sin\theta$, $2\tilde{\varepsilon} = r^2 - k_F^2 < 0$, the following expression for the current

$$\mathbf{j}_z = \frac{\hat{z}}{8\pi^2} \int_0^{k_F} r^2\, dr \int_{\pi/2}^{3\pi/2} \cos\theta\, d\theta \left| f_0\left(\frac{\operatorname{tg}\theta}{H}\right) \right|^2$$

$$- \frac{\hat{z}}{8\pi^2} \int_0^{k_F} r^2\, dr \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta \left| f_0\left(\frac{\operatorname{tg}\theta}{H}\right) \right|^2$$

$$= \frac{1}{4} \operatorname{curl}(\rho \mathbf{l}) - \frac{1}{2} \rho \mathbf{l}(\mathbf{l}\cdot\operatorname{curl}\mathbf{l}). \qquad (1.21)$$

It can be seen from here that the momentum carried by the fermions which occupy the anomalous branch of the spectrum is exactly equal to the anomalous current (3) in the texture under consideration.

We show that the remaining levels do not contribute to the total current (1.21), i.e., complete compensation of the currents of the non-anomalous levels occurs. For this purpose we consider the expressions of j_z for the two possible signs of H and the two values of the momentum k_z from which the current operator (1.18) is formed

$$j_z(H, k_z) = \sum_{s,E_s<0} k_z |u_s|^2 - \sum_{s,E_s>0} k_z |v_s|^2, \quad s = n, k_z, k_y, \qquad (1.22a)$$

$$j_z(-H, -k_z) = \sum_{s^*,E_{s^*}<0} (-k_z) |u_{s^*}|^2 - \sum_{s^*,E_{s^*}>0} (-k_z) |v_{s^*}|^2.$$

$$s^* = n, -k_z, k_y. \qquad (1.22b)$$

The currents (1.22a) and (1.22b) are equal, since the Bogolyubov operator (1.7) is invariant under the simultaneous substitution $Hk_z = B \to (-H)(-k_z) = B$.

On the other hand, since the current is odd under coordinate inversion, $j_z(-H) = -j_z(H)$, it follows that

$$\frac{1}{2} [j_z(H) - j_z(-H)] = \frac{1}{2} [j(H, p_z) - j(H, -p_z)]$$

$$= \frac{1}{2} \Big\{ \sum_{\substack{E<0 \\ n\neq 0}} p_z |u|^2$$

$$- \sum_{\substack{E>0 \\ n\neq 0}} p_z |v|^2 - \sum_{\substack{E<0 \\ n\neq 0}} p_z |u|^2 + \sum_{\substack{E>0 \\ n\neq 0}} p_z |v|^2$$

$$+ \sum_{E_0<0} p_z |u_0|^2 - \sum_{E_0>0} p_z |v_0|^2$$

$$+ \sum_{E_0>0} p_z |u_0(-p_z)|^2 - \sum_{E_0<0} p_z |v_0(-p_z)|^2 \Big\}$$

$$= \sum_{E_0<0} p_z |u_0|^2 - \sum_{E_0>0} p_z |v_0|^2 = j_z. \qquad (1.23a)$$

In obtaining this formula we have taken account of

$$|v_0(-p_z)|^2 = |u_0(-p_z)|^2 = |u_0(p_z)|^2 = |v_0(p_z)|^2.$$

Now, on account of the condition $E_{p_z} = E_{-p_z}$ for $n \neq 0$

and $E_{p_z} = -E_{-p_z}$ for $n=0$ [see Eq. (1.12)], only the terms with $n=0$ survive in the sum (1.23a), whence, noting that $|u_0|^2 = |v_0|^2$, we obtain the formula we are after:

$$j_z = \sum_{\substack{n=0\\E<0}} p_z |u|^2 - \sum_{\substack{n=0\\E>0}} p_z |v|^2. \tag{1.23b}$$

Let us consider the density of states for $E=0$. As can be seen from Eqs. (1.1) and (1.12), $E=0$ corresponds to the value $p_0 = 0$, i.e., to the case when a minimum of the first type (1.9a) is realized. But this case had been considered in Refs. 4 and 6, and is equivalent to the limit in which the Bogolyubov equation coincides with the Dirac equation in 2 + 1 dimensions, in an external magnetic field $\mathbf{B} = \text{curl } \mathbf{l}$. Therefore the density of states coincides with that for a Landau level and equals

$$N(0) = \sum_s |\psi(x=0)|^2 \delta(E) = |\mathbf{l}\cdot\text{curl } \mathbf{l}|/(2\pi)^2. \tag{1.24}$$

However, for $|\varepsilon| > a^2$, when the approximation (1.9b) of expanding around the minimum is valid, the density of states increases substantially and equals

$$N(\varepsilon) = (|\varepsilon|/\alpha^2)\,|\mathbf{l}\cdot\text{curl } \mathbf{l}|/(2\pi)^2. \tag{1.25}$$

2. THE CASE OF A TEXTURE WITH curl l⊥l

In this case the Bogolyubov equations (1.1) can be solved exactly. We choose the basis $\mathbf{e}_1$, $\mathbf{e}_2$ in the form

$$\mathbf{e}_1 = \hat{x}, \quad \mathbf{e}_2 = \hat{y} + \hat{z} T_\perp z, \quad \text{curl } \mathbf{l}|_{\mathbf{r}=0} = T_\perp \hat{x}, \text{curl } \mathbf{l} \perp \mathbf{l}. \tag{2.1}$$

Then, separating the dependence on the conserved quantum numbers p_x and p_y:

$$\chi = \chi \exp(ip_x x + ip_y y), \tag{2.2}$$

and changing to the momentum representation in p_z: $-i\partial/\partial z) \to p_z$, $z \to i\partial/\partial p_z$, we can rewrite the Bogolyubov equations (1.1) in the form

$$\Big\{\tau_3\Big(\frac{p_x^2+p_y^2+p_z^2}{2} - \mu\Big) + \alpha\tau_1 p_x - \alpha\tau_2\Big[p_y - T_\perp\Big(i\frac{\partial}{\partial p_z}p_z + p_z i\frac{\partial}{\partial p_z}\Big)\Big]\Big\}\chi = E\chi. \tag{2.3}$$

In Eq. (2.3) we effect the following change of variables:

$$\chi = \varphi |p_z|^{-1/2}, \quad \ln(|p_z|/k_F) = \xi, \quad p_z = ek_F e^\xi = e|p_z|, \tag{2.4}$$

where $e = \pm 1$, depending on the sign of p_z. Then the Bogolyubov operator in Eq. (2.3) can be rewritten in the form

$$H_B = \tau_3[\mu(e^{2\xi}-1) + \varepsilon_\perp] + \alpha\tau_1 p_x - \alpha\tau_2(p_y - T_\perp p_\xi), \tag{2.5}$$

where $\varepsilon_\perp = (p_x^2 + p_y^2)/2$, $p_\xi = i\partial/\partial\xi$. Separating the p_y-dependence

$$\varphi = \varphi \exp(ip_y \xi / T_\perp), \tag{2.6}$$

and carrying out the rotation:

$$\varphi \to \varphi \exp(iu\tau_2/2), \quad \hat{H} \to \exp(-iu\tau_2/2)\hat{H}_B \exp(iu\tau_2/2), \quad u = \pi/2, \tag{2.7}$$

we obtain the Hamiltonian which we are going to investigate:

$$\hat{H}\varphi = \{\alpha p_x \tau_3 - \tau_1[\varepsilon_\perp + \mu(e^{2\xi}-1)] + \alpha\tau_2 T_\perp p_\xi\}\varphi = E\varphi. \tag{2.8}$$

Note that no approximations have been made in the derivation of Eq. (2.8).

We determine the ground state of the Hamiltonian (2.8), state which is already anomalous. To this end we consider the equation

$$[\varepsilon_\perp + \mu(e^{2\xi}-1)]\varphi_0 \pm \alpha T_\perp \partial\varphi_0/\partial\xi = 0. \tag{2.9}$$

Its solution is

$$\varphi_0 = \varphi(0)\exp\Big\{\mp\frac{1}{\alpha T_1}\Big[(\varepsilon_\perp - \mu)\xi + \frac{\mu}{2}e^{2\xi}\Big]\Big\}. \tag{2.10}$$

In order to satisfy the normalizability condition for φ_0 for $\xi \to \pm\infty$, it is necessary to require that $\varepsilon_\perp - \mu < 0$; in addition, a normalizable solution of Eq. (2.9) depends on the sign of $\alpha T_\perp$. Thus, from Eqs. (2.8)–(2.10) we obtain the spectrum represented in Fig. 2:

$$H\varphi = E_0\varphi, \quad E_0 = -\alpha p_x \theta(T_\perp) + \alpha p_x \theta(-T_\perp), \tag{2.11a}$$

$$\varphi = \theta(T_\perp)\begin{pmatrix}0\\ \varphi_0\end{pmatrix} + \theta(-T_\perp)\begin{pmatrix}\varphi_0\\ 0\end{pmatrix}. \tag{2.11b}$$

Before calculating the vacuum current we show how the existence of the anomalous branch of the spectrum (2.11) follows from the Atiyah-Singer index theorem. We consider the spectral flow of the Hamiltonian (2.8) as a function of the parameter p_x, with $-\infty < p_x < +\infty$ (the fact that the limits are infinite is not essential in our case, the only important thing is that p_x changes sign as it changes adiabatically). We show that the number of eigenvalues of the Hamiltonian (2.8) which for some value $p_x = p_x^*$ intersect the axis $E=0$ is equal to one, and therefore is the anomalous level (2.11) which we have found. It follows immediately from this, as can be seen from (2.11a), that $p_x^* = 0$. For this we consider the index of the operator $H(p_x)$ (the index is calculated in Appendex B):

$$\eta[H(p_x)] = -\text{sgn}(T_\perp p_x). \tag{2.12}$$

Then the number of levels crossing the value $E=0$ as p_x varies is given by the equation

$$\frac{1}{2}\{\eta[H(p_x)] - \eta[H(-p_x)]\} - \frac{1}{2}\int dp_x \frac{d\eta}{dp_x} = -\text{sgn}(T_\perp p_x).$$

Remembering that the index η is by definition the difference between the number of levels with positive and negative energy, we obtain

$$\eta[H(p_x)] = \sum_{E(p_x)>0} 1 - \sum_{E(p_x)<0} 1 = -\text{sgn}(T_\perp p_x), \tag{2.13}$$

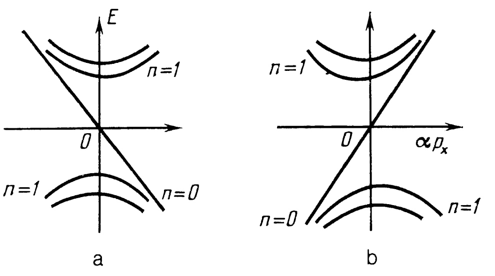

FIG. 2. The spectrum of the operator (2.8) as a function of αp_x for $T_\perp > 0$ (a) and $T_\perp < 0$ (b). The level with $n=0$ is anomalous.

as follows also from the exact solution (2.11). From the calculation of the index it also follows that the anomalous level (2.11) which we have obtained is the only one for the given operator (2.8).

Let us find the current transported by the fermions on the anomalous level. Since the anomalous level violates the symmetry under $p_x \mapsto -p_x$, an uncompensated current will flow parallel to the external field, curl $\mathbf{l} \| \hat{x}$. It follows from Eq. (1.7) that

$$j_x = \sum_{E_0<0} \sum_{e,p_x,p_y} |u(z=0,p_x,p_y)|^2 p_x - \sum_{E_0>0} \sum_{e,p_x,p_y} |v(z=0,p_x,p_y)|^2 p_x, \qquad (2.14)$$

where, taking account of Eqs. (2.4), (2.6), (2.7) (we have assumed that $T_\perp > 0$; the case $T_\perp < 0$ is similar):

$$\begin{pmatrix} u(z=0) \\ v(z=0) \end{pmatrix} = \frac{e}{\sqrt{2}} N^{-1} \frac{k_F^{1/2}}{2\pi} \int d\xi \exp\left(\frac{\xi}{2} + \frac{ip_y\xi}{T_\perp}\right) \varphi_0(\xi, p_x, p_y). \qquad (2.15)$$

The normalization factor N^{-1} is

$$N^{-1} = \sum_{e=\pm 1} \int \frac{dp_z}{2\pi} |\varphi_0(p_z)|^2 = \int \frac{d\xi}{2\pi} |\varphi_0|^2 = \frac{\text{const}}{2\pi} \left[\frac{\pi\alpha T_\perp}{2(\mu-\varepsilon_\perp)}\right]^{1/2} \qquad (2.16)$$

[the integral in (2.16) was computed by means of the saddle-point method, since $\alpha T_\perp \ll |\varepsilon_\perp - \mu|$ holds for practicaly all values of p_x, p_y]. A similar calculation of the integral in (2.15) yields

$$|u_0(z=0)|^2 = |v_0(z=0)|^2 = \frac{k_F}{(2\pi)^{1/2}} \left(\frac{\alpha T_\perp}{\mu-\varepsilon_\perp}\right)^{1/2} \times \exp\left[\xi_0 - \frac{\alpha p_y^2}{2T_\perp(\mu-\varepsilon_\perp)}\right], \qquad (2.17)$$

where $\xi_0 = \ln(1-\varepsilon_\perp/\mu)^{1/2}$

Substituting (2.17) into (2.14) and introducing the coordinates $p_x = r\sin\theta$, $p_y = r\cos\theta$, $r < k_F$, we obtain

$$\mathbf{j} = \hat{x} \frac{1}{2\pi^2} \int_0^{k_F} \int_0^{\pi} r^2\, dr\, d\theta \sin\theta \frac{k_F}{(2\pi)^{1/2}} \left(\frac{\alpha T_\perp}{\mu-\varepsilon_\perp}\right)^{1/2} \times \exp\left[\xi_0 - \frac{\alpha r^2\cos^2\theta}{2T_\perp(\mu-\varepsilon_\perp)}\right] = \frac{1}{4}\rho T_\perp \hat{x} = \frac{1}{2}\operatorname{rot}\frac{\rho \mathbf{l}}{2}, \qquad (2.18)$$

where we have taken account of $\nabla\rho = 0$ (the case $\nabla\rho \neq 0$ is considered in the following section).

We show that the total current is carried by the fermions of the anomalous level. For this we consider, e.g., $j_x(T_\perp)$. The equations (2.8) have the symmetry $\varphi \to \tau_1\varphi$, $p_x \to -p_x$, $T_\perp \to -T_\perp$, $E \to E$, and in addition, the Hamiltonian (2.8) has the formal symmetry $p_x \to -p_x$, $T_\perp \to T_\perp$, $E \to E$, $\varphi \to \varphi$, since the solution (2.8) is also a solution of the equation $H^2\varphi = E^2\varphi$. However, this symmetry is broken by the level with $n = 0$, i.e., the anomalous level, for which it is necessary that $E \to -E$, whence by analogy with Eq. (1.23a) it follows that

$$j_x(T_\perp) = \frac{1}{2}[j_x(T_\perp) - j_x(-T_\perp)]$$
$$= \frac{1}{2}\Big\{ \sum_{E_{p_x}<0}^{n\neq 0} p_x[|u(p_x)|^2 + |u(-p_x)|^2] - \sum_{E_{p_x}>0}^{n\neq 0} p_x[|v(p_x)|^2 + |v(-p_x)|^2] + \sum_{E_{f_x}<0} p_x|u_0|^2 - \sum_{E_{p_x}>0} p_x|v_0|^2$$
$$- \sum_{E_{-p_x}>0} (-p_x)|u_0(-p_x)|^2 + \sum_{E_{-p_x}<0} (-p_x)|v_0(-p_x)|^2 \Big\}$$
$$= \frac{1}{2}\left(\sum_{E_0<0} 2p_x|u|^2 - \sum_{E_0>0} 2p_x|v|^2\right) = j_x(T_\perp). \qquad (2.19)$$

In the derivation of Eq. (2.19) we have taken into account the fact that $E_{p_x} = E_{-p_x}$ for $n \neq 0$ and $E_{p_x} = -E_{-p_x}$ for $n = 0$. In addition, the sum in (2.10) with $n \neq 0$ cancels by virtue of $|u_0(p_x)|^2 = |u_0(-p_x)|^2$, $|v_0(p_x)|^2 = |v_0(-p_x)|^2$ and an even function is integrated over an odd interval.

We now consider the density of states with $E = 0$. We arrive at the result derived by a different method in Refs. 2 and 4:

$$N(0) = \sum_{p_x,p_y} |\chi(z=0)|^2 \delta(E_0) = \frac{N_0}{\Delta_0}\frac{v_F}{2} |[\mathbf{l}\cdot\text{curl}\,\mathbf{l}]|, \qquad (2.20)$$

where $N_0 = k_F/\pi^2$ is the density of states at the Fermi level in an ideal gas. We note that the existence of an anomalous level follows from the Atiyah-Singer index theorem for the Bogolyubov operator, a theorem which makes essential use (see Appendix A) of the fact that when one of the parameters of the Hamiltonian changes adiabatically, the anomalous level changes the sign of the energy, and thus the level crosses the axis $E = 0$. Consequently, the existence of a finite density of states for $T = 0$ follows directly from the index theorem, on account of (1.24) and (2.20).

3. THE CASE $\nabla\mu \neq 0$

In this section we consider the case when the vector $\mathbf{l}$ is parallel to $\hat{z}$ and is spatially homogeneous, but $\nabla\mu \neq 0$. This case can be solved exactly, and as in the preceding case, the Bogolyubov equations admit an anomalous solution which leads to the appearance of an uncompensated current in the vacuum of such a "texture."

We consider the case when $\mu = \mu(x)$. Let $\mathbf{e}_1 = \hat{x}$, $\mathbf{e}_2 = \hat{y}$, $\mathbf{l} = \hat{z}$. Rewriting the Bogolyubov equations in the momentum representation $p_z \to p$, $x \to i\partial/\partial p$ and separating the dependence on p_x, p_y, as we did in Eq. (2.2), we obtain

$$\hat{H}\chi = E\chi, \quad \hat{H} = \tau_3(\varepsilon + p^2/2 + \nabla\mu\,\partial/i\partial p) + \alpha\tau_1 p - \alpha\tau_2 p_y, \qquad (3.1)$$

where we have set $\varepsilon = (p_y^2 + p_z^2)/2 - \mu\ (x=0)$ and retained only the terms linear in $\nabla\mu$ in the Hamiltonian. Indeed, as in Ref. 6, the characteristic scale of the wave function is much smaller than the characteristic scale of the texture, and therefore the approximation of constant electric field is valid, analogous to the approximation of constant magnetic field, used in Sec. 1.2.

The unitary transformation

$$\chi \to \varphi \exp(i\tau_1\pi/4), \quad H \to \exp(-i\tau_1\pi/4) H \exp(i\tau_1\pi/4) \qquad (3.2)$$

leads to

$$\hat{H}\varphi=E\varphi,\quad \hat{H}=-\alpha p_y\tau_3+\alpha\tau_1 p-\tau_2(\varepsilon+p^2/2+E\partial/i\partial p),\qquad (3.3)$$

where we have introduced the notation $\nabla\mu = E$. The anomalous solution of Eq. (3.3) can be obtained exactly:

$$\varphi_0=\varphi(0)\exp\left[\mp\frac{1}{E}\left(\frac{\alpha p^2}{2}\pm i\varepsilon p\pm i\frac{p^3}{6}\right)\right].\qquad (3.4)$$

The choice of sign in Eq. (3.4) is dictated by normalization requirements and depends on the sign of E:

$$\varphi=\theta(E)\begin{pmatrix}0\\ \varphi_0\end{pmatrix}+\theta(-E)\begin{pmatrix}\varphi_0\\ 0\end{pmatrix},\quad E_0\varphi=\hat{H}\varphi,\qquad (3.5a)$$

$$E_0=\alpha p_y\theta(E)-\alpha p_y\theta(-E).\qquad (3.5b)$$

We show that the anomalous level that we have found is unique. For this we apply the index theorem to the operator (3.3). We consider the spectral flow of the operator (3.3) as a function of $t\equiv p_y$, $-\infty<p_y<+\infty$. A calculation of the η-invariant of the operator (3.3) is given in Appendix B (in our case $E=E_\perp$):

$$\eta[\hat{H}(p_y)]=\mathrm{sgn}(E_\perp p_y),\qquad (3.6)$$

from which we obtain the number of levels which change sign under an adiabatic change of p_y:

$$\tfrac{1}{2}\{\eta[\hat{H}(p_y=+\infty)]-\eta[\hat{H}(p_y=-\infty)]\}=\mathrm{sgn}(E_\perp p_y),$$

$$d\eta/dp_y=0.\qquad (3.7)$$

On the other hand, it can be seen that the solution (3.5) satisfies the condition (3.7), and consequently the anomalous solution that we have found is the unique anomalous solution for the Hamiltonian (3.3).

Let us compute the current transported by the fermions of the anomalous level. We assume that $E>0$; the case $E<0$ is treated similarly. Since the solution (3.5) violates the symmetry $p_y\to -p_y$, the current flows along the direction $\hat{y}$:

$$j_y=\sum_{\substack{p_y,p_z\\ E_0<0}} p_y|u|^2-\sum_{\substack{p_y,p_z\\ E_0>0}} p_y|v|^2,\qquad (3.8)$$

where, taking account of Eqs. (3.2) and (3.4),

$$|u|^2=|v|^2=\tfrac{1}{2}|\varphi_0(x=0)|^2.\qquad (3.9)$$

Introducing the new coordinates $r\sin\theta=p_y$, $r\cos\theta=p_z$, $\varepsilon=r^2/2-\mu$, we have

$$\varphi_0(x=0)=\frac{N^{1/2}}{2\pi}\varphi(0)\int dp\exp\left(-\frac{\alpha p^2}{2E}-i\frac{\varepsilon p}{E}-i\frac{p^3}{6E}\right),\qquad (3.10a)$$

$$N^{-1}=|\varphi(0)|^2\int\frac{dp}{2\pi}\exp\left(-\frac{\alpha p^2}{E}\right)=\frac{|\varphi(0)|^2}{2\pi}\left(\frac{\pi E}{\alpha}\right)^{1/2}.\qquad (3.10b)$$

Then

$$j_y=\int_0^{k_F}\frac{r^2\,dr}{4\pi^2}\int_0^{2\pi}d\theta\sin\theta\frac{N|\varphi(0)|^2}{4\pi^2}\int_{-\infty}^{+\infty}dp\,dq\exp\left[-\frac{\alpha(p^2+q^2)}{2E}-\frac{i\varepsilon(p-q)}{E}-\frac{i(p^3-q^3)}{6E}\right]=-\frac{Ek_F}{4\pi^2}=-\frac{\nabla\rho}{4}=\frac{1}{4}[\nabla\rho,1]_y.\qquad (3.11)$$

Here we have taken into account the fact that $\rho=k_F^3/3\pi^2$.

Thus for $\nabla\mu\neq 0$ an uncompensated momentum appears in the ground state, momentum which is carried by the fermions occupying the anomalous level. For this level the existence of this momentum is a consequence of the violation of the symmetry $E\to -\varepsilon$, or, what is the same, $p_y\to -p_y$.

Let us prove that it is exactly the anomalous level that carries the uncompensated vacuum momentum. For this we note that $j_y(E)=-j_y(-E)$. Since the current is odd under coordinate inversions $p_y\to -p_y$, it follows similarly to (2.19) that the only level that contributes to $j_y(E,p_y)-j_y(E,-p_y)=2j_y(E,p_y)$ is the level for which $E_0(p_y)=-E_0(-p_y)$, i.e., the anomalous level (3.5).

We note that Eq. (3.3) can be solved by a method analogous to the one discussed in Sec. 1, i.e., to make use of an expansion around the minimum of the potential $U=[(\varepsilon+p^2/2)^2+(\alpha p)^2]^{1/2}$, which also leads to the result (3.11) for the current.

We now show that the contribution to the current for $\nabla\mu\neq 0$ comes just from $\nabla\mu\perp 1$. To this end we consider the general case:

$$\hat{H}=\tau_3(\hat{p}^2/2-\mu-E_\perp x-E_\parallel z)+\alpha\tau_1\hat{p}_x-\alpha\tau_2\hat{p}_y,\qquad (3.12)$$

where $E_\perp=\nabla_x\mu$, $E_\parallel=\nabla_z\mu$. We carry out in Eq. (3.12) the change of variables

$$\tilde{E}_\perp x+\tilde{E}_\parallel z=m,\quad \tilde{E}_{\perp,\parallel}=E_{\perp,\parallel}/E,$$

$$-\tilde{E}_\parallel x+\tilde{E}_\perp z=n,\quad E=(E_\parallel^2+E_\perp^2)^{1/2}.\qquad (3.13)$$

Then Eq. (3.12) can be rewritten in the form

$$\hat{H}=\tau_3(p_y^2/2+p_m^2/2+p_n^2/2-\mu-mE)+\alpha\tau_1(p_m\tilde{E}_\perp-p_n\tilde{E}_\parallel)-\alpha\tau_2 p_y,\qquad (3.14)$$

from which, by analogy with Eq. (3.2), we obtain the Hamiltonian

$$\hat{H}\varphi=[-\alpha p_y\tau_3+\alpha\tau_1(p_m\tilde{E}_\perp-p_n\tilde{E}_\parallel)-\tau_3(\varepsilon-mE)]\varphi,\qquad (3.15)$$

where $\varepsilon=(p_y^2+p_n^2+p_m^2)/2$. After the obvious substitutions

$$\varphi\to\varphi\exp(imE_\parallel/E_\perp),\quad m\to m-\varepsilon/E$$

we find the Hamiltonian

$$\hat{H}=-\alpha p_y\tau_3+\alpha\tau_1 p_m\tilde{E}_\perp+\tau_3 mE.\qquad (3.16)$$

The index for this Hamiltonian is computed in Appendix B and equals $\eta(H)=\mathrm{sgn}(E_\perp p_y)$. Thus, the Hamiltonian (3.12) has an anomalous level only for $E_\perp\neq 0$, and on account of the arguments above, it is the one responsible for the vacuum current j_y, i.e., j_y depends only on $E_\perp=(\nabla\mu\times 1)_y$.

Since it follows from the Atiyah-Singer index theorem that the anomalous level (3.5) does not have a gap, the density of states for $E_0=0$ will be different from zero:

$$N(0)=2\sum|\chi(x=0)|^2\delta(E_0)=\frac{1}{8\pi^2\Delta_0}|\nabla\mu|.\qquad (3.17)$$

4. THE CASE OF AN ARBITRARY TEXTURE

In this section we consider the current carried by the quasiparticles of the anomalous level in an arbitrary texture, and show that it coincides with the current (2). We set

$$\mathbf{e}_1=\hat{x}-az(\hat{y}\cos\varphi+\hat{z}\sin\varphi),\qquad (4.1a)$$

$$\mathbf{e}_2=\hat{y}\sin\varphi-\hat{z}\cos\varphi, \quad (4.1b)$$

$$\mathbf{l}=\hat{y}\cos\varphi+\hat{z}\sin\varphi+az\hat{x}, \quad (4.1c)$$

$$\operatorname{curl}\mathbf{l}=a\hat{y}. \quad (4.1d)$$

In this case curl l is direction along the $\hat{y}$ axis and at the point $z=0$ the vector forms an angle φ with curl l. We separate, as before, the dependence on the "good" quantum numbers p_x, p_y:

$$\chi\to\chi\exp(ip_xx+ip_yy). \quad (4.2)$$

In the Bogolyubov equation (1.1) we transform to the momentum representation in the variable p_z: $\hat{p}_z\to p_z$, $z\to i\partial/\partial p_z$, and make the substitution

$$p=p_z\operatorname{ctg}\varphi+p_y\operatorname{ctg}^2\varphi, \quad p_z=p\operatorname{tg}\varphi-p_y\operatorname{ctg}\varphi, \quad (4.3)$$

after which we define, similarly to (2.4),

$$\ln(|p|/k_F)=\xi, \quad \operatorname{sgn}p=e=\pm1, \quad p=ek_Fe^{\xi}, \quad (4.4a)$$

$$\chi=\varphi k_F^{-1/2}\exp(-\xi/2-ip_x\xi/a\sin\varphi). \quad (4.4b)$$

This yields the one dimensional Bogolyubov equation:

$$H\varphi=[\varepsilon(\xi)\tau_3-W(\xi)\tau_2-i\tilde{a}d\tau_1/d\xi]\varphi=E\varphi, \quad (4.5)$$

where

$$\varepsilon(\xi)=p_x^2/2+{}^1/_2(p_y/\sin\varphi)^2+{}^1/_2k_F^2(e^{2\xi}\operatorname{tg}^2\varphi-1)-ek_Fp_ye^{\xi},$$

$$W(\xi)=\alpha p_y/\sin\varphi-e\alpha k_Fe^{\xi}\sin\varphi, \quad \tilde{a}=a\sin\varphi. \quad (4.6)$$

The spectrum of the Hamiltonian cannot be determined exactly; however, one can make use of the method explained in Sec. 1 and solve an approximate problem. For this purpose we consider, as in Sec. 1, the minima of the potential

$$U=(\varepsilon^2+W^2)^{1/2}. \quad (4.7)$$

We assume that for a sufficiently weak texture $a\to0$ the low-lying states of the Hamiltonian (4.5) are well localized near the minimum of the potential (4.7) in which the particle with mass $\sim1/a$ is moving. In this case we expand ε and W near the minimum point $\xi=\xi_0$:

$$\varepsilon(\xi)\approx\varepsilon(\xi_0)+(\partial\varepsilon/\partial\xi)_{\xi_0}(\xi-\xi_0),$$

$$W(\xi)\approx W(\xi_0)+(\partial W/\partial\xi)_{\xi_0}(\xi-\xi_0). \quad (4.8)$$

After this approximation the spectrum determination problem can be solved exactly. We determine the ground state of (4.5) for the values of ε and W from Eq. (4.8). For this we carry out a unitary transformation

$$\tilde{\varphi}=\exp(i\tau_1\beta/2)\varphi, \quad \tilde{H}=\exp(-i\tau_1\beta/2)H\exp(i\tau_1\beta/2). \quad (4.9)$$

We set

$$\operatorname{tg}\beta=\frac{W(\xi_0)}{\varepsilon(\xi_0)}, \quad \sin\beta=\frac{W(\xi_0)}{[\varepsilon^2(\xi_0)+W^2(\xi_0)]^{1/2}},$$

$$\cos\beta=\frac{\varepsilon(\xi_0)}{[\varepsilon^2(\xi_0)+W^2(\xi_0)]^{1/2}}. \quad (4.10)$$

Then Eq. (4.5) can be written in the form

$$\tilde{H}\tilde{\varphi}=(\tilde{\varepsilon}\tau_3-\tilde{W}\tau_2-i\tilde{a}\tau_1d/d\xi)\tilde{\varphi}=E\tilde{\varphi}, \quad (4.11)$$

where

$$\tilde{\varepsilon}=(\varepsilon^2+W^2)^{1/2}, \quad \tilde{W}=(\partial W/\partial\xi)_{\xi_0}(\xi-\xi_0)/\cos\beta=M(\xi-\xi_0).$$

This immediately leads to a solution for (4.11) with $n=0$:

$$\tilde{\varphi}=\begin{pmatrix}\theta(M\tilde{a})f_0(\xi-\xi_0)\\ \theta(-M\tilde{a})f_0(\xi-\xi_0)\end{pmatrix}, \quad E=\tilde{\varepsilon}\operatorname{sgn}(M\tilde{a}), \quad (4.12)$$

$$f_0=(\pi\lambda)^{-1/4}\exp[-(\xi-\xi_0)^2/2\lambda^2], \quad \lambda=|\tilde{a}/M|^{1/2}, \quad (4.13)$$

where f_0 is the normalized harmonic oscillator wave function (it is easy to see that then φ and χ will also be normalized).

We calculate the current carried by the fermions on the level with $n=0$. From Eq. (1.18) we have:

$$\mathbf{j}=\sum_s\mathbf{k}_s[\theta(-E_s)|u_s|^2-\theta(E_s)|v_s|^2], \quad (4.14)$$

where $E_s=\tilde{\varepsilon}\operatorname{sgn}(M\tilde{a})$ for $n=0$ and s denotes a complete set of quantum numbers. Consequently, taking into account Eq. (4.12), we obtain

$$M\tilde{a}>0, \quad |v_0|^2=|f_0|^2\sin^2(\beta/2),$$
$$M\tilde{a}<0, \quad |u_0|^2=|f_0|^2\sin^2(\beta/2), \quad (4.15)$$

from which it follows that

$$\mathbf{j}=\sum_s\mathbf{k}_s[-\operatorname{sgn}(M\tilde{a})\sin^2(\beta/2)]|f_0|^2, \quad (4.16)$$

$$|f_0|^2=\frac{k_F|\operatorname{tg}\varphi|}{2\pi}\left|\int\exp\left(\frac{\xi}{2}-i\frac{p_x\xi}{a\sin\varphi}\right)f_0(\xi-\xi_0)d(\xi-\xi_0)\right|^2$$
$$=\frac{k_F|\operatorname{tg}\varphi|}{\pi^{1/2}}\lambda\exp\left(\xi_0+\frac{\lambda^2}{4}-\frac{\lambda p_x^2}{a^2\sin^2\varphi}\right). \quad (4.17)$$

We note that, as can be seen easily $\tan\beta\gg1$, hence $\sin^2(\beta/2)=1/2$ for practically all values of p_x, p_y. Then the expression for the current becomes, after integration with respect to p_x,

$$\mathbf{j}=-\frac{1}{8\pi^2}\int dk_y\mathbf{k}_s\operatorname{sgn}(M\tilde{a})|k_F\operatorname{tg}\varphi e^{\xi_0}a\sin\varphi|. \quad (4.18)$$

We introduce the momenta $k_\parallel$ and $k_\perp$ (see Fig. 3):

$$k_\parallel=p_z\sin\varphi+p_y\cos\varphi=k_F\cos(\varphi-\omega), \quad (4.19a)$$

$$k_Fe^{\xi_0}\operatorname{tg}\varphi=e(p_z+p_y\operatorname{ctg}\varphi), \quad (4.19b)$$

$$\operatorname{sgn}(M\tilde{a})=-\operatorname{sgn}(ak_\parallel k_\perp p_z), \quad (4.19c)$$

$$p_y=k_F\cos\omega. \quad (4.19d)$$

In the derivation of (4.19b) we have taken account of Eq. (4.3), and in the derivation of Eq. (4.19c) we have considered Eqs. (4.6) and (4.11). Then the expression for the current can be finally written making use of the fact that all the momenta are close to the Fermi momentum:

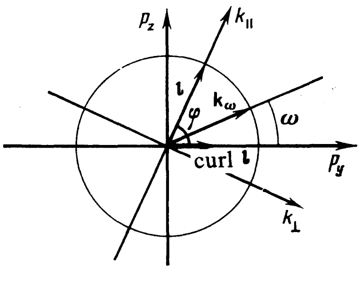

FIG. 3. A section of the Fermi sphere for $k_x=0$. The vector $\mathbf{k}_s$ is on a circumference of radius k_F; ω is the angle defining the direction of the vector $\mathbf{k}_s$ along this circle.

$$\mathbf{j}=\frac{a}{8\pi^2}\int \mathbf{k}_s\, dp_y\, \mathrm{sgn}\,(p_z k_\perp)\, k_\parallel = -\frac{a}{8\pi^2}\int \mathbf{k}_\omega k_F{}^2(-\sin\omega)\cos(\varphi-\omega)\,\mathrm{sgn}\,(k_z k_\perp)\, d\omega$$

$$=-\frac{a}{8\pi^2}k_F{}^3\int_0^\pi d\omega \mathbf{k}_\omega(-\sin\omega)\cos(\varphi-\omega)\,\mathrm{sgn}\,(\varphi-\omega)$$

$$=-{}^1/_4\rho a\hat{y}\cos 2\varphi a-{}^1/_4\rho a\hat{z}\sin 2\varphi={}^1/_4\rho\,\mathrm{rot}\,\mathbf{l}-{}^1/_2\rho\mathbf{l}(\mathbf{l}\,\mathrm{rot}\,\mathbf{l}). \tag{4.20}$$

Thus, we have obtained the result that the current transported by the anomalous level (4.12) is indeed the total current flowing in the texture under consideration. We note that in the derivation of (4.20) we have integrated with respect to ω from 0 to π, since this corresponds to one value of the charge e; if one takes into account two possible values of e one must double the magnitude of the current, but then one must take into account the possible values of e also in the normalization of the wave function (4.13), which leads to the previous result (4.20) [see also Eqs. (2.14)–(2.16)].

We now apply the index theorem to elucidate the question of whether there exist anomalous levels for the original Hamiltonian (4.5). For this we consider the index of the operator H as a function of p_x (the computation of the index can be found in Appendix B):

$$\eta[H(p_x)]=2\,\mathrm{sgn}\,(a\sin\varphi\cos\varphi)\theta[-\varepsilon(W=0)]. \tag{4.21}$$

We note that $\eta(H)$ is odd under the transformation $a\to -a$, $\varphi\to\varphi+\pi$, which means the substitution $\mathbf{l}\to-\mathbf{l}$, as well as under $a\to -a$, which means curl $\mathbf{l}\to -$ curl $\mathbf{l}$. The current is also odd under these operations. The number of energy levels of the Hamiltonian which change sign as p_x varies from 0 to ∞, or, in other words, as $\varepsilon(W=0)$ varies from $-\mu$ to ∞ (one may assume $k_y=0$) is

$$^1/_2[\eta(H_{\varepsilon=+\infty})-\eta(H_{\varepsilon=-\mu})]$$
$$=-\mathrm{sgn}\,(a\sin\varphi\cos\varphi). \tag{4.22}$$

Thus, in the case under consideration, there exists a gap-free anomalous level, and its approximate value has been found to be (4.12). Similar to the preceding, assuming $E_n(a)\approx E_n(-a)$ for $n\neq 0$, and $E_0(a)=-E_0(-a)$ for $n=0$ (in case one uses the symmetries of the approximate Hamiltonian these assertions are exact) one can show that the main contribution to the current comes from the anomalous level (4.12).

CONCLUSION

In this paper we have considered the Bogolyubov equations for quasiparticles in the ^{3}He-A in the presence of a texture and have shown that the spectrum of the Bogolyubov equation has an anomalous branch, i.e., a branch which violates the symmetry under spatial inversion and carries an uncompensated quasiparticle momentum in the vacuum, (3). The momentum carried by the femions in other branches of the spectrum is compensated, totally or approximately, up to terms of higher order in the gradients. In those cases when the non-anomalous levels could, in principle, lead to an uncompensated current, this current has a nontopological nature, in distinction from the anomalous current (3).

In a certain case, e.g., when the momenta are close to the Fermi momentum, the Bogolyubov operator is analogous to the Dirac operator in the presence of an external field, the role of which is played by the texture of the vector $\mathbf{l}$ (Ref. 6). In this case the spectrum of the Bogolyubov operator reproduces the chiral anaomaly of quantum electrodyynamics (in 2 + 1 dimensions). At the same time, if one determines the total vacuum current, to which portions of the anomalous level which are remote from the Fermi surface contributed, this analogy no longer exits. Nevertheless, in solving the problem of determination of the spectrum, exactly or approximately, we find that the spectrum exhibits, as before, an anomalous branch, leading to a current flowing in the vacuum state of ^{3}He-A, current which equals the momentum of the fermions which are in the anomalous level.

The reason for the existence of an anomaly in the quasiparticle spectrum follows from the topology of the spectrum of the Bogolyubov operator and is related to the Atiyah-Singer index theorem (cf. Appendix A), in the same manner as in quantum electrodynamics. We note that the existence of an anomalous branch was proved rigorously for the original Bogolyubov operator. Since, in addition, the gapless Fermi fluid which occupies the anomalous branch has a finite density of states $N(E=0)$, this implies a finite density of the normal component $\rho_n(T=0)=N(0)k_F^2$ in any texture in ^{3}He-A, including the case when the texture $\mathbf{l}$ is absent, but there is only a gradient of the chemical potential in the fluid.

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APPENDIX A

In this Appendix we consider the Atiyah-Singer index theorem. As follows from the discussion of the Bogolyubov operator for the case of different textures, when we are solving the problem of determining the quasiparticle spectrum we always deal (either right away, or after an appropriate change of variables) with a one-dimensional quantum-mechanics problem for the determination of the eigenvalues of some operator. For this case use is made of a version of the Atiyah-Singer index theorem applied to Hamiltonians defined over space of odd dimension $d=2n+1$. In our case $d=1$. The good quantum numbers, i.e., the eigenvalues of operators which commute with the Hamiltonian, may be regarded as parameters.

After these remarks we reproduce a formulation of the theorem following Ref. 9. We consider a one-dimensional manifold R and on it a family of Hamiltonians H_t, depending on a parameter t, $-\infty<t<+\infty$. Assume that for $t=\pm\infty$ the operator H_t has no zero modes. We analyze the change of the spectrum of H_t as t varies adiabatically from $-\infty$ to $+\infty$. Assume that some eigenvalues of the Hamiltonian

$$H_t\psi_t{}^m=\lambda_t{}^m\psi_t{}^m, \tag{A1}$$

change sign as t varies, and consequently, for certain values t_i, $i=1,\ldots,N$ they intersect the line $\lambda=0$. The Atiyah-Singer index theorem asserts that to the number of eigenvalues of the Hamiltonian H_t which change sign as t varies, or in other words, to the spectral flow of H_t, one can associate the index of some Dirac operator D in a space of dimension $d=2n+2=2$, and this association is bijective. Consider the operator D on $\mathbb{R}\times\mathbb{R}$:

$$D=i\sigma_1\partial/\partial t+\sigma_2\otimes H_t, \tag{A2}$$

where σ_i are the Pauli matrices and $\otimes$ denotes the direct product.

Indeed, assume that some eigenvalue λ_t^m changes sign at $t=t_i$; then one can construct in the adiabatic approximation the zero mode of the operator D:

$$(i\sigma_1\partial/\partial t+\sigma_2\otimes H_t)f(t)\otimes\psi_t^m=0, \tag{A3}$$

where

$$f(t)=\exp\Big(-\int^t \sigma_3\lambda^m(t')\,dt'\Big)\,\nu\,, \tag{A4}$$

where ν is a constant spinor; in order that (A4) should be normalizable one must require ν to be an eigenvector of the matrix σ_i with eigenvalue ± 1, depending on the direction of the sign change of $\lambda^m(t)$. Since $\sigma\otimes 1$ anticommutes with D, and on account of what was said before, one can classify the zero modes of (A3) according to their chirality, and consequently one can define the index of the operator D as the difference between the numbers of zero modes of D having different chiralities (Ref. 10):

$$-\operatorname{ind} D=\sum_i \pm 1. \tag{A5}$$

We introduce the Atiyah-Patodi-Singer index for the Hamiltonian:

$$\eta(H)=\lim_{s\to 0}\frac{2}{\Gamma[(s+1)/2]}\int_0^{+\infty} dy\,y^s\,\mathrm{Tr}[H\exp(-y^2H^2)]. \tag{A6}$$

Here Tr is the total trace, to be understood as

$$\mathrm{Tr}[\hat Q]=\mathrm{tr}\sum_n \Phi_n^*(x)\hat Q\Phi_n(x),$$

where Φ_n is a complete set of functions and tr is the trace over spinor indice. The index $\eta(H)$ is a topological invariant which measures the spectal asymmetry of H_t. Indeed, from (A1) it follows that

$$\eta(H)=\sum_\lambda \mathrm{sgn}(\lambda_t^m)=\sum_{\lambda>0}1-\sum_{\lambda<0}1. \tag{A7}$$

Consider the variation of $\eta(H_t)$ in the vicinity of t_i. It can be seen from Eq. (A7) that $\eta(H_t)$ changes discontinuously with a jump of ± 2, depending on the direction of the sign change of λ_t^m. On account of Eqs. (A3)–(A5) the index of the operator D is proportional to the number of jumps of $\eta(H)$:

$$-\operatorname{ind} D=\frac{1}{2}\sum_{t_i}[\eta(H_{t_i+})-\eta(H_{t_i-})]. \tag{A8}$$

On the other hand, the number of discrete jumps of $\eta(H)$ corresponding to a passage of λ through zero can be determined as the difference between the total variation of $\eta(H_t)$ for adiabatic variation of t and the continuous variation of η. If $\eta(H)$ varies continuously there exists a formula[9]

$$\frac{d\eta}{dt}=\frac{2}{\pi^{1/2}}\lim_{M\to\infty}\mathrm{Tr}\left[\frac{1}{M}\frac{\partial H}{\partial t}\exp\Big(-\frac{H^2}{M^2}\Big)\right], \tag{A9}$$

from which

$$-\operatorname{ind} D=\frac{1}{2}[\eta(H_{t=+\infty})-\eta(H_{t=-\infty})]-\frac{1}{2}\sum_i\int_{t_i}^{t_{i+1}}\frac{d\eta}{dt}dt. \tag{A10}$$

In fact we shall calculate the quantity in the right-hand side of Eq. (A10), which equals the number of levels of the operator H which intersect the axis $\lambda=0$. In our case H is the Bogolyubov operator and t is one of its quantum numbers. The levels which contribute to (A10) will be designated as anomalous. In the computation we shall follow the Fujikawa[10] method.

APPENDIX B

This Appendix contains the calculation of the index η for actual Hamiltonians in specific textures.

1. The texture with curl $\mathbf{l}\|\mathbf{l}$. In this case it follows from Eq. (A1) that

$$\eta[H(\tilde\varepsilon)]=\frac{2}{\Gamma(^1/_2)}\int_0^{+\infty}dy\,\mathrm{Tr}\Big\{\Big[\tau_3\Big(\tilde\varepsilon+\frac{p^2}{2}\Big)+\alpha\tau_1 p+\alpha\tau_2 Bi\frac{\partial}{\partial p}\Big]\exp(-y^2H^2)\Big\}$$

$$=\frac{2}{\pi^{1/2}}\int_0^{+\infty}dy\int\frac{dk}{2\pi}\int dp\int dp'\sum_n\Phi_n(p)\times\Big[\tau_3\Big(\tilde\varepsilon+\frac{p^2}{2}\Big)+\alpha\tau_1 p\Big]$$
$$\times\exp\{-y^2[(\tilde\varepsilon+p^2/2)^2+\alpha^2p^2-\alpha^2B^2\partial^2/\partial p^2+\alpha^2\tau_3 B-\alpha pB\tau_1]\}\Phi_n^*(p')\exp[-ik(p-p')]$$

$$=\frac{1}{\pi|\alpha B|}\int_0^{+\infty}\frac{dy}{y}y^2\int dp'\,dp\,\mathrm{tr}\sum_n\Phi_n(p)\times\Big[-\tau_3^2\Big(\tilde\varepsilon+\frac{p^2}{2}\Big)\alpha^2B+\tau_1^2\alpha^2p^2B\Big]\Phi_n(p')$$
$$\times\exp\Big\{-y^2\Big[\Big(\tilde\varepsilon+\frac{p^2}{2}\Big)^2+\alpha^2p^2\Big]-\frac{(p-p')^2}{4\alpha^2B^2y^2}\Big\}=\frac{2\alpha^2B}{\pi|\alpha B|}\int_0^{+\infty}dy\,y\int dp$$
$$\times\Big(-\tilde\varepsilon+\frac{p^2}{2}\Big)\exp\Big\{-y^2\Big[\Big(\tilde\varepsilon+\frac{p^2}{2}\Big)^2+\alpha^2p^2\Big]\Big\}$$
$$=\frac{\alpha^2B}{\pi|\alpha B|}\int dp\frac{-\tilde\varepsilon+p^2/2}{(\tilde\varepsilon+p^2/2)^2+\alpha^2p^2}=2\theta(-\tilde\varepsilon)\,\mathrm{sgn}\,B. \tag{B1}$$

In the calculation of the last integral we have taken into account (cf. Ref. 9) the fact that $\tilde\varepsilon$ may take on both positive and negative values. It is not hard to see that the continuous variation of η is

$$\frac{d\eta}{d\tilde\varepsilon}=\lim_{M\to\infty}\frac{1}{M}\mathrm{Tr}\frac{dM}{d\tilde\varepsilon}\exp\Big(-\frac{H^2}{M^2}\Big)\sim\lim_{M\to\infty}\frac{1}{M}=0.$$

2. The texture with curl $\mathbf{l}\perp\mathbf{l}$. The computation of the index is analogous to (B1), so we omit some intermediate steps:

$$\eta(H) = \frac{2}{\pi^{1/2}} \int_0^{+\infty} dy\, \mathrm{Tr}\{\tau_3 \alpha p_x \exp[-y^2(\hat{\varepsilon}^2 + (\alpha p_x)^2 + (\alpha T_\perp p_\xi)^2 + 2\alpha T_\perp \mu e^{2\xi}\tau_3)]\}$$

$$= \frac{2}{\pi^{1/2}} \alpha p_x \int_0^{+\infty} dy\, \mathrm{Tr}\{\tau_3^2(-y^2) 2\alpha T_\perp \mu e^{2\xi} \times \exp[-y^2(\hat{\varepsilon}^2 + (\alpha p_x)^2 + (\alpha T_\perp p_\xi)^2)]\}$$

$$= \frac{4\mu(-\alpha p_x)\alpha T_\perp}{|\alpha T_\perp|} \int_0^{+\infty} dy\, y\, \mathrm{Tr}\{e^{2\xi} \exp[-y^2(\hat{\varepsilon}^2 + (\alpha p_x)^2)]\}$$

$$= \frac{2(-\alpha p_x)\alpha T_\perp \mu}{|\alpha T_\perp| \pi} \times \int_0^{+\infty} dz \int_{-\infty}^{+\infty} d\xi \exp\{2\xi - z[(\varepsilon_\perp + \mu(e^{2\xi}-1))^2 + (\alpha p_x)^2]\}$$

$$= \frac{(-\alpha p_x)\alpha T_\perp}{\pi |\alpha p_x| |\alpha T_\perp|} \operatorname{arctg}\left(\frac{x}{|\alpha p_x|}\right)\Bigg|_{\varepsilon_\perp - \mu}^{\infty} = -\mathrm{sgn}(T_\perp p_x). \quad \text{(B2)}$$

In the derivation of (B2) we have assumed that $\varepsilon_\perp - \mu \gg \alpha p_x$, e.g., for sufficiently small $|p_x|$, although, as can be seen from the exact solution (2.11) Eq. (B2) is valid in the general case too. One can show analogus to the derivation of (B1a) that the continuous part of the variation of η vanishes:

$$\frac{d\eta}{dp_x} = \lim_{M\to\infty} \frac{1}{M} \mathrm{Tr} \frac{dH}{dp_x} \exp\left(-\frac{H^2}{M^2}\right) \sim \lim_{M\to\infty} \frac{1}{M} = 0.$$

As can be seen from the exact solution (2.11), the topological structure of the spectrum does not change under variations δp_x for arbitrary $p_x \neq 0$.

3. The case $\nabla\mu \neq 0$. We calculate the index of the operator (3.16). Proceeding as in the derivation of (B1) and (B2), we have

$$\eta = \frac{2}{\pi^{1/2}} \int dy\, \mathrm{Tr}\{-\alpha\tau_3 p_y \exp[-y^2(\alpha^2 p_y^2 + \alpha^2 k^2 \tilde{E}_\perp^2 + m^2(E_\parallel^2 + E_\perp^2) + \alpha E_\perp \tau_3)]\}$$

$$= \frac{\alpha^2 p_y E_\perp}{|\alpha \tilde{E}_\perp| \pi} \int_0^{+\infty} dy\, y^2\, \mathrm{Tr}\{\tau_3^2 \exp[-y^2(\alpha^2 p_y^2 + m^2 |E|^2)]\}$$

$$= \frac{\alpha^2 p_y^2 E_\perp}{|\alpha \tilde{E}_\perp \alpha p_y| |E|} = \mathrm{sgn}(E_\perp p_y). \quad \text{(B3)}$$

One can show that $d\eta/dt = 0$.

4. The case of an arbitrary texture. We calculate the index of the operator (4.5):

$$\eta = \frac{2}{\pi^{1/2}} \int_0^{+\infty} dy\, \mathrm{Tr}[H \exp(-y^2 H^2)]$$

$$= \frac{1}{\pi} \mathrm{sgn}\,\tilde{a} \int_{-\infty}^{+\infty} dp_z \frac{\varepsilon W'_{p_z} - W \varepsilon'_{p_z}}{\varepsilon^2 + W^2}. \quad \text{(B4)}$$

We rewrite the integrand in the form

$$\frac{\varepsilon\, dW - W\, d\varepsilon}{\varepsilon^2 + W^2} = -\mathrm{Im} \frac{dz}{z}, \quad \varepsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2} - \mu,$$

$$W = \alpha(p_y \sin\varphi - p_z \cos\varphi),$$

where $z = W + i\varepsilon = |z| e^{i\chi}$. Now (B4) will take the form

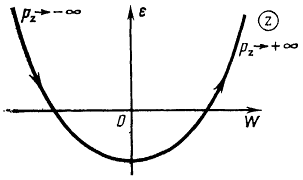

FIG. 4. The trajectory of the point $z = z(p_z) = W(p_z) + i\varepsilon(p_z)$ in the complex plane of the variable $z = W + i\varepsilon$.

$$\eta = -\frac{1}{\pi} \mathrm{Im}\{\ln z|_{p_z=-\infty}^{p_z=+\infty}\} \mathrm{sgn}\,\tilde{a} = -\frac{1}{\pi} \Delta(\arg z) \mathrm{sgn}\,\tilde{a}, \quad \text{(B5)}$$

where $\Delta(\arg z)$ is the increment of the argument of z as p_z varies from $-\infty$ to $+\infty$. In the complex plane this increment is given by the variation of the argument of a point as it moves along a parabola implicitly refined by $[\varepsilon(p_z), W(p_z)]$, Fig. 4.

It is clear from the figure that $\Delta(\arg z) \neq 0$ only when the poarabola circumscribes the coordinate origin, i.e., when the point $\varepsilon(W=0)$ of the parabola is situated below the point $\varepsilon = 0$, i.e., $\varepsilon(W=0) < 0$. Making use of the equations of Sec. 4 it is easy to see that $\varepsilon(W=0) = p_x^2/2 - p_y^2/2 \cos^2\varphi - \mu$. Now it is obvious that $\Delta(\arg z)$ changes sign depending on the direction of motion along the parabola. The direction of motion is given by $\mathrm{sgn}(\partial W/\partial p_z)$, hence

$$\Delta(\arg z) = 2\pi\theta[-\varepsilon(W=0)]\, \mathrm{sgn}(\partial W/\partial p_z).$$

We finally obtain for $\eta(H)$

$$\eta(H) = 2\theta[-\varepsilon(W=0)]\, \mathrm{sgn}(a \sin\varphi \cos\varphi). \quad \text{(B6)}$$

Thus, it is particularly transparent from the calculation above that $\eta(H)$ is a topological characteristic of the spectrum. Indeed, for any deformation of the path shown in Fig. 4 which does not cross the point $\varepsilon = 0$, $W = 0$, the result for the index $\eta(H)$ remains unchanged. Moreover, in the derivation of (B6) it was clear that in order that an anomalous level should exist it is necessary that some energy formed out of good quantum numbers should become negative (see Sec. 1).

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