New expansion for the critical indices of the theory of Anderson localization

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A model of a disordered system is considered in which the Fourier components of the potential are random quantities obeying a certain hierarchy: The greatest of them are of order V, the next greatest are of order βV, and so on. As $\beta \to 1$ such a model goes over into the usual Anderson model. For small β an expansion in the parameter β is constructed for the critical index ν of the localization length for arbitrary dimensionality d; for $\beta \sim 1$ the expansion is found to be in qualitative agreement with the results for $d = 2 + \varepsilon$ and $d = \infty$; for $\beta \to 0$ the value of ν coincides with the value for incommensurate systems. The conductivity index s is related to ν by the usual scaling relation $s = \nu(d-2)$. The renormalization-group equations contain random parameters, and so the Anderson-transition point is determined not by a fixed point but by a stationary point of the renormalization-group transformation. This ensures agreement between the scaling and the recently discussed large fluctuations of the conductivity.

The central problem of the theory of localization is the elucidation of the critical behavior of the conductivity σ and of the localization length ξ of the wave functions near the Anderson transition.[1-3] For energies E near the mobility edge E_c it is customary to describe the dependences $\sigma(E)$ and $\xi(E)$ by power laws:

$$\sigma \propto (E-E_c)^s, \quad \xi \propto (E_c-E)^{-\nu}, \tag{1}$$

while admitting the value 0 for the indices s and ν; $s=0$ obtains in the case of a Mott minimum conductivity.[4] At present the values of the indices s and ν are known in two cases: a) for space of dimensionality $d = 2 + \varepsilon$ (Refs. 5–7):

$$\nu = 1/\varepsilon + O(\varepsilon^2), \quad s = 1 + O(\varepsilon^3), \tag{2}$$

b) for $d = \infty$ (a Bethe lattice)[8,9]:

$$\nu = {}^1/_2, \quad s = 0 \tag{3}$$

(for a discussion of the apparent contradiction between the results of Refs. 8 and 9, see Ref. 10). The results for $d = 2 + \varepsilon$ are in agreement with the one-parameter scaling hypothesis of Ref. 11; in particular, the following relation between the indices s and ν is valid:

$$s = \nu(d-2). \tag{4}$$

For $d = \infty$ the relation (4) is not fulfilled, and therefore the results (2) and (3) cannot be made consistent without the introduction of a special dimensionality d_{c2} separating them; we shall call this the upper critical dimensionality (the lower critical dimensionality $d_{c1} = 2$). For d_{c2} the values 4, 6, 8, and ∞ have been suggested (Refs. 11–14); in view of the difficulties that the $2 + \varepsilon$ theory has recently experienced,[15,16] the value $d_{c2} = 2$ is also possible. As a result, it becomes unclear how σ and ξ should be expected to behave in the three-dimensional case.

In the situation that has been created it becomes essential to develop methods that make it possible to consider spaces of arbitrary dimensionality d; such a method is proposed below. We shall define models that it is natural to call hierarchical (by analogy with the hierarchical models of Dyson[17]); they are characterized by a certain parameter β, and go over into the usual Anderson models as $\beta \to 1$. For $\beta \ll 1$ we succeed in constructing an expansion of the critical indices in the parameter β, and one can attempt to extrapolate the results of this expansion to $\beta = 1$.

On the other hand, as $\beta \to 0$ the models under consideration go over into periodic models, thereby effecting a smooth interpolation between maximally disordered and ordered systems. Therefore, values of β differing from unity can also be of physical interest. This is confirmed by the fact that as $\beta \to 0$ the values of the indices coincide with those for incommensurate systems,[18] for which a scaling theory of localization was constructed earlier by the author[19]; the latter theory has now received mathematical justification.[20,21]

1. IDEA OF THE METHOD AND PRINCIPAL RESULTS

In the models under consideration the Anderson transition already exists in the one-dimensional case. We shall explain the idea of the method using the example of a one-dimensional chain described by the discrete Schrödinger equation

$$J(a_{l+1}+a_{l-1}) + V_l a_l = E a_l. \tag{5}$$

The potential V_l will be specified in the momentum representation:

$$V_l = \sum_q V(q) e^{iql}. \tag{6}$$

For complete specification of the potential on a chain of 2^N atoms it is sufficient to specify its Fourier components for values of q of the form $2\pi l/2^N$. We shall do this as follows:

$$V(0) = V(2\pi) = 0, \quad V(\pi) = V, \quad V(\pi/2) \sim V(3\pi/2) \sim \beta V,$$
$$V(\pi/4) \sim V(3\pi/4) \sim V(5\pi/4) \sim V(7\pi/4) \sim \beta^2 V,$$

etc. The most interesting model is that in which all the $V(q)$ have Gaussian distributions,[1)] with

$$\langle V(q) \rangle = 0, \quad \langle V^*(q) V(q') \rangle = 0, \quad q \neq q', \tag{7}$$
$$\langle |V(q)|^2 \rangle = (\beta^n V)^2 \text{ for } q = (2l-1)\pi/2^n, \; l = 1, 2, \ldots, 2^n.$$

It is obvious that as $\beta \to 1$ such a model goes over into

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the usual Anderson model with a Gaussian distribution of levels. The extrapolation $\beta \to 1$ should be performed before the thermodynamic limit is taken, since otherwise the model is not defined. Since the critical exponents will be determined from the renormalization-group equations describing the relationship of the parameters for finite systems, the correct order of the limits is achieved automatically.

That the case of small β is indeed simple for analysis will be shown by constructing the zeroth approximation for Eq. (5) with the potential

$$V_l = V\sum_{n=0}^{\infty}\beta^n\varphi(l/2^n), \quad \varphi(x)=\begin{cases} 1, & -1/2+2\pi l \leqslant x < 1/2+2\pi l \\ -1, & 1/2+2\pi l \leqslant x < 3/2+2\pi l. \end{cases} \tag{8}$$

With complete neglect of the potential V_l the spectrum of Eq. (5) has the form $\varepsilon(k) = 2J\cos k$, and is a band of width $\sim J$. We separate out from the potential the largest Fourier component $V(\sigma)$:

$$V_l = V(-1)^l + V_l', \tag{9}$$

and for the start of the calculation we neglect the potential V_l'. Then Eq. (5) describes a periodic system and its spectrum consists of two bands:

$$\varepsilon_{1,2}(k) = \pm[V^2 + 2J^2(1+\cos 2k)]^{1/2}$$

$$\approx \pm[V + (J^2/V)\cos 2k + \ldots]. \tag{10}$$

We have performed the expansion assuming that $V \gg J$. Thus, to within a trivial scale transformation, the spectrum of each of the bands coincides with the spectrum of the original band; the spacing between the bands is $\sim V$, and their width is $\sim J^2/V \ll V$. In the potential it is now necessary to take into account the remaining part V_l', whose amplitude is $\sim \beta V$ and is small in comparison with the spacing between the bands; therefore, in the zeroth approximation we can regard the bands as independent and write for each of them its own Schrödinger equation.

By analogy with the derivation of Eq. (1) from the continuous Schrödinger equation, we introduce Wannier functions describing the distribution of the amplitudes near a pair of sites, and seek the wavefunction of the whole system in the form of a superposition of Wannier functions with expansion coefficients b_m (m is the label of a pair). For the quantities b_m we set up an equation analogous to (5):

$$J'(b_{m+1} + b_{m-1}) + u_m b_m = E b_m. \tag{11}$$

For $V \gg J$ the Wannier functions of the first band are localized mainly at even sites, and those of the second band at odd sites, and therefore u_m (the average value of V_l' for the corresponding Wannier function) coincides with V_{2m}' or V_{2m+1}', respectively. As is clear from (8), the potential V_l' taken at even or odd points coincides with the original potential V_l, but with the constant V replaced by $V' = \beta V$. The relationship of J' to the parameters of Eq. (5) is established from the condition that, in the absence of V_l', the spectrum of (11) coincide with (10). As a result, (11) has the form of Eq. (5) in which, in place of V and J, new parameters V' and J' appear:

$$J' = \pm J^2/2V, \quad V' = \beta V. \tag{12}$$

The quantity b_m can be regarded as the amplitude for finding an electron at the mth pair of sites. Thus, in going from (5) to (11) we have reduced the description: The wavefunctions are coarsened on scales of two interatomic distances, rather than one as in Eq. (5). Continuing this reduction of the description further, in place of (12) we obtain

$$J^{(n+1)} = \pm J^{(n)2}/2V^{(n)}, \quad V^{(n+1)} = \beta V^{(n)}, \tag{13}$$

where $J^{(n)}$ and $V^{(n)}$ are the parameters of Eq. (5) at the nth stage of this procedure. The fact that the coarsened wavefunctions satisfy Eq. (5) with changed coefficients implies the existence of scaling: Upon change of the parameters of Eq. (5), the envelope of its wavefunctions changes while remaining self-similar.

Since the form of the wavefunctions of Eq. (5) depends only on the ratio[2] $|J/V|$, it is convenient to introduce the quantity

$$g^{(n)} = |J^{(n)}/V^{(n)}|, \tag{14}$$

which is the Thouless parameter[11] for a block of size 2^n. From (13) it is easy to obtain for $g^{(n)}$ the renormalization-group equation

$$g^{(n+1)} = g^{(n)2}/2\beta, \quad g^{(0)} = J/V, \tag{15}$$

which has the same functional form as that postulated in Ref. 11. Equation (15) has a fixed point $g_c = 2\beta$; for $g_0 > g_c$ the ratio $|J^{(n)}/V^{(n)}|$ increases, and for large n the scattering potential $V_l^{(n)}$ becomes unimportant; for $g_0 < g_c$ the ratio $|J^{(n)}/V^{(n)}|$ decreases, and for $n \gg 1$ we can neglect the overlap integral $J^{(n)}$. The first case corresponds to delocalized states, and the second case to localized states, i.e., the point

$$g^{(0)} = J/V = 2\beta \tag{16}$$

is the Anderson-transition point. For small β near the transition we have $J^{(n)} \ll V^{(n)}$, which justifies the assumption made.

Linearizing (15) about g_c,

$$g^{(n+1)} - g_c = 2(g^{(n)} - g_c), \tag{17}$$

and taking into account that the unit of length is doubled with each step, we find that the characteristic scale on which the deviation of $g^{(n)}$ becomes of order unity is

$$\xi \propto |g^{(0)} - g_c|^{-1}. \tag{18}$$

In the region of localized states, ξ coincides with the localization length, and, consequently, for the index ν we obtain

$$\nu = 1, \quad \beta \to 0, \tag{19}$$

this completes the construction of the zeroth approximation.

For the Gaussian model (7) the first of Eqs. (13) is preserved, but the second is valid only in a statistical sense: $V^{(n+1)}$ has the same distribution as $aV^{(n)}$ with a certain $a \sim 1$. Changing the unit of measurement of energy at the nth step in such a way as to make all the $D^{(n)}$ statistically equivalent, in place of (15) we obtain

$$g^{(n+1)} = \frac{1}{2a\beta}\left|\frac{V^{(n)}}{V^{(n+1)}}\right| g^{(n)2}. \tag{20}$$

The renormalization-group equation acquires an important qualitative feature: A random parameter appears in it. Therefore, the Anderson-transition point is determined not

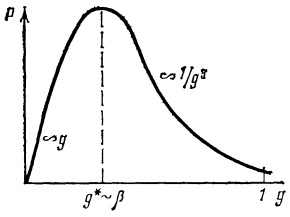

FIG. 1.

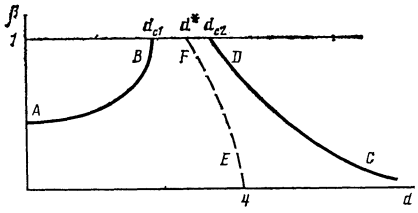

FIG. 2.

by a fixed point, but by a stationary point of Eq. (20), i.e., by the condition that the sequence $G^{(n)}$ is a stationary random process. The distribution of $g^{(n)}$ at the stationary point is shown in Fig. 1. In order of magnitude, $g^{(n)}$ coincides with the total conductivity of a block of size 2^n (Ref. 11) in units of $e^2/\hbar$. Consequently, the conductivity fluctuations at the mobility edge turn out to be of the order of the average conductivity, in agreement with the qualitative conclusions of Ref. 22 (the author of Ref. 22, however, studied the standard deviation of $g^{(n)}$, which, as is clear from Fig. 1, can be significantly greater than the width of the distribution). Fluctuations of $g^{(n)}$ do not impede the existence of scaling, and, moreover, do not change the result (19).

The result (19) turns out to be valid for an arbitrary dimensionality d of space. Starting with this result, we can construct an expansion of the exponent ν in the hierarchy parameter β; in this paper we calculate the first nonvanishing correction:

$$\nu=1-(2/3\ln 2)2^d(d-4)\beta^2\ln(1/\beta)+O(\beta^2). \qquad (21)$$

In the calculation with this accuracy the renormalization-group transformation becomes substantially more complicated: In Eq. (11) overlap integrals between next-nearest neighbors appear, the potential u_m becomes nonlocal, and, in addition, the two bands (10) become coupled (in the d-dimensional case one has to take into account coupling 2^d bands). Despite the constructive many-parameter character of the renormalization-group thus obtained, the final results do not contradict the hypothesis of one-parameter scaling; in particular, the relation (4) between the exponents s and ν is preserved. It is evident that by an appropriate change of variables the renormalization group can be transformed to an explicitly one-parameter form.

The extrapolation $\beta\to 1$ is most reliable in that region in which the correction to the zeroth approximation is small, i.e., for $d\cong 4$. Therefore, the main prediction that can be made for $\beta=1$ is

$$\nu=1 \text{ for } d\approx 4. \qquad (22)$$

From the $2+\varepsilon$ theory ($\nu=1/\varepsilon$) and the numerical calculations of Ref. 23 ($\nu=1.2\pm 0.3$ for $d=3$) it follows that the value of d for which $\nu=1$ is close to 3. In view of the logarithmic accuracy of formula (21), such a discrepancy can scarcely be regarded as important. The decrease of ν with increase of d predicted by formula (21) is also in qualitative agreement with the results (2) and (3).

The postulated phase diagram in the (d, β) plane is depicted in Fig. 2. For small values of d there exists a singular line AB, above which all states are localized: For large values of d the singular line CD bounds the region of one-parameter scaling—above this line the relation (4) is violated. The large coefficient of $\beta^2\ln(1/\beta)$ in formula (21) suggests a small radius of convergence of the expansion in β for large values of d; it is natural to suppose that it is bounded by the line CD, the equation of which for $d\to\infty$ is $\beta\sim 2^{-d/2}$. Near the line EF, on which $\nu=1$, the radius of convergence of the β-expansion is a maximum, and in the neighborhood of $d^*\cong 3$ the region of one-parameter scaling emerges onto the line $\beta=1$. The curves AB and CD intersect with the straight line $\beta=1$ at the points d_{c1} and d_{c2}, which must be identified with the lower and upper critical dimensionalities. On the line AB the exponent ν becomes infinite. If the proposed picture is correct, calculation of further terms of the expansion (21) will make it possible to obtain a satisfactory theory for the three-dimensional case.

It is possible to define hierarchical models in which the site energies have distributions with infinite variance (Sec. 3). In this case the coefficients of $\beta^2\ln(1/\beta)$ in (21) changes; in particular, the value of d at which it vanishes turns out to be smaller, and it is possible that this implies a lowering of d_{c1} and d_{c2}.

2. DEFINITION OF THE MODELS

We shall consider models describable by a discrete Schrödinger equation:

$$J\sum_{i=1}^{d}(a_{\mathbf{m}+\mathbf{e}_i}+a_{\mathbf{m}-\mathbf{e}_i})+V_{\mathbf{m}}a_{\mathbf{m}}=Ea_{\mathbf{m}}, \qquad (23)$$

where $\mathbf{m}$ labels the sites of a d-dimensional cubic lattice, and $\mathbf{e}_i$ ($i=1, 2, \ldots, d$) are the basis vectors of the lattice; for the following it is convenient not to fix the direction of $\mathbf{e}_i$, since the choice of $\mathbf{e}_i$ will always be clear from the context. We shall specify the potential V_m in the momentum representation:

$$V_{\mathbf{m}}=\sum_{\mathbf{q}}V(\mathbf{q})e^{i\mathbf{q}\mathbf{m}}. \qquad (24)$$

It is convenient to assume that the thermodynamic limit is taken over a sequence of systems having the shape of a d-dimensional cube with side of 2^N atoms; then the momentum space can be assumed to be discrete, and the components of the vector $\mathbf{q}$ take only values of the form $\pi l/2^{n-1}$, with $l=0, 1, \ldots, 2^n-1$.

We introduce in momentum space a set of points Γ_p ($p=0, 1, 2, \ldots$) as a set of vectors of the form

$$\mathbf{q}=(\pi l_1/2^{p-1}, \pi l_2/2^{p-1}, \ldots, \pi l_d/2^{p-1}), \quad l_i=0, 1, \ldots, 2^p-1.$$

The set of points contained in Γ_{p+1} but not in Γ_p will be called Ω_p:

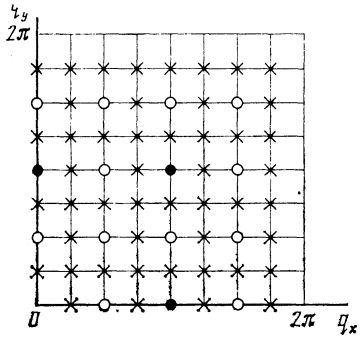

FIG. 3. Sets Ω_n for the two-dimensional case: ●) Ω_0; ○) Ω_1; ×) Ω_2.

$$\Omega_p=\Gamma_{p+1}/\Gamma_p.$$

The sets Ω_p for the two-dimensional case are shown in Fig. 3. We note two indentities that follow directly from the given definition:

$$\sum_{\mathbf{q}\in\Gamma_n}\sum_{\mathbf{k}\in\Gamma_p} f(\mathbf{q}+\mathbf{k}/2^n)=\sum_{\mathbf{q}\in\Gamma_{p+n}} f(\mathbf{q}), \tag{25}$$

$$\sum_{\mathbf{q}\in\Omega_p} f(\mathbf{q})=\sum_{\mathbf{q}\in\Omega_{p-n}}\sum_{\mathbf{a}\in\Gamma_n} f(\mathbf{q}/2^n+\mathbf{a}). \tag{26}$$

The Fourier component with $\mathbf{q}=0$ in (24) can be assumed to be equal to zero, since it leads only to a shift in the energy origin. We subject the remaining Fourier components to the hierarchy

$$V(\mathbf{q})\sim\beta^p V \text{ for } \mathbf{q}\in\Omega_p. \tag{27}$$

For the main part of our account we do not need any assumptions about the properties of $V(\mathbf{q})$, apart from the condition $\beta\ll 1$. Concrete quantitative results will be obtained for the Gaussian model, in which all the $V(\mathbf{q})$ have independent Gaussian distributions, with

$$\langle V(\mathbf{q})\rangle=0, \quad \langle V^*(\mathbf{q})V(\mathbf{q}')\rangle=0, \quad \mathbf{q}\neq\mathbf{q}';$$
$$\langle|V(\mathbf{q})|^2\rangle=(\beta^n V)^2, \quad \mathbf{q}\in\Omega_n. \tag{28}$$

As $\beta\to 1$ such a model goes over into the Anderson model with a Gaussian distribution of levels. In Sec. 8 we define models in which the distribution of levels has an infinite variance.

We denote the analogs in coordinate space of the sets Γ_p and Ω_p by γ_p and ω_p:

$$\gamma_p\colon \mathbf{m}=(m_1, m_2, \ldots, m_d), \quad m_i=0, 1, \ldots, 2^p-1;$$
$$\omega_p=\gamma_{p+1}/\gamma_p.$$

For the following it is convenient to define the π-periodization operator $\hat{Q}_\pi^{\mathbf{k}}$, which transforms a function $f(\mathbf{k})$ that is 2π-periodic in all its components into a π-periodic function:

$$\hat{Q}_\pi^{\mathbf{k}} f(\mathbf{k})=\frac{1}{2^d}\sum_{\mathbf{a}\in\Gamma_1} f(\mathbf{k}+\mathbf{a}).$$

We shall often find it convenient to write sums over all $\mathbf{q}$ in the form

$$\sum_{\mathbf{q}}\ldots\to\sum_{p=0}^{\infty}\sum_{\mathbf{q}\in\Omega_p}\ldots.$$

3. FORMAL CONSTRUCTION OF THE RENORMALIZATION GROUP

1. We shall elucidate the general structure of the renormalization group for arbitrary β. The calculations are conveniently performed in the momentum representation, in which Eq. (23) has the form

$$\Big(\sum_{i=1}^{d} 2J\cos k_i\Big)a(\mathbf{k})+\sum_{p=0}^{\infty}\sum_{\mathbf{q}\in\Omega_p} V(\mathbf{q})a(\mathbf{k}+\mathbf{q})=Ea(\mathbf{k}). \tag{29}$$

At the nth step of the renormalization group we retain in (29) the Fourier components with $p=0, 1, \ldots, n-1$; since $V(0)=0$, the equation takes the form

$$\Big(\sum_{i=1}^{d} 2J\cos k_i\Big)a(\mathbf{k})+\sum_{\mathbf{q}\in\Gamma_n} V(\mathbf{q})a(\mathbf{k}+\mathbf{q})=Ea(\mathbf{k}).$$

Shifting the argument $\mathbf{k}$ by all possible $\mathbf{k}_0\in\Gamma_n$ and taking into account that $\mathbf{q}+\mathbf{k}_0$, after reduction to the interval $(0, 2\pi)$, belongs to Γ_n, we obtain a system of 2^{nd} equations for the 2^{nd} quantities $a(\mathbf{k}+\mathbf{k}_0)$, $\mathbf{k}_0\in\Gamma_n$. Solving this system we find the eigenvalues $\varepsilon_s(\mathbf{k})$ and eigenfunctions $a_s(\mathbf{k})$, $s=1, 2, \ldots, 2^{nd}$. We shall seek the solution of (29) in the form

$$a(\mathbf{k})=\sum_{s=1}^{2^{nd}} b_s(\mathbf{k})a_s(\mathbf{k}), \quad b_s(\mathbf{k}+\mathbf{k}_0)=b_s(\mathbf{k}), \quad \mathbf{k}_0\in\Gamma_n. \tag{30}$$

Substituting into (29), using the orthogonality relations for $a_s(\mathbf{k})$, the relation (26), and the $\pi/2^{n-1}$-periodicity of the spectra $\varepsilon_s(\mathbf{k})$:

$$\varepsilon_s(\mathbf{k}+\mathbf{k}_0)=\varepsilon_s(\mathbf{k}), \quad \mathbf{k}_0\in\Gamma_n, \tag{31}$$

introducing the notation

$$V_{ss'}(\mathbf{k},\mathbf{q})=\sum_{\mathbf{k}_0,\mathbf{q}_0\in\Gamma_n} V\Big(\frac{\mathbf{q}}{2^n}+\mathbf{q}_0\Big)a_s^*\Big(\frac{\mathbf{k}}{2^n}+\mathbf{k}_0\Big)$$
$$\times a_{s'}\Big(\frac{\mathbf{k}+\mathbf{q}}{2^n}+\mathbf{k}_0+\mathbf{q}_0\Big) \tag{32}$$

and performing the scale transformation

$$b_s(\mathbf{k})=\tilde{b}_s(2^n\mathbf{k}), \quad \varepsilon_s(\mathbf{k})=\tilde{\varepsilon}_s(2^n\mathbf{k}), \quad 2^n\mathbf{k}=\tilde{\mathbf{k}}, \tag{33}$$

we obtain, omitting the tildes,

$$b_s(\mathbf{k})[\varepsilon_s(\mathbf{k})-E]+\sum_{s'=1}^{2^{nd}}\sum_{p=0}^{\infty}\sum_{\mathbf{q}\in\Omega_p} V_{ss'}(\mathbf{k},\mathbf{q})b_{s'}(\mathbf{k}+\mathbf{q})=0 \tag{34}$$

(below, in this section, an analogous transformation is carried out in more detail). The meaning of these transformations was explained in Sec. 1 for a simpler example: We have eliminated those Fourier components of the potential with period smaller than 2^n (in the definition (32) of $V_{ss'}$, only those $V(\mathbf{q})$ with $q\in\Omega_p$ for $p\geqslant n$ appear), we have reduced the description by going over to the amplitudes $b_s(\mathbf{k})$, coarsening the wave functions on the scales 2^n, and we have performed the scale transformation (33), increasing the unit of measurement of length by a factor of 2^n.

Equation (34) is the Schrödinger equation at the nth step of the renormalization group. In appearance it is analogous to the original equation (29), differing from it in the following features: a) It describes not one but 2^{nd} coupled bands; b) each band has a spectrum of the general form

$\varepsilon_s(\mathbf{k})$, in place of the simple cosinusoidal spectrum of Eq. (29); c) the potential in the coordinate representation is nonlocal, as is manifested in the dependence of its Fourier transform $V_{ss'}(\mathbf{k},\mathbf{q})$ on $\mathbf{k}$. The latter two circumstances are easily taken into account by extending the functional form of the equation, but the first presents a more substantial difficulty, since the number of bands increases with each step of the renormalization group. If the coupling between all these bands were important, the renormalization group could not have fixed points and we could not speak of any scaling. However, it is clear intuitively that only a few bands near the energy level under investigation are important. Therefore, a natural algorithm is as follows: We confine ourselves to a finite number N of bands, near the energy value being studied, discarding all the other bands that arise in the renormalization-group transformations; we find a fixed point and the critical indices; if with increase of N the values of the indices converge to a constant limit, the scheme adopted is correct; but if this does not happen, there is no scaling in the system.

In the hierarchical models studied below the numbers of bands are restricted in a formally rigorous manner in the small parameter β: In zeroth order and first order in β it is sufficient to confine ourselves to one band, while in calculations to order β^2 one must retain the 2^d bands that are nearest in the hierarchy.

2. We derive formulas relating the parameters of the nth and $(n+1)$th steps of the renormalization group, while retaining N coupled bands in (34). In accordance with what has been said, the Schrödinger equation at the nth step of the renormalization group has the form

$$a_s(\mathbf{k})\,[\varepsilon_s^{(n)}(\mathbf{k})-E]+\sum_{s'=1}^{N}\sum_{p=0}^{\infty}\sum_{\mathbf{q}\in\Omega_p} V_{ss'}^{(n)}(\mathbf{k},\mathbf{q})\,a_{s'}(\mathbf{k}+\mathbf{q})=0, \tag{35}$$

where the quantities $V_{ss'}$ are expressed in terms of Fourier components of the original potential [compare with (32)]:

$$V_{ss'}^{(n)}(\mathbf{k},\mathbf{q})=\sum_{\mathbf{q}_0\in\Gamma_n} A_{ss'}^{(n)}\left(\mathbf{k},\frac{\mathbf{q}}{2^n}+\mathbf{q}_0\right)V\left(\frac{\mathbf{q}}{2^n}+\mathbf{q}_0\right). \tag{36}$$

It is necessary to find an algorithm for calculating $\varepsilon_s^{(n+1)}(\mathbf{k})$ and $A_{ss'}^{(n+1)}(\mathbf{k},\mathbf{q})$ from known $\varepsilon_s^{(n)}(\mathbf{k})$ and $A_{ss'}^{(n)}(\mathbf{k},\mathbf{q})$. We retain in (35) the leading terms in the hierarchy:

$$a_s(\mathbf{k})\,[\varepsilon_s^{(n)}(\mathbf{k})-E]+\sum_{s'=1}^{N}\sum_{\mathbf{q}\in\Omega_0} V_{ss'}^{(n)}(\mathbf{k},\mathbf{q})\,a_{s'}(\mathbf{k}+\mathbf{q})=0. \tag{37}$$

Shifting the argument $\mathbf{k}$ by all possible $\mathbf{k}_0\in\Gamma_1$, we obtain a system of 2^dN equations for the 2^dN quantities $a_s(\mathbf{k}+\mathbf{k}_0)$ ($s=1,2,\ldots,N$ and $\mathbf{k}_0\in\Gamma_1$). Let $\mu_\nu(\mathbf{k})$ and $\mathbf{l}_\nu(\mathbf{k})$ be the eigenvalues and eigenvectors of (37) ($\nu=1,2,\ldots,2^dN$); the $\mathbf{l}_\nu(\mathbf{k})$ are columns $\|c_{s\nu}(k+k_0)\|$ with components labeled by the indices $s=1,2,\ldots,N$ and $\mathbf{k}_0\in\Gamma_1$. We shall seek the solution of (35) in the form

$$a_s(\mathbf{k})=\sum_{\nu=1}^{2^dN} b_\nu(\mathbf{k})\,c_{s\nu}(\mathbf{k}),\quad b_\nu(\mathbf{k}+\mathbf{k}_0)=b_\nu(\mathbf{k}),\quad \mathbf{k}_0\in\Gamma_1. \tag{38}$$

Substituting this into (35) and making use of the orthogonality relations

$$\sum_{s=1}^{N}\sum_{\mathbf{k}_0\in\Gamma_1} c_{s\nu}^*(\mathbf{k}+\mathbf{k}_0)\,c_{s\nu'}(\mathbf{k}+\mathbf{k}_0)=\delta_{\nu\nu'}, \tag{39}$$

we obtain

$$[\mu_\nu(\mathbf{k})-E]\,b_\nu(\mathbf{k})+\sum_{s,s'=1}^{N}\sum_{p=0}^{\infty}\sum_{\mathbf{q}\in\Omega_{p+1}}\sum_{\nu'=1}^{2^dN}\sum_{\mathbf{k}_0\in\Gamma_1} b_{\nu'}(\mathbf{k}+\mathbf{q})\,V_{ss'}^{(n)}(\mathbf{k}+\mathbf{k}_0,\mathbf{q})\,c_{s\nu}^*(\mathbf{k}+\mathbf{k}_0)\,c_{s'\nu'}(\mathbf{k}+\mathbf{q}+\mathbf{k}_0)=0.$$

Applying formula (26) with $n=1$ to the summation over $\mathbf{q}$, making the replacement $\mathbf{k}\to\mathbf{k}/2$, and introducing the notation

$$\varepsilon_\nu^{(n+1)}(\mathbf{k})=\mu_\nu(\mathbf{k}/2), \tag{40}$$

$$V_{\nu\nu'}^{(n+1)}(\mathbf{k},\mathbf{q})=\sum_{s,s'=1}^{N}\sum_{\mathbf{a},\mathbf{b}\in\Gamma_1} V_{ss'}^{(n)}\left(\frac{\mathbf{k}}{2}+\mathbf{a},\frac{\mathbf{q}}{2}+\mathbf{b}\right)\times c_{s\nu}^*\left(\frac{\mathbf{k}}{2}+\mathbf{a}\right)c_{s'\nu'}\left(\frac{\mathbf{k}+\mathbf{q}}{2}+\mathbf{a}+\mathbf{b}\right),\quad a_\nu(\mathbf{k})=b_\nu(\mathbf{k}/2), \tag{41}$$

we obtain

$$[\varepsilon_\nu^{(n+1)}(\mathbf{k})-E]\,a_\nu(\mathbf{k})+\sum_{\nu'=1}^{2^dN}\sum_{p=0}^{\infty}\sum_{\mathbf{q}\in\Omega_p} V_{\nu\nu'}^{(n+1)}(\mathbf{k},\mathbf{q})\,a_{\nu'}(\mathbf{k}+\mathbf{q})=0.$$

Retaining in the sum over ν' the N bands closest to the energy level being studied, we obtain the Schrödinger equation of the $(n+1)$st step of the renormalization group. From (36) and (41) with the use of (25) we obtain the relation between $A_{ss'}^{(n+1)}$ and $A_{ss'}^{(n)}$:

$$A_{\nu\nu'}^{(n+1)}(2\mathbf{k},\mathbf{q})=\sum_{s,s'=1}^{N}\sum_{\mathbf{a}\in\Gamma_1} A_{ss'}^{(n)}(\mathbf{k}+\mathbf{a},\mathbf{q})\,c_{s\nu}^*(\mathbf{k}+\mathbf{a})\,c_{s'\nu'}(\mathbf{k}+\mathbf{a}+2^n\mathbf{q}). \tag{42}$$

The formulas (40) and (42) solve the problem of the determination of $A_{ss'}^{(n+1)}(\mathbf{k},\mathbf{q})$ and $\varepsilon_s^{(n)}(\mathbf{k})$.

3. It is convenient to carry Eq. (37) over into the coordinate representation. We set

$$a_s(\mathbf{k})=\sum_{\mathbf{m}\in\gamma_1} d_s^{\mathbf{m}}(\mathbf{k})\,e^{i\mathbf{k}\mathbf{m}},\quad d_s^{\mathbf{m}}(\mathbf{k}+\mathbf{k}_0)=d_s^{\mathbf{m}}(\mathbf{k}),\quad \mathbf{k}_0\in\Gamma_1.$$

For $D_s^{\mathbf{m}}(\mathbf{k})$ it is not difficult to obtain the eigenvalue equation

$$\sum_{s'=1}^{N}\sum_{\mathbf{m}'\in\gamma_1} H_{ss'}^{\mathbf{m}\mathbf{m}'}\,d_{s'}^{\mathbf{m}'}(\mathbf{k})=E d_s^{\mathbf{m}}(\mathbf{k}), \tag{43}$$

where we have introduced the notation

$$H_{ss'}^{\mathbf{m}\mathbf{m}'}=[\delta_{ss'}\varepsilon_s^{\mathbf{m}'-\mathbf{m}}(\mathbf{k})+W_{\mathbf{m}'}^{ss'}(\mathbf{k};\mathbf{m}'-\mathbf{m})]\exp[i\mathbf{k}(\mathbf{m}'-\mathbf{m})], \tag{44}$$

where

$$\varepsilon_s^{\mathbf{m}}(\mathbf{k})\equiv\frac{1}{2^d}\sum_{\mathbf{k}_0\in\Gamma_1}\varepsilon_s^{(n)}(\mathbf{k}+\mathbf{k}_0)\exp(i\mathbf{k}_0\mathbf{m}),$$

$$W_{\mathbf{n}}^{ss'}(\mathbf{k};\mathbf{m})\equiv\frac{1}{2^d}\sum_{\mathbf{q}\in\Omega_0}\sum_{\mathbf{k}_0\in\Gamma_1} V_{ss'}^{(n)}(\mathbf{k}+\mathbf{k}_0,\mathbf{q})\exp(i\mathbf{k}_0\mathbf{m}+i\mathbf{q}\mathbf{n}). \tag{45}$$

If $d_{s\nu}^{\mathbf{m}}$ is the normalized eigenvector (43), the normalized

$c_{sv}(\mathbf{k})$ are determined by the formula

$$c_{sv}(\mathbf{k})=\frac{1}{2^{d/2}}\sum_{\mathbf{m}\in\gamma_1} d_{sv}{}^{\mathbf{m}}(\mathbf{k})\,e^{i\mathbf{k}\mathbf{m}}.$$

The convenience of Eq. (43) in comparison with (37) lies in the fact that in hierarchical models with $\beta\ll 1$ the nondiagonal matrix elements of the Hamiltonian $\overline{H}$ in(43) are small in comparison with the spacings between its levels, and this makes it possible to use ordinary perturbation theory (in the Brillouin–Wigner form) to calculate its eigenvalues E_v and unnormalized eigenfunctions φ_v ($v=(\mathbf{m},s)$):

$$E_{v_0}=H_{v_0v_0}+\sum_{v'\neq v_0}\frac{H_{v_0v'}H_{v'v_0}}{E_{v_0}-H_{v'v'}}+\sum_{v',v''\neq v_0}\frac{H_{v_0v'}H_{v'v''}H_{v''v_0}}{(E_{v_0}-H_{v'v'})(E_{v_0}-H_{v''v''})}+\ldots, \tag{46}$$

$$(\varphi_v)_{v_0}=\delta_{vv_0}+\sum_{v'\neq v_0}\frac{H_{v'v_0}}{E_{v_0}-H_{v'v'}}\delta_{vv'}+\sum_{v',v''\neq v_0}\frac{H_{v'v''}H_{v''v_0}}{(E_{v_0}-H_{v'v'})(E_{v_0}-H_{v''v''})}\delta_{vv'}+\ldots.$$

We formulate the final algorithm for passing from the nth step to the $(n+1)$st step of the renormalization group, performing certain transformations and changes of notation on the way.

1) At the nth step the quantities $\varepsilon_s^{(n)}(\mathbf{k})$ and $A_{ss'}^{(n)}(\mathbf{k},\mathbf{q})$ ($s,s'=1,2,\ldots,N$) are known.

2) We use formula (36) to calculate $V_{ss'}^{(n)}(\mathbf{k},\mathbf{q})$ and formula (45) to calculate $\varepsilon_s^{\mathbf{m}}(\mathbf{k})$ and $W_{\mathbf{n}}^{ss'}(\mathbf{k};\mathbf{m})$ ($\mathbf{m},\mathbf{n}\in\gamma_1$), after which we construct the Hamiltonian $\overline{H}$ from its matrix elements:

$$\overline{H}_{ss'}^{\mathbf{m}\mathbf{m}'}=\delta_{ss'}\varepsilon_s^{\mathbf{m}'-\mathbf{m}}(\mathbf{k})+W_{\mathbf{m}}^{ss'}(\mathbf{k};\mathbf{m}'-\mathbf{m}). \tag{47}$$

3) The quantities $E_v(\mathbf{k})$ and $T_{vv'}(\mathbf{k})$ are given by the following formal series ($v=(\mathbf{m},s)$, $v'=(\mathbf{m}',s')$, etc., where $\mathbf{m},\mathbf{m}',\ldots\in\gamma_1$, $s,s',\ldots=1,2,\ldots,N$):

$$T_{v'v}=\frac{\overline{H}_{v'v}}{E_v-\overline{H}_{v'v'}}+\sum_{v''\neq v}\frac{\overline{H}_{v'v''}\overline{H}_{v''v}}{(E_v-\overline{H}_{v'v'})(E_v-\overline{H}_{v''v''})}+\ldots, \tag{48}$$

$$E_v=\overline{H}_{vv}+\sum_{v'\neq v}\frac{\overline{H}_{vv'}\overline{H}_{v'v}}{E_v-\overline{H}_{v'v'}}+\sum_{v',v''\neq v}\frac{\overline{H}_{vv'}\overline{H}_{v'v''}\overline{H}_{v''v}}{(E_v-\overline{H}_{v'v'})(E_v-\overline{H}_{v''v''})}+\ldots$$

4) We calculate the quantity

$$\bar{c}_{sv_0}(\mathbf{k})=\Big[\delta_{ss_0}+\sum_{v'\neq v_0}T_{v'v_0}\delta_{ss'}\Big]\Big(1+\sum_{v'\neq v_0}|T_{v'v_0}|^2\Big)^{-1/2}. \tag{49}$$

5) The parameters of the $(n+1)$th step are determined by the formulas

$$\varepsilon_v^{(n+1)}(\mathbf{k})=E_v(\mathbf{k}/2), \tag{50}$$

$$A_{vv'}^{(n+1)}(2\mathbf{k},\mathbf{q})=\exp(2^n i\mathbf{q}\mathbf{m}')\,\bar{Q}_\pi{}^{\mathbf{k}}\sum_{ss'=1}^{N}\exp[i\mathbf{k}(\mathbf{m}'-\mathbf{m})]\times A_{ss'}^{(n)}(\mathbf{k},\mathbf{q})\,\bar{c}_{sv}{}^{*}(\mathbf{k})\,\bar{c}_{s'v'}(\mathbf{k}+2^n\mathbf{q}). \tag{51}$$

6) From the 2^dN values of the index v we retain the N values corresponding to the bands closest to the energy value of interest.

4. THE TRUNCATED RENORMALIZATION GROUP

In order to make the origin of the correction to the index ν more transparent, in this section we shall define a truncated renormalization group, postponing the construction of the full renormalization group to Sec. 6. As is clear from Sec. 3, the renormalization-group transformation leads to the appearance, in the Schrödinger equation of the nth step, of terms that are not contained in the original equation (23); in the framework of the truncated renormalization group defined here, such terms will simply be discarded. This procedure makes it possible to obtain exact values for the indices, at least to the investigated order β^2; the point is that the additional terms that appear in the equations of the full renormalization group lead only to a renormalization of the transition point (Sec. 6). Unfortunately, we see no possibility of proving this except by constructing the full renormalization group.

We shall formulate the algorithm of the truncated renormalization group.

1) In Eq. (35) we retain one band and in all the expressions omit the index s.

2) At the nth step of the renormalization group, $\varepsilon^{(n)}(\mathbf{k})$ and $A^{(n)}(\mathbf{k},\mathbf{q})$ have the form

$$\varepsilon^{(n)}(\mathbf{k})=\sum_{i=1}^{d}2J_i^{(n)}\cos k_i,\quad A^{(n)}(\mathbf{k},\mathbf{q})=R^{(n)}(\mathbf{q}). \tag{52}$$

3) The Hamiltonian $\overline{H}$ is determined by the matrix elements ($\mathbf{m}\in\gamma_1$)

$$\overline{H}_{\mathbf{m}\mathbf{m}}=W_{\mathbf{m}}^{(n)}\equiv\sum_{\mathbf{q}\in\Omega_n}R^{(n)}(\mathbf{q})\,V(\mathbf{q})\exp(2^n i\mathbf{q}\mathbf{m}),$$

$$\overline{H}_{\mathbf{m}'\mathbf{m}}=\sum_{i=1}^{d}2J_i^{(n)}\delta_{\mathbf{m}',\mathbf{m}+\mathbf{e}_i}\cos k_i,\quad \mathbf{m}'\neq\mathbf{m}. \tag{53}$$

4) The functions $E_{\mathbf{m}}(k)$ and $T_{\mathbf{m}'\mathbf{m}}(\mathbf{k})$ are obtained from (48) by the replacements $v\to\mathbf{m}$, $v'\to\mathbf{m}'$, etc. For definiteness, from the 2^d bands that arise we shall choose the band corresponding to the site $\mathbf{m}=0$. The function $c(\mathbf{k})$ is determined by the expression

$$c(\mathbf{k})=\Big[1+\sum_{\mathbf{m}'\neq 0}T_{\mathbf{m}'0}\Big]\Big(1+\sum_{\mathbf{m}'\neq 0}|T_{\mathbf{m}'0}|^2\Big)^{-1/2}. \tag{54}$$

5) Having expanded the functions $A^{(n+1)}(\mathbf{k},\mathbf{q})$ and $\varepsilon^{(n+1)}(\mathbf{k})$ in Fourier series with respect to $\mathbf{k}$, we retain in the former case the zeroth harmonic, and in the latter case the first harmonics, obtaining the parameters of the $(n+1)$th step:

$$J_i^{(n+1)}=\frac{1}{(2\pi)^d}\int E_0(\mathbf{k}/2)\cos k_i\,d\mathbf{k},$$

$$R^{(n+1)}(\mathbf{q})=R^{(n)}(\mathbf{q})\,\frac{1}{(2\pi)^d}\int c^{*}(\mathbf{k})\,c(\mathbf{k}+2^n\mathbf{q})\,d\mathbf{k} \tag{55}$$

(the integration is over the Brillouin zone, $0\leqslant k_i\leqslant 2\pi$).

We write out the first four terms of the perturbation-theory series for $E_0(\mathbf{k})$:

$$E_0=W_0+\sum_{ij}\frac{2J_i\cos k_i 2J_j\cos k_j}{E_0-W_{e_i}}\delta_{e_i+e_j,0}+\sum_{ijl}\frac{2J_i\cos k_i 2J_j\cos k_j 2J_l\cos k_l}{(E_0-W_{e_i})(E_0-W_{e_i+e_j})}\delta_{e_i+e_j+e_l,0}+\sum_{ijlm}\frac{2J_i\cos k_i\,2J_j\cos k_j\,2J_l\cos k_l\,2J_m\cos k_m}{(E_0-W_{e_i})(E_0-W_{e_i+e_j})(E_0-W_{e_i+e_j+e_l})}\delta_{e_i+e_j+e_l+e_m,0}, \tag{56}$$

where $\mathbf{e}_i$, $\mathbf{e}_i+\mathbf{e}_j$, $\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_l\in\omega_1$, and the summation runs from 1 to d; the index n indicating the number of the renormalization-group step will be omitted in the intermediate expressions. To each nonvanishing term of the sum (56) there corresponds a path over the sides of the d-dimensional cube—a path that emerges from some vertex of the cube and returns to this vertex at the end (but not earlier). Selecting the nonvanishing terms

$$E_0=W_0+\sum_i\frac{4J_i^2\cos^2 k_i}{E_0-W_{e_i}}+\sum_i\sum_{j\neq i}\frac{4J_i^2\cos^2 k_i\,4J_j^2\cos^2 k_j}{(E_0-W_{e_i})(E_0-W_{e_i+e_j})(E_0-W_{e_j})}+\sum_i\sum_{j\neq i}\frac{4J_i^2\cos^2 k_i\,4J_j^2\cos^2 k_j}{(E_0-W_{e_i})^2(E_0-W_{e_i+e_j})}$$

and eliminating E_0 iteratively from the right-hand side, we obtain

$$E_0(\mathbf{k})=W_0+\sum_i\frac{4J_i^2\cos^2 k_i}{W_0-W_{e_i}}-\sum_i\frac{16J_i^4\cos^4 k_i}{(W_0-W_{e_i})^3}+\sum_i\sum_{j\neq i}8J_i^2J_j^2\cos^2 k_i\cos^2 k_j\times\frac{(2W_0-W_{e_i}-W_{e_j})(W_0+W_{e_i+e_j}-W_{e_i}-W_{e_j})}{(W_0-W_{e_i})^2(W_0-W_{e_j})^2(W_0-W_{e_i+e_j})}. \tag{57}$$

The sums appearing in (54) will be needed in second order:

$$\sum_{m\neq 0}T_{m0}=\sum_i\frac{2J_i\cos k_i}{W_0-W_{e_i}}+\sum_i\sum_{j\neq i}\frac{2J_i\cos k_i\,2J_j\cos k_j}{(W_0-W_{e_i})(W_0-W_{e_i+e_j})},\qquad \sum_{m\neq 0}|T_{m0}|^2=\sum_i\frac{4J_i^2\cos^2 k_i}{(W_0-W_{e_i})^2}. \tag{58}$$

Substituting (58) into (54), and (54) and (57) into (55), and introducing the notation

$$g_i^{(n)}=|J_i^{(n)}/(W_0^{(n)}-W_{e_i}^{(n)})|, \tag{59}$$

we obtain the system of renormalization-group equations in the form

$$g_i^{(n+1)}=\left|\frac{W_0^{(n)}-W_{e_i}^{(n)}}{W_0^{(n+1)}-W_{e_i}^{(n+1)}}\right|\left(g_i^{(n)3}-4g_i^{(n)4}+\sum_{j\neq i}2Q_{ij}^{(n)}g_i^{(n)2}g_j^{(n)2}\right), \tag{60a}$$

$$R^{(n+1)}(\mathbf{q})=R^{(n)}(\mathbf{q})\left[1-\sum_i 2g_i^{(n)2}(1-\cos 2^n q_i)\right], \tag{60b}$$

$$W_m^{(n)}=\sum_{\mathbf{q}\in\Omega_n}R^{(n)}(\mathbf{q})V(\mathbf{q})\exp(2^n i\mathbf{q}\mathbf{m}), \tag{60c}$$

where we have introduced the notation

$$Q_{ij}^{(n)}=(2W_0^{(n)}-W_{e_i}^{(n)}-W_{e_j}^{(n)})(W_0^{(n)}+W_{e_i+e_j}^{(n)}-W_{e_i}^{(n)}-W_{e_j}^{(n)})[(W_0^{(n)}-W_{e_i}^{(n)})(W_0^{(n)}-W_{e_i+e_j}^{(n)})]^{-1}.$$

5. INVESTIGATION OF THE RENORMALIZATION-GROUP EQUATIONS

1. In the zeroth approximation, Eqs. (60) take the form

$$g_i^{(n+1)}=\left|\frac{Z_{e_i}^{(n)}}{Z_{e_i}^{(n+1)}}\right|g_i^{(n)2},\qquad Z_{e_i}^{(n)}=\sum_{\mathbf{q}\in\Omega_n}V(\mathbf{q})[1-\exp(2^n i\mathbf{q}\mathbf{e}_i)] \tag{61}$$

($R^{(n)}(\mathbf{q})=1$ in view of the initial condition $R^{(0)}(\mathbf{q})=1$). These equations can be analyzed without any assumptions about the properties of the potential V, except for the strong-hierarchy assumption (27). The rigorous meaning of the latter is as follows: There exists a number $\tilde\beta\ll 1$ ($\tilde\beta\sim\beta$), such that after the scale transformation

$$Z_{e_i}^{(n)}\to Z_{e_i}^{(n)}\tilde\beta^n,\qquad J_i^{(n)}\to J_i^{(n)}\tilde\beta^n \tag{62}$$

all the quantities $Z_{e_i}^{(n)}$ are of the same order, with typical value W. As a result of the transformation (62) the first of Eqs. (61) takes the form

$$g_i^{(n+1)}=\tilde\beta^{-1}|Z_{e_i}^{(n)}/Z_{e_i}^{(n+1)}|g_i^{(n)2}. \tag{63}$$

Its solution is

$$g_i^{(n)}=\tilde\beta\left[\frac{|Z_{e_i}^{(0)}|^{1/8}}{|Z_{e_i}^{(1)}|^{1/4}|Z_{e_i}^{(2)}|^{1/8}\ldots|Z_{e_i}^{(n-1)}|^{1/2^n}|Z_{e_i}^{(n)}|^{1/2^n}}\frac{g_i^{(0)}}{\tilde\beta}\right]^{2^n}=\left(\frac{J_i^{(0)}}{\tilde\beta W_i}\right)^{2^n}\tilde\beta\frac{|Z_{e_i}^{(n+1)}|^{1/4}|Z_{e_i}^{(n+2)}|^{1/8}\ldots}{|Z_{e_i}^{(n)}|^{1/2}}, \tag{64}$$

where we have introduced the notation

$$W_i\equiv\prod_{n=0}^{\infty}|Z_{e_i}^{(n)}|^{1/2^{n+1}}. \tag{65}$$

Since all the $Z_{e_i}^{(n)}\sim W$, we have $W_i\sim W$. The condition

$$J_i^{(0)}=\tilde\beta W_i \tag{66}$$

determines the transition point, and when it is fulfilled each term of the sequence

$$g_i^{c(n)}=\tilde\beta\frac{X_i^{(n)1/2}}{|Z_{e_i}^{(n)}|^{1/2}},\qquad X_i^{(n)}\equiv|Z_{e_i}^{(n+1)}|^{1/2}|Z_{e_i}^{(n+2)}|^{1/4}\ldots \tag{67}$$

is a quantity of the order of $\tilde\beta$. Linearizing (53) about $g_i^{c(n)}$,

$$\delta g_i^{(n+1)}=2\tilde\beta^{-1}|Z_{e_i}^{(n)}/Z_{e_i}^{(n+1)}|g_i^{c(n)}\delta g_i^{(n)},$$

and making the substitution $\delta g_i^{(n)}=g_i^{c(n)}x_i^{(n)}$, we obtain

$$x_i^{(n+1)}=2x_i^{(n)}. \tag{68}$$

Since all the $g_i^{c(n)}$ are of the same order, the rate of growth of $x_i^{(n)}$ coincides with the rate of growth of $\delta g_i^{(n)}$; thus, (68) is a generalization of (17). From (68) the result (19) for the index ν follows in an obvious way.

In random hierarchical models the quantities W_i do not have determinate values. Each of them has approximately the same distribution as $W_m^{(n)}$, and so the localization edges with respect to the constants $J_i^{(0)}$ are random. In particular, they are different for two close energy values. Therefore, for a fixed realization of the potential V_m and fixed $J_i^{(0)}$, localized and delocalized states are intermingled in the spectrum of Eq. (23) in a random manner (an analogous situation

obtains in incommensurate systems[19]). This situation is unstable against the switching on of an arbitrarily small perturbation of general form,[4] since the latter will lead to mixing of localized states with delocalized states and to the destruction of the localization. This is connected with the trivial circumstance that a perturbation of general form destroys the postulated hierarchy.

The mobility edges are determined by d conditions of the form (66). In a physical formulation of the problem the parameters of the model are subjected to additional conditions, ensuring the simultaneous fulfillment of the conditions (66) for all i upon variation of a certain external parameter (such as the impurity concentration or Fermi level). For $\beta \sim 1$ the analogs of the quantities W_i should have determinate values, and these additional conditions are ensured by the cubic symmetry: $J_1^{(0)} = J_2^{(0)} = \ldots = J_d^{(0)}$, and $W_1 = W_2 = \ldots = W_d$. For $\beta \ll 1$ these conditions should have a more complicated and artificial character.

2. We shall calculate the correction to the zeroth-order value of the index ν for the Gaussian model (28). We introduce the quantity

$$D_n(\mathbf{q}) = \sum_{\mathbf{a}\in\Gamma_n} \left| R^{(n)}\left(\frac{\mathbf{q}}{2^n} + \mathbf{a}\right)\right|^2. \tag{69}$$

With the aid of (60b) and (25), for this quantity it is easy to obtain the recursion relation

$$D_{n+1}(\mathbf{q}) = \sum_{\mathbf{b}\in\Gamma_1} D_n\left(\frac{\mathbf{q}}{2} + \mathbf{b}\right) \times\left\{1 - 4\sum_i g_i^{(n)2}\left[1 - \cos\left(\frac{\mathbf{q}_i}{2} + \mathbf{b}_i\right)\right]\right\}. \tag{69a}$$

In view of the initial condition $D_0(\mathbf{q}) = 1$ it is found that the quantity $D_n(\mathbf{q})$ is independent of $\mathbf{q}$ and satisfies the recursion relations

$$D_{n+1} = D_n 2^d\left(1 - 4\sum_i g_i^{(n)2}\right). \tag{70}$$

The variances of the quantities $W_{\mathbf{m}}^{(n)}$ appearing in (60) are expressed in terms of the quantity D_n: $\langle|W_{\mathbf{m}}^{(n)}|^2\rangle = (2^d - 1)(\beta^n V)^2 D_n$ [in the derivation one uses (25)].

We introduce the quantity

$$U_{\mathbf{m}}^{(n)}(\mathbf{q}) = \sum_{\mathbf{a}\in\Gamma_n} R^{(n)}\left(\frac{\mathbf{q}}{2^n} + \mathbf{a}\right) V\left(\frac{\mathbf{q}}{2^n} + \mathbf{a}\right)\exp[i\mathbf{m}(\mathbf{q} + 2^n\mathbf{a})], \tag{71}$$

where $\mathbf{m}$ is an arbitrary vector. For $\mathbf{m} = 0$ the quantity $U_{\mathbf{m}}^{(n)}(\mathbf{q})$ coincides with the potential $V^{(n)}(\mathbf{q})$ that appears in the Schrödinger equation (35) of the nth step of the truncated renormalization group; the quantities $U_{\mathbf{m}}^{(n)}(\mathbf{q})$ with $\mathbf{m} \neq 0$ are required in Sec. 6. The variances of $U_{\mathbf{m}}^{(n)}(\mathbf{q})$ are also expressed in terms of D_n:

$$\langle|U_{\mathbf{m}}^{(n)}(\mathbf{q})|^2\rangle = (\beta^{n+p}V)^2 D_n, \quad \mathbf{q}\in\Omega_p.$$

Performing on the energy a scale transformation that equalizes the variances of the quantities $W_{\mathbf{m}}^{(n)}$ and $U_{\mathbf{m}}^{(n)}(\mathbf{q})$ with different n:

$$W_{\mathbf{m}}^{(n)} \to W_{\mathbf{m}}^{(n)}\beta^n D_n^{1/2}, \quad J_i^{(n)} \to J_i^{(n)}\beta^n D_n^{1/2}, \quad U_{\mathbf{m}}^{(n)}(\mathbf{q}) \to U_{\mathbf{m}}^{(n)}(\mathbf{q})\beta^n D_n^{1/2}, \tag{72}$$

we obtain (60a) in the form

$$g_i^{(n+1)} = \frac{1}{\tilde\beta}\left|\frac{Z_{\mathbf{e}_i}^{(n)}}{Z_{\mathbf{e}_i}^{(n+1)}}\right|\left(g_i^{(n)2} - 2g_i^{(n)4} + 2g_i^{(n)2}\sum_{j\neq i}\varkappa_{ij}^{(n)} g_j^{(n)2}\right), \tag{73}$$

where

$$\varkappa_{ij}^{(n)} = 1 + \frac{(Z_{\mathbf{e}_i}^{(n)} + Z_{\mathbf{e}_j}^{(n)})(Z_{\mathbf{e}_i}^{(n)} + Z_{\mathbf{e}_j}^{(n)} - Z_{\mathbf{e}_i+\mathbf{e}_j}^{(n)})}{Z_{\mathbf{e}_i}^{(n)} Z_{\mathbf{e}_i+\mathbf{e}_j}^{(n)}},$$
$$Z_{\mathbf{m}}^{(n)} \equiv W_0^{(n)} - W_{\mathbf{m}}^{(n)}, \quad \tilde\beta = 2^{d/2}\beta. \tag{74}$$

Linearizing Eq. (73) about the transition point, $g_i^{(n)} = g_i^{c(n)} + \delta g_i^{(n)}$, and making the substitution $\delta g_i^{(n)} = g_i^{c(n)} x_i^{(n)}$, we obtain the following equation, which refines (68):

$$x_i^{(n+1)} = (2 - 4g_i^{c(n)2})x_i^{(n)} + \sum_{j\neq i} 4g_j^{c(n)2}\varkappa_{ij}^{(n)} x_j^{(n)}. \tag{75}$$

(After the scale transformation (72), to the necessary accuracy the quantities $Z_{\mathbf{e}_i}^{(n)}$ can be regarded as independent of $g_i^{(n)}$.) It is sufficient to know the sequence $g_i^{(n)}$ in the zeroth approximation (67); in view of the fact that $Z_{\mathbf{e}_i}^{(n)}$ with different n are independent, the distribution of $g_i^{c(n)}$ is stationary. It is calculated from (67) using the rules of probability theory:

$$P(g) = \int_{-\infty}^{\infty} dZ_{\mathbf{e}_i}\int_0^{\infty} dX_i P_Z(Z_{\mathbf{e}_i}) P_X(X_i)\,\delta\left(g - \tilde\beta\frac{X_i^{1/2}}{|Z_{\mathbf{e}_i}|^{1/2}}\right) = \frac{4\tilde\beta^2}{g^3}\int_0^{\infty} dX_i X_i P_X(X_i) P_Z\left(\frac{\tilde\beta^2 X_i}{g^2}\right).$$

From this it is easy to find the asymptotic forms (see Fig. 3)

$$P(g) = \begin{cases} (4\tilde\beta^2/g^3)P_Z(0)\overline{X} \sim \tilde\beta^2/g^3, & g \gg \tilde\beta, \\ (2g/\tilde\beta^2)P_X(0)\,\overline{|Z|} \sim g/\tilde\beta^2, & g \ll \tilde\beta. \end{cases} \tag{76}$$

3. The analysis of Eq. (75) is complicated by the mutual dependence of the $g_i^{c(n)}$ with different n and by the divergence of the averages $\overline{(g_i^c)^2}$ and $\overline{\varkappa_{ij}(g_j^c)^2}$. We make the substitution $x_i^{(n)} = 2^n y_i^{(n)}$ and write (75) in vector form:

$$\mathbf{y}^{(n+1)} = (1 + \hat A_n)\mathbf{y}^{(n)}, \tag{77}$$

where $\hat{\mathbf{A}}_n$ is a random matrix. Equations of this form are studied in the theory of one-dimensional localization[24,25]; they are characterized by an exponential-growth exponent describing the increase of $\mathbf{y}^{(n)}$ for initial conditions of general form. This is the exponent of interest for us, since it characterizes the rate at which the system moves away from the stationary point.

To eliminate the mutual dependence of the matrices $\hat A_n$ we shall integrate Eq. (77) N times ($1 \ll N \ll 1/\tilde\beta^2$):

$$\mathbf{y}^{(n+N)} = (1 + \hat B_n)\mathbf{y}^{(n)}, \quad \hat B_n \equiv \sum_{m=0}^{N-1}\hat A_{n+m}.$$

The correlations between $X_i^{(n)}$ and $Z_{\mathbf{e}_i}^{(n)}$ with different n fall off exponentially:

$$\overline{X_i^{(n)}X_j^{(n+l)}} - \overline{X_i^{(n)}}\,\overline{X_j^{(n+l)}} \propto 1/2^l,$$
$$\overline{X_i^{(n)}Z_{\mathbf{e}_j}^{(n+l)}} - \overline{X_i^{(n)}}\,\overline{Z_{\mathbf{e}_j}^{(n+l)}} \propto 1/2^l,$$

and for large N the matrices $\hat{B}_n$ become independent. Introducing the distribution function $P_n\{y_i\}$ of the components of $\mathbf{y}^{(n)}$ and the distribution function $P_B\{B_{ij}\}$ of the elements of the matrices $\hat{B}_n$, we obtain from (75)

$$P_{n+N}\{y_i\} = \int\dots\int\Big[\prod_{i=1}^{d}\delta(y_i-\tilde{y}_i-B_{ij}\tilde{y}_j)\Big]P_B\{B_{ij}\}P_n\{\tilde{y}_i\} \times\Big[\prod_{i=1}^{d}d\tilde{y}_i\Big]\Big[\prod_{ij=1}^{d}dB_{ij}\Big]. \quad (78)$$

In view of the divergence of the averages $\overline{B}_{ij}$, in place of $P_B\{B_{ij}\}$ it is convenient to use the characteristic function

$$\chi_B\{\theta_{ij}\} = \int\dots\int P_B\{B_{ij}\}\exp(iB_{ij}\theta_{ij})\prod_{ij}dB_{ij}. \quad (79)$$

In lowest order in $\tilde{\beta}$ the characteristic function has the form (see below)

$$\chi_B\{\theta_{ij}\} = 1+2iNg^2\Big[-\sum_i\theta_{ii}+\varkappa\sum_i\sum_{j\neq i}\theta_{ij}\Big]+O(\tilde{\beta}^2), \quad (80)$$

where

$$g^2=\tilde{\beta}^2\ln(1/\tilde{\beta}), \quad \varkappa={}^1/_3, \quad (81)$$

which can be rewritten in the form

$$\chi_B\{\theta_{ij}\}=\exp\{i\overline{B}_{ij}\theta_{ij}\}+O(\tilde{\beta}^2),$$
$$\overline{B}_{ii}=-2Ng^2, \quad \overline{B}_{ij}=2N\varkappa g^2, \quad i\neq j.$$

Substituting into (78) and (79), we obtain

$$P_{n+N}\{y_i\}=P_n\{y_i-\overline{B}_{ij}y_j\}.$$

The same result is obtained if in (75) we replace $g_i^{c(n)}$ and $\varkappa_{ij}^{(n)}$ by constant values:

$$x_i^{(n+1)}=2(1-2g^2)x_i^{(n)}+\sum_{j\neq i}4\varkappa g^2x_j^{(n)}. \quad (82)$$

Putting $x_i^{(n)}=x_i\lambda^n$, we obtain an eigenvalue equation, which can be solved by Fourier transformation with respect to the variable i. The largest eigenvalue

$$\lambda_{max}=2-4g^2+4\varkappa g^2(d-1)$$

determines the desired growth exponent of Eq. (75). The critical index ν is expressed in terms of $\lambda_{\max}$ as follows:

$$\nu=\frac{\ln 2}{\ln\lambda_{max}}\approx 1-\frac{2\varkappa g^2}{\ln 2}\Big(d-1-\frac{1}{\varkappa}\Big). \quad (83)$$

Substituting (81), we obtain the final expression (21). We note that to refine the formula (21), i.e., to calculate the term $O(\beta^2)$, the expansion (75) is no longer sufficient.

4. It remains to obtain an expression for the characteristic function (80). By the rules of probability theory,

$$\chi_B\{\theta_{ij}\}=\int\dots\int\exp\Big\{2i\tilde{\beta}^2\sum_i\sum_{m=1}^{N}\frac{X_i^{(m)}}{|Z_{\mathbf{e}_i}^{(m)}|} \times\Big(-\theta_{ii}+\sum_{j\neq i}\varkappa_{ji}^{(m)}\theta_{ji}\Big)\Big\}P\{W_1^{(m)}\}\prod_{n=1}^{\infty}\prod_{1\in\gamma_1}dW_1^{(n)}.$$

Expanding the exponential in a series, we obtain a sum of integrals, of which some converge, giving a contribution of order $\tilde{\beta}^2$, and the others diverge logarithmically. The latter can be calculated with logarithmic accuracy by cutting off the divergences at the limit of applicability of the expansion. The calculations can be carried out more rigorously if we transform the exponential using the scheme

$$e^{ix+iy}=1+(e^{ix}-1)+(e^{iy}-1)+(e^{ix}-1)(e^{iy}-1)$$

and estimate the resulting integrals. As a result, we obtain

$$\chi_B\{\theta_{ij}\}=1+4i\tilde{\beta}^2\ln(1/\tilde{\beta})\sum_{n=1}^{N}\sum_i\int\dots\int X_i^{(n)}\Big(-2\theta_{ii} +\sum_{j\neq i}2\theta_{ji}\frac{Z_{\mathbf{e}_j}^{(n)}}{Z_{\mathbf{e}_i+\mathbf{e}_j}^{(n)}}\Big)\delta(Z_{\mathbf{e}_i}^{(n)})P\{W_0^{(n)};Z_{\mathbf{m}}^{(n)}\} \times\prod_{n=1}^{\infty}\Big(dW_0^{(n)}\prod_{\mathbf{m}\in\omega_1}dZ_{\mathbf{m}}^{(n)}\Big)+O(\tilde{\beta}^2).$$

Integrating over all the variables except $Z_{\mathbf{e}_i}$, $Z_{\mathbf{e}_j}$, and $Z_{\mathbf{e}_i+\mathbf{e}_j}$, we obtain

$$\chi_B\{\theta_{ij}\}=1+4i\tilde{\beta}^2\ln(1/\tilde{\beta})N\overline{X} \times\sum_i\sum_{j\neq i}\int dZ_{\mathbf{e}_i}dZ_{\mathbf{e}_j}dZ_{\mathbf{e}_i+\mathbf{e}_j}P(Z_{\mathbf{e}_i},Z_{\mathbf{e}_j},Z_{\mathbf{e}_i+\mathbf{e}_j}) \times\Big(-2\theta_{ii}+2\theta_{ji}\frac{Z_{\mathbf{e}_j}}{Z_{\mathbf{e}_i+\mathbf{e}_j}}\Big)\delta(Z_{\mathbf{e}_i})+O(\tilde{\beta}^2). \quad (84)$$

The joint distribution of $Z_{\mathbf{e}_i}$, $Z_{\mathbf{e}_j}$, and $Z_{\mathbf{e}_i+\mathbf{e}_j}$, is Gaussian and is fully determined by the second moments:

$$\overline{Z_{\mathbf{e}_i}^2}=\overline{Z_{\mathbf{e}_j}^2}=\overline{Z_{\mathbf{e}_i+\mathbf{e}_j}^2}=2^{d+1}V^2,$$
$$\overline{Z_{\mathbf{e}_i}Z_{\mathbf{e}_j}}=\overline{Z_{\mathbf{e}_i}Z_{\mathbf{e}_i+\mathbf{e}_j}}=\overline{Z_{\mathbf{e}_j}Z_{\mathbf{e}_i+\mathbf{e}_j}}=2^dV^2.$$

They are calculated using the definition (60c) of $W_{\mathbf{m}}^{(n)}$, with allowance for the scale transformation (72) that was performed. Hence,

$$P(Z_{\mathbf{e}_i},Z_{\mathbf{e}_j},Z_{\mathbf{e}_i+\mathbf{e}_j}) =\frac{1}{2(2\pi)^{3/2}}\exp\Big\{-\frac{1}{8}[3(Z_{\mathbf{e}_i}^2+Z_{\mathbf{e}_j}^2+Z_{\mathbf{e}_i+\mathbf{e}_j}^2) -2(Z_{\mathbf{e}_i}Z_{\mathbf{e}_j}+Z_{\mathbf{e}_i}Z_{\mathbf{e}_i+\mathbf{e}_j}+Z_{\mathbf{e}_j}Z_{\mathbf{e}_i+\mathbf{e}_j})]\Big\}.$$

The choice of scale has no significance if $\overline{X}$ is calculated with the aid of the same distribution. As a result, we obtain

$$\chi_B\{\theta_{ij}\}=1+4i\tilde{\beta}^2\ln\Big(\frac{1}{\tilde{\beta}}\Big)\frac{N\overline{X}}{2\pi^{1/2}}\sum_i\Big(-2\theta_{ii}+\frac{2}{3}\sum_{j\neq i}\theta_{ij}\Big) +O(\tilde{\beta}^2).$$

Calculating $\overline{X}$,

$$\overline{X}=\prod_{n=1}^{\infty}\frac{1}{2\pi^{1/2}}\int_{-\infty}^{\infty}|Z|^{1/2^n}e^{-Z^2/4}dZ=2\sqrt{\pi}\prod_{n=1}^{\infty}\frac{\Gamma(1/2+1/2^n)}{\Gamma({}^1/_2)} =\frac{\pi^{1/2}}{2}$$

(we use properties of the gamma function—see Ref. 26), we obtain (80).

6. THE FULL RENORMALIZATION GROUP

The only purpose of this section is to substantiate the truncated renormalization group used in Sec. 4; therefore, this section can be omitted without detriment to the understanding of the main text.

1. We must realize in its complete form the algorithm formulated in Sec. 3. To derive the renormalization-group equations to order β^2 it is sufficient to confine ourselves to the 2^d-band approximation; therefore, in the formulas of Sec. 3 we set $N = 2^d$.

We shall explain the scheme of manipulations with the band index s. At the nth step of the renormalization group the index s labels 2^d bands; as a result of allowance for further Fourier components of the potential, each band is split into 2^d subbands, labeled by the index $\mathbf{m}$; the complete labeling is implemented by the index $\nu = (\mathbf{m}, s)$. After this we select one of the initial bands (for definiteness, the band with $s = 0$), and designate its subbands to be bands of the $(n+1)th$ step of the renormalization group. Thus, the index s coincides with the index $\mathbf{m}$ of the preceding step, and for the following it is convenient to regard s as a vector index.

We shall consider an expression of the form

$$a\exp(im_1k_1+im_2k_2+\ldots+im_dk_d), \quad a\sim\beta^n, \quad m=\sum_i|m_i| \tag{85}$$

and call it extremal when $n = m$, nonextremal when $n > m$, and superextremal when $n < m$. Introducing the energy scale $W^{(n)} \sim W^{00(n)}(0;0)$, we consider the quantities

$$[\varepsilon_s^{(n)}(\mathbf{k})-\langle\varepsilon_s^{(n)}\rangle_0]/W^{(n)}, \quad A_{ss'}^{(n)}(\mathbf{k},\mathbf{q})/R^{(n)}(\mathbf{q}), \tag{86}$$

where $R^{(n)}(\mathbf{q}) = \langle A_{00}^{(n)}(\mathbf{k},\mathbf{q})\rangle_0$, and the symbol $\langle\ldots\rangle_0$ denotes the zeroth Fourier component with respect to $\mathbf{k}$. We shall prove the following lemma: In the Fourier series of the quantities (86) there are no superextremal terms. The proof is performed by induction on n. For $n = 0$,

$$A_{ss'}^{(0)}=\delta_{s0}\delta_{s'0}, \quad \varepsilon^{(0)}(\mathbf{k})=\sum_i 2J_i\cos k_i, \quad W^{(0)}\sim V$$

and the assertion is obvious. Assuming that the assertion is true for some n, we shall prove it for $n + 1$. It follows from the induction assumption and formulas (45), (36), and (47) that superextremal terms are absent in the Fourier series of the quantities

$$\overline{H}_{\nu\nu'}(\mathbf{k})/W^{(n)} \quad (\nu\neq\nu'), \quad [\overline{H}_{\nu\nu}(\mathbf{k})-\langle\overline{H}_{\nu\nu}\rangle_0]/W^{(n)}.$$

The series for $T_{\nu\nu'}$ (48), after iterative elimination of E_ν from the right-hand side, consists of terms of the form

$$\frac{\overline{H}_{\nu\nu'}(\mathbf{k})}{\overline{H}_{\nu\nu}(\mathbf{k})-\overline{H}_{\nu'\nu'}(\mathbf{k})}\frac{\overline{H}_{\nu'\nu''}(\mathbf{k})}{\overline{H}_{\nu\nu}(\mathbf{k})-\overline{H}_{\nu''\nu''}(\mathbf{k})}\cdots$$

Transforming the denominators in accordance with the scheme

$$\overline{H}_{\nu\nu}(\mathbf{k})-\overline{H}_{\nu'\nu'}(\mathbf{k})=[\langle\overline{H}_{\nu\nu}\rangle_0-\langle\overline{H}_{\nu'\nu'}\rangle_0]\Big\{1+\frac{W^{(n)}}{\langle\overline{H}_{\nu\nu}\rangle_0-\langle\overline{H}_{\nu'\nu'}\rangle_0}\times\frac{[\overline{H}_{\nu\nu}(\mathbf{k})-\langle\overline{H}_{\nu\nu}\rangle_0]-[\overline{H}_{\nu'\nu'}(\mathbf{k})-\langle\overline{H}_{\nu'\nu'}\rangle_0]}{W^{(n)}}\Big\},$$

we find that superextremal terms are absent in $T_{\nu\nu'}(\mathbf{k})$, and hence also in $\overline{c}_{s\nu}(\mathbf{k})$. Dividing (51) by $R^{(n)}(\mathbf{q})$, we find that under the operator $\hat{Q}_\pi^{\mathbf{k}}$ is a sum of terms of the form

$$a\exp[i(m_1+\alpha_1)k_1+i(m_2+\alpha_2)k_2+\ldots+i(m_d+\alpha_d)k_d],$$

$$m\leqslant n, \quad \alpha_i=0,\pm1,$$

since $\mathbf{m}', \mathbf{m}\in\gamma_1$. The operator $\hat{Q}_\pi^{\mathbf{k}}$ eliminates the odd $m_i+\alpha_i$, and the scale transformation in $\mathbf{k}$ halves them. As a result $A_{ss'}^{(n+1)}(\mathbf{k},\mathbf{q})/R^{(n)}(\mathbf{q})$ consists of terms of the form (85) with m_i' in place of m_i, where

$$m_i'=(m_i+\alpha_i)/2, \quad \alpha_i=0,\pm1, \quad m_i+\alpha_i- \text{ is even}. \tag{87}$$

It is obvious that $|m_i'|<|m_i|$, and therefore $m'\leqslant m$. Taking into account that $R^{(n+1)}(\mathbf{q})\sim R^{(n)}(\mathbf{q})$, we find that there are no superextremal terms in $A_{ss'}^{(n+1)}(\mathbf{k},\mathbf{q})/R^{(n+1)}(\mathbf{q})$.

Analogously, the quantity $(E_\nu(\mathbf{k})-\langle\overline{H}_{\nu\nu}\rangle_0)/W^{(n)}$ consists of terms of the type (85) with $m\leqslant n$; here all m_i are even in view of (31). By virtue of (50), $\varepsilon_\nu^{(n+1)}(\mathbf{k})$ is a sum of terms of the form of (85) with m_i' in place of m_i, where $m_i'=m_i/2$, and a certain constant term. If $m\neq0$, then $m'\neq m-1$ and the quantity $(\varepsilon_\nu^{(n+1)}(\mathbf{k})-\langle\varepsilon_\nu\rangle_0^{(n+1)})/W^{(n+1)}$, where $W^{(n+1)}\sim\beta W^{(n)}$, does not contain superextremal terms. The lemma is proved.

The lemma shows that $\varepsilon_s^{(n)}(\mathbf{k})$ and $A_{ss'}^{(n)}(\mathbf{k},\mathbf{q})$ can be expanded in Fourier series with respect to $\mathbf{k}$ with coefficients that decrease with increase of the parameter β. For our purposes the first few terms will be sufficient:

$$\varepsilon_s^{(n)}(\mathbf{k})=\text{const}+\sum_i 2J_i^{(n)}\cos k_i+\sum_i 2J_{ii}^{(n)}\cos 2k_i+\sum_i\sum_{j\neq i}2J_{ij}^{(n)}\cos k_i\cos k_j+O(\beta^5),$$

$$A_{ss}^{(n)}(\mathbf{k},\mathbf{q})=aR^{(n)}(\mathbf{q})\Big[1+\sum_i\alpha_i\cos k_i+\sum_i\rho_i\sin k_i+\sum_i\gamma_i\cos 2k_i+\sum_i\zeta_i\sin 2k_i+\sum_{i,j\neq i}\varphi_{ij}\cos k_i\sin k_i+\sum_{i,j\neq i}\delta_{ij}\cos k_i\cos k_j+\sum_{i,j\neq i}\sigma_{ij}\sin k_i\sin k_j+O(\beta^6)\Big], \tag{88}$$

$$A_{s+\mathbf{e}_i,s}^{(n)}(\mathbf{k},\mathbf{q})=R^{(n)}(\mathbf{q})[\xi_i+\eta_i\exp(i\mathbf{k}\mathbf{e}_i)+O(\beta^3)],$$

$$A_{ss'}^{(n)}(\mathbf{k},\mathbf{q})\sim\beta^2R^{(n)}(\mathbf{q}), \quad \mathbf{s}'-\mathbf{s}\neq0,\mathbf{e}_i,$$

$$J_i^{(n)}/W^{(n)}\sim\beta, \quad J_{ii}^{(n)}/W^{(n)}\sim J_{ij}^{(n)}/W^{(n)}\sim\beta^3, \quad a\sim1,$$

$$\alpha_i\sim\rho_i\sim\beta^2, \quad \gamma_i\sim\zeta_i\sim\delta_{ij}\sim\varphi_{ij}\sim\sigma_{ij}\sim\beta^4, \quad \xi_i\sim\eta_i\sim\beta.$$

The form of the expansions (88) is a consequence of the form of the right-hand sides of (50) and (51) that were established in the proof of the lemma. From (88) there follow estimates for the matrix elements of $\overline{H}$:

$$W_{\mathbf{m}}^{ss}(\mathbf{k};0)-W_{\mathbf{m}'}^{ss}(\mathbf{k};0)\sim W, \quad \mathbf{m}\neq\mathbf{m}',$$

$$W_{\mathbf{m}}^{ss}(\mathbf{k};0)-W_{\mathbf{m}}^{ss}(0;0)\sim\beta^4W$$

$$W_{\mathbf{m}}^{ss}(\mathbf{k};\mathbf{l})\sim\begin{cases}\beta^2W, & \mathbf{l}=\mathbf{e}_i\\ \beta^4W, & \mathbf{l}=0,\mathbf{e}_i\end{cases},$$

$$W_{\mathbf{m}}^{ss'}(\mathbf{k};\mathbf{l})\sim\begin{cases}\beta W, & \mathbf{l}=0,\mathbf{e}_i, \quad \mathbf{s}'-\mathbf{s}=\mathbf{e}_i\\ \beta^3W, & \mathbf{l}\neq0,\mathbf{e}_i, \quad \mathbf{s}'-\mathbf{s}=\mathbf{e}_i,\\ \beta^2W, & \mathbf{s}'-\mathbf{s}\neq0,\mathbf{e}_i\end{cases} \tag{89}$$

$$\varepsilon_s{}^0(\mathbf{k})-\varepsilon_{s'}{}^0(\mathbf{k})\sim W/\beta, \quad \mathbf{s}\neq\mathbf{s}', \quad \varepsilon_s{}^{\mathbf{m}}(\mathbf{k})\sim\begin{cases}\beta W, & \mathbf{m}=\mathbf{e}_i\\ \beta^3W, & \mathbf{m}\neq0,\mathbf{e}_i\end{cases}.$$
$$\varepsilon_s{}^0(\mathbf{k})-\varepsilon_s{}^0(0)\sim\beta^3W,$$

2. We shall need expressions for $A_{00}^{(n)}(\mathbf{k},\mathbf{q})$ and $A_{\mathbf{s}+\mathbf{e}_i,\mathbf{s}}^{(n)}(\mathbf{k},\mathbf{q})$ in which α_i and ρ_i are calculated to order $\sim\beta^3$, the quantities γ_i and ζ_i are calculated to order $\sim\beta^4$, and the quantities ξ_i and η_i are calculated to order $\sim\beta$; the coefficients φ_{ij}, δ_{ij}, and σ_{ij} are not needed at all. These expressions are obtained by iterative solution of Eq. (51)[3]:

$$
\begin{aligned}
A_{00}^{(n)}(\mathbf{k},\mathbf{q}) &= R^{(n)}(\mathbf{q})\Big\{1-\sum_i g_i^{(n-1)2}\,[\cos k_i+\cos(k_i+2^n q_i)\\
&-2\cos(k_i+2^{n-1}q_i)]\\
&-\sum_i g_i^{(n-1)}\,g_i^{(n-2)2}\,[\cos k_i(1-2\cos(2^{n-2}q_i)\\
&+2\cos(2^{n-1}q_i)-2\cos(3\cdot 2^{n-2}q_i)\\
&+\cos(2^n q_i))+\sin k_i(2\sin(2^{n-2}q_i)\\
&-2\sin(2^{n-1}q_i)+2\sin(3\cdot 2^{n-2}q_i)\\
&-\sin(2^n q_i))-S_i^{(n)}[\cos k_i+\cos(k_i+2^n q_i)\\
&-2\cos(k_i+2^{n-1}q_i)]]\\
&+\sum_i 2g_i^{(n-1)4}\Big[\frac{11}{8}\cos 2k_i-\frac{3}{2}\cos(2k_i+2^{n-1}q_i)\\
&+\frac{1}{4}\cos(2k_i+2^n q_i)-\frac{3}{2}\cos(2k_i+3\cdot 2^{n-1}q_i)\\
&+\frac{11}{8}\cos(2k_i+2^{n+1}q_i)\Big]\Big\},
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
A_{\mathbf{s},\mathbf{s}+\mathbf{e}_i}^{(n)}(\mathbf{k},\mathbf{q}) &= R^{(n)}(\mathbf{q})\,\frac{J_i^{(n-1)}}{W_{\mathbf{s}}^{(n-1)}-W_{\mathbf{s}+\mathbf{e}_i}^{(n-1)}}\exp[2^{n-1}i\mathbf{q}(\mathbf{s}+\mathbf{e}_i)]\\
&\times\{\exp(i\mathbf{k}\mathbf{e}_i)[1-\exp(2^{n-1}i\mathbf{q}\mathbf{e}_i)]+[1-\exp(-2^{n-1}i\mathbf{q}\mathbf{e}_i)]\};
\end{aligned}
$$

where

$$
\begin{aligned}
S_i^{(n)} &= [W_0^{(n-1)}+W_{\mathbf{e}_i/2}^{(n-1)}-W_{\mathbf{e}_i/4}^{(n-1)}-W_{-\mathbf{e}_i/4}^{(n-1)}]\\
&\times[W_0^{(n-1)}-W_{\mathbf{e}_i/2}^{(n-1)}]^{-1}.
\end{aligned}
$$

We shall describe the procedure of the calculations. The solution of (51) is sought in the form

$$
A_{\mathbf{ss}'}^{(n)}(\mathbf{k},\mathbf{q}) = R_{\mathbf{s}}^{(n)}(\mathbf{q})\exp(2^{n-1}i\mathbf{q}\mathbf{s})\,[\delta_{\mathbf{ss}'}+\Phi_{\mathbf{ss}'}^{(n)}(\mathbf{k},\mathbf{q})], \tag{91}
$$

where $\langle\Phi_{\mathbf{ss}'}\rangle_0=0$, $\Phi_{\mathbf{ss}}\sim\beta^2$, $\Phi_{\mathbf{ss}'}\sim\beta$, $\mathbf{s}\neq\mathbf{s}'$. To calculate $\Phi_{\mathbf{ss}'}$ in lowest order in β we do not need to know the matrix elements $\overline{H}_{\mathbf{ss}'}^{\mathbf{mm}'}$ with $\mathbf{s}\neq\mathbf{s}'$, and for $\overline{H}_{\mathbf{ss}'}^{\mathbf{mm}'}$ the approximation (53) is sufficient. Equation (51) for $\nu=\nu'=0$ is brought to the form

$$
\begin{aligned}
\Big[\frac{R^{(n+1)}(\mathbf{q})}{R^{(n)}(\mathbf{q})}-1\Big]+\Phi_{00}^{(n+1)}(2\mathbf{k},\mathbf{q}) &= \hat{Q}_\pi^{\mathbf{k}}\Phi_{00}^{(n)}(\mathbf{k},\mathbf{q})\\
&-\sum_i 2g_i^{(n)2}\,[\cos k_i-\cos(k_i+2^n q_i)]
\end{aligned}
$$

and is solved by the substitution

$$
\begin{aligned}
\Phi_{00}^{(n)}(\mathbf{k},\mathbf{q}) &= -\sum_i g_i^{(n-1)2}[\cos\kappa_i(1-2\cos 2^{n-1}q_i+\cos 2^n q_i)\\
&+\sin k_i(2\sin 2^{n-1}q_i-\sin 2^n q_i)],
\end{aligned}
$$

which gives $A_{00}^{(n)}(\mathbf{k},\mathbf{q})$ to order β^2; for $R^{(n)}(\mathbf{q})$ we obtain the recursion relation (60b). For $\nu=\nu'$ (51) gives

$$
\begin{aligned}
\Phi_{\mathbf{ss}'}^{(n+1)}(2\mathbf{k},\mathbf{q}) &= \exp[2^n i\mathbf{q}(\mathbf{s}'-\mathbf{s})]\,\hat{Q}_\pi^{\mathbf{k}}\exp[i\mathbf{k}(\mathbf{s}'-\mathbf{s})]\\
&\times\Big[\sum_{\mathbf{s}''\neq\mathbf{s}}\frac{[\varepsilon_0^{\mathbf{s}''-\mathbf{s}}(\mathbf{k})]^{(n)}}{W_{\mathbf{s}}^{(n)}-W_{\mathbf{s}''}^{(n)}}+\sum_{\mathbf{s}''\neq\mathbf{s}'}\frac{[\varepsilon_0^{\mathbf{s}''-\mathbf{s}'}(\mathbf{k}+2^n\mathbf{q})]^{(n)}}{W_{\mathbf{s}'}^{(n)}-W_{\mathbf{s}''}^{(n)}}\Big]
\end{aligned}
$$

whence follows the expression for $A_{\mathbf{s},\mathbf{s}+\mathbf{e}_i}^{(n)}(\mathbf{k},\mathbf{q})$.

For the calculations of $A_{00}^{(n)}(\mathbf{k},\mathbf{q})$ to order β^3 we need the following expressions for the matrix elements:

$$
\begin{aligned}
&[W_0^{00}(\mathbf{k};\mathbf{e}_i)]^{(n)}\\
&\quad=-g_i^{(n-1)2}[\cos\mathbf{k}\mathbf{e}_i(W_0^{(n)}+W_{\mathbf{e}_i}^{(n)}-W_{\mathbf{e}_i/2}^{(n)}-W_{-\mathbf{e}_i/2}^{(n)})\\
&\qquad-i\sin\mathbf{k}\mathbf{e}_i(W_{\mathbf{e}_i/2}^{(n)}-W_{-\mathbf{e}/2}^{(n)})],\\
&[W_0^{\mathbf{e}_i 0}(\mathbf{k};0)]^{(n)}=g_i^{(n-1)}(W_{\mathbf{e}_i/2}^{(n)}-W_0^{(n)}),\\
&W_0^{\mathbf{e}_i 0(n)}(\mathbf{k};\mathbf{e}_i)=g_i^{(n-1)}\,(W_{-\mathbf{e}_i/2}^{(n)}-W_0^{(n)})\,e^{-i\mathbf{k}\mathbf{e}_i},
\end{aligned}
\tag{92}
$$

which can be obtained by substitution of the previously found approximation for $A_{\mathbf{ss}'}^{(n)}(\mathbf{k},\mathbf{q})$ into the formula

$$
\begin{aligned}
&[W_{\mathbf{n}}^{\mathbf{ss}'}(\mathbf{k};\mathbf{m})]^{(n)}\\
&\quad=\frac{1}{2^d}\sum_{\mathbf{k}_0\in\Gamma_1}\exp(i\mathbf{k}_0\mathbf{m})\sum_{\mathbf{q}\in\Omega_n}A_{\mathbf{ss}'}^{(n)}(\mathbf{k}+\mathbf{k}_0,\mathbf{q})\,V(\mathbf{q})\exp(2^n i\mathbf{q}\mathbf{n}),
\end{aligned}
\tag{93}
$$

which follows from (36) and (45) with the use of (26).

The calculation of the terms $\sim\beta^4$ in (90) is simplified considerably if we note that these terms can arise only from the extremal terms in the right-hand side of (51). Therefore, in the series (48) we can omit all terms containing interband transitions, since here at least one of the energy differences in the denominators is of order $W^{(n)}/\beta$, implying that the corresponding term is nonextremal. For the same reason, in (91) we can omit the nonextremal quantities $\Phi_{\mathbf{ss}}$ and, for $H_{\mathbf{ss}}^{\mathbf{mm}'}$, confine ourselves to the approximation (53). Furthermore, in series of the type (56) we must take into account only terms with $i=j=l=m$, since only they can lead to the appearance in the right-hand side of (51) of the terms $\cos(4k_i+\varphi)$ of interest to us. Therefore, for the quantities that we need we obtain

$$
\sum_{\mathbf{m}\neq 0}T_{\mathbf{m}0}=\sum_i\frac{2J_i\cos k_i}{E_0-W_{\mathbf{e}_i}},\qquad \sum_{\mathbf{m}\neq 0}|T_{\mathbf{m}0}|^2=\sum_i\frac{4J_i^2\cos^2 k_i}{(E_0-W_{\mathbf{e}_i})^2}
$$

$$
E_0=W_0+\sum_i\frac{4J_i^2\cos^2 k_i}{E_0-W_{\mathbf{e}_i}},
$$

whence

$$
\begin{aligned}
\bar{c}_{00}(\mathbf{k}) &= 1+\sum_i(z_i-{}^1/_2 z_i^2-{}^3/_2 z_i^3+{}^{11}/_8 z_i^4),\\
z_i &= 2J_i\cos k_i/(W_0-W_{\mathbf{e}_i}).
\end{aligned}
$$

It remains to substitute into (51) and select the appropriate terms.

Substituting (90) into formula (93), we obtain the necessary matrix elements in the form

$$[W_0^{00}(\mathbf{k};\mathbf{e}_i)]^{(n)} = \cos k_i [H_i^{(n)} g_i^{(n-1)2} + G_i^{(n)} g_i^{(n-1)} g_i^{(n-2)2}]$$
$$+ i \sin \mathbf{k}\mathbf{e}_i [P_i^{(n)} g_i^{(n-1)2} + T_i^{(n)} g_i^{(n-1)} g_i^{(n-2)2}] + O(\beta^4),$$
$$[W_0^{00}(\mathbf{k};0)]^{(n)} = W_0^{(n)} + \sum_i 2F_i^{(n)} g_i^{(n-1)4} \cos 2k_i,$$
$$[W_0^{0\mathbf{e}_i}(\mathbf{k};\mathbf{e}_i)]^{(n)} = g_i^{(n-1)} e^{i\mathbf{k}\mathbf{e}_i} [W_{\mathbf{e}_i/2}^{(n)} - W_{\mathbf{e}_i}^{(n)}], \quad (94)$$
$$W_{\mathbf{e}_i}^{0\mathbf{e}_i}({}^{(n)}\mathbf{k};0) = g_i^{(n-1)} [W_{-\mathbf{e}_i/2}^{(n)} - W_{\mathbf{e}_i}^{(n)}]$$

[the others are written out in (92)], where we have introduced the notation

$$H_i^{(n)} = -(W_0^{(n)} + W_{\mathbf{e}_i}^{(n)} - W_{\mathbf{e}_i/2}^{(n)} - W_{-\mathbf{e}_i/2}^{(n)}),$$
$$P_i^{(n)} = W_{\mathbf{e}_i/2}^{(n)} - W_{-\mathbf{e}_i/2},$$
$$G_i^{(n)} = -S_i^{(n)} H_i^{(n)} - (W_0^{(n)} + W_{\mathbf{e}_i}^{(n)} - W_{\mathbf{e}_i/4}^{(n)} - W_{-\mathbf{e}_i/4}^{(n)}$$
$$+ W_{\mathbf{e}_i/2}^{(n)} + W_{-\mathbf{e}_i/2}^{(n)} - W_{3\mathbf{e}_i/4}^{(n)} - W_{-3\mathbf{e}_i/4}^{(n)}), \quad (95)$$
$$T_i^{(n)} = -S_i^{(n)} P_i^{(n)} + (W_{\mathbf{e}_i/4}^{(n)} - W_{-\mathbf{e}_i/4}^{(n)} - W_{\mathbf{e}_i/2}^{(n)} + W_{-\mathbf{e}_i/2}^{(n)}$$
$$+ W_{3\mathbf{e}_i/4}^{(n)} - W_{-3\mathbf{e}_i/4}^{(n)}),$$
$$F_i^{(n)} = {}^{11}/_4 W_0^{(n)} + {}^1/_4 W_{\mathbf{e}_i}^{(n)} - {}^3/_2 W_{\mathbf{e}_i/2}^{(n)} - {}^3/_2 W_{-\mathbf{e}_i/2}^{(n)}.$$

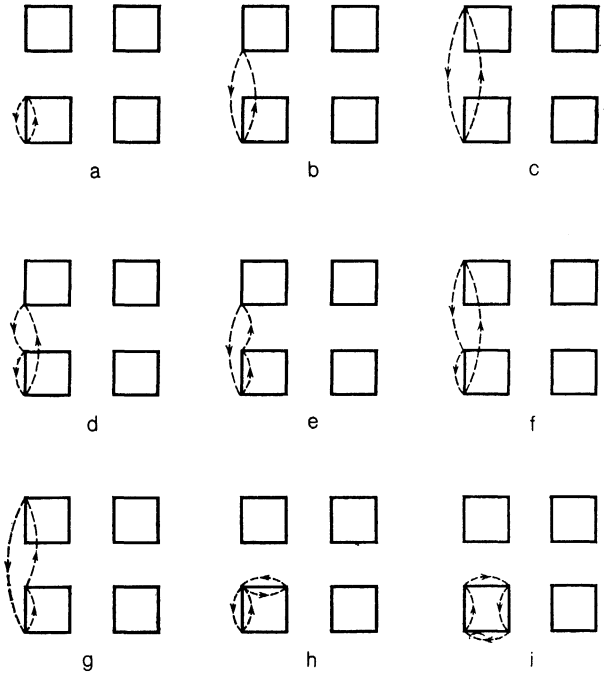

FIG. 4. Processes giving a contribution to $E_0(\mathbf{k})$ to order β^4 (for $d = 2$). The small squares positioned at the vertices of the large squares correspond to bands of the nth step of the renormalization group and are labeled by the index $\mathbf{s}$; their vertices correspond to the subbands, labeled by the index $\mathbf{m}$, arising in the nth step. Transitions $(\mathbf{s},\mathbf{m}) \to (\mathbf{s}',\mathbf{m}')$ of the following form are taken into account: a) $\mathbf{m}' - \mathbf{m} = \mathbf{e}_i$, $\mathbf{s}' - \mathbf{s} = 0$; b) $\mathbf{m}' - \mathbf{m} = 0$, $\mathbf{s}' - \mathbf{s} = \mathbf{e}_i$; c) $\mathbf{m}' - \mathbf{m} = \mathbf{e}_i$, $\mathbf{s}' - \mathbf{s} = \mathbf{e}_i$. In the d-dimensional case the small and large squares are replaced by d-dimensional cubes.

3. We shall write out the expression for $E_0(\mathbf{k})$ to order β^4; for this it is necessary to take into account the processes shown in Fig. 4. These are selected with the aid of the estimates (89):

$$E_0(\mathbf{k}) = \varepsilon_0^0(\mathbf{k}) + W_0^{00}(\mathbf{k};0) + \sum_i \frac{|\varepsilon_0^0(\mathbf{k}) + W_0^{00}(\mathbf{k};\mathbf{e}_i)|^2}{W_0 - W_{\mathbf{e}_i}} - \sum_{ij} \frac{[\varepsilon_0^{\mathbf{e}_i}(\mathbf{k})\varepsilon_0^{\mathbf{e}_j}(\mathbf{k})]^2}{(W_0 - W_{\mathbf{e}_i})^2 (W_0 - W_{\mathbf{e}_j})}$$
$$+ \sum_i \frac{\varepsilon_0^{\mathbf{e}_i}(\mathbf{k})[W_0^{\mathbf{e}_i 0}(\mathbf{k};0) W_0^{0\mathbf{e}_i}(\mathbf{k};\mathbf{e}_i) + W_{\mathbf{e}_i}^{0\mathbf{e}_i}(\mathbf{k};0) W_0^{\mathbf{e}_i 0}(\mathbf{k};\mathbf{e}_i) + \text{c.c.}]}{(W_0 - W_{\mathbf{e}_i})(\varepsilon_0^0 - \varepsilon_{\mathbf{e}_i}^0)}$$
$$+ \sum_i \sum_{j \neq i} \frac{[\varepsilon_0^{\mathbf{e}_i}(\mathbf{k})\varepsilon_0^{\mathbf{e}_j}(\mathbf{k})]}{(W_0 - W_{\mathbf{e}_i})(W_0 - W_{\mathbf{e}_i + \mathbf{e}_j})} \left[\frac{1}{W_0 - W_{\mathbf{e}_i}} + \frac{1}{W_0 - W_{\mathbf{e}_j}}\right].$$

We have omitted the contributions of the diagrams b and c, since, in view of the relation

$$|W_{\mathbf{n}}^{\mathbf{ss}'}(\mathbf{k};\mathbf{m})|^2 = \text{const}(\mathbf{k}) + O(\beta^4), \quad \mathbf{s} \neq \mathbf{s}',$$

which follows from (88), they turn out to be independent of $\mathbf{k}$. Substituting $\varepsilon_{\mathbf{s}}(\mathbf{k})$ in the form (88) and the matrix elements in the form (92), (94), we obtain the law of transformation of the spectrum:

$$J_i^{(n+1)} = J_{ii}^{(n)} + (W_0^{(n)} - W_{\mathbf{e}_i}^{(n)})[g_i^{(n)2} - 4g_i^{(n)4}$$
$$+ \sum_{j \neq i} 2Q_{ij}^{(n)} g_i^{(n)2} g_j^{(n)2})] + [H_i^{(n)} + C_i^{(n)}] g_i^{(n)} g_i^{(n-1)2}$$
$$+ \left[F_i^{(n)} + \frac{H_i^{(n)2} - P_i^{(n)2}}{4(W_0^{(n)} - W_{\mathbf{e}_i}^{(n)})}\right] g_i^{(n-1)4} + G_i^{(n)} g_i^{(n)} g_i^{(n-1)} g_i^{(n-2)2},$$
$$J_{ii}^{(n+1)} = -(W_0^{(n)} - W_{\mathbf{e}_i}^{(n)}) g_i^{(n)4},$$
$$J_{ij}^{(n+1)} = (W_0^{(n)} - W_{\mathbf{e}_i}^{(n)}) Q_{ij}^{(n)} g_i^{(n)2} g_j^{(n)2}, \quad (96)$$

where we have introduced the notation

$$C_i^{(n)} = [(W_{\mathbf{e}_i/2}^{(n)} - W_0^{(n)})(W_{\mathbf{e}_i/2}^{(n)} - W_{\mathbf{e}_i}^{(n)}) + (W_{-\mathbf{e}_i/2}^{(n)} - W_0^{(n)})$$
$$\times (W_{-\mathbf{e}_i/2}^{(n)} - W_{\mathbf{e}_i}^{(n)})](W_0^{(n-1)} - W_{\mathbf{e}_i}^{(n-1)})^{-1}.$$

Eliminating $J_{ii}^{(n)}$ from the first equation and iteratively transforming the fourth-order terms, we obtain the basic renormalization-group equation:

$$g_i^{(n+1)} = \frac{W_0^{(n)} - W_{\mathbf{e}_i}^{(n)}}{W_0^{(n+1)} - W_{\mathbf{e}_i}^{(n+1)}} u^{(n)} \left[g_i^{(n)2} - 4g_i^{n(4)} + \sum_{j \neq i} 2Q_{ij}^{(n)} g_i^{(n)2} g_j^{(n)2}\right], \quad (97)$$

where

$$u^{(n)}=1+\frac{H_i^{(n)}+C_i^{(n)}-W_0^{(n)}+W_{\mathbf{e}_i}^{(n)}}{W_0^{(n-1)}-W_{\mathbf{e}_i}^{(n-1)}}+\frac{G_i^{(n)}}{W_0^{(n-2)}-W_{\mathbf{e}_i}^{(n-2)}}+\frac{4F_i^{(n)}(W_0^{(n)}-W_{\mathbf{e}_i}^{(n)})+H_i^{(n)2}-P_i^{(n)2}}{4(W_0^{(n-1)}-W_{\mathbf{e}_i}^{(n-1)})^2}.$$

Going over to the modulus of $g_i^{(n)}$ and denoting it by the same symbol, we obtain an equation that differs from Eq. (60a) of the truncated renormalization group in that the coefficient $u^{(n)}$ is present. It is easy to see that $u^{(n)}=1+O(\beta)$, and the derivatives of $u^{(n)}$ with respect to $g_i^{(m)}$ are of order β^2; under variation with respect to $g_i^{(m)}$ the quantity $u^{(n)}$ can be regarded as constant. As a result, the coefficient $u^{(n)}$ does not appear in the linearized equations (75), and in the equation for $g_i^{c(n)}$ the coefficient $u^{(n)}$ can be replaced by unity.

The complete system of renormalization-group equations consists of Eq. (97), the evolution equation (60b) for $R^{(n)}(\mathbf{q})$, the definitions (60c) and (36) of $W_{\mathbf{m}}^{(n)}$ and $V_{ss'}^{(n)}(\mathbf{k},\mathbf{q})$, and the expressions (90) for the quantities $A_{ss'}^{(n)}(\mathbf{k},\mathbf{q})$. It is not difficult to verify that all the quantities appearing in the Schrödinger equation (35) of the nth step can be expressed in terms of $g_i^{(n)}$, $W_{\mathbf{m}}^{(n)}$, and $U_{\mathbf{m}}^{(n)}(\mathbf{q})$ with $\mathbf{m}=\mathbf{0}$, $\mathbf{e}_i$, $\pm\mathbf{e}_i/2$, etc. The quantities $W_{\mathbf{m}}^{(n)}$ and $U_{\mathbf{m}}^{(n)}(\mathbf{q})$ have Gaussian distributions; their variances, as a result of the scale transformation (72), do not depend on n, and the pair correlation coefficients are functions only of $g_i^{(n)}$. Consequently, provided that the $g_i^{(n)}$ are stationary, all the parameters of Eq. (35) are stationary, and their deviation from the stationary point is determined by the deviation of $g_i^{(n)}$ from $g_i^{c(n)}$. Thus, the role of the Thouless parameter $g_i^{(n)}$ for the scaling theory of localization[11] is justified to order β^2.

The rate at which $g_i^{(n)}$ moves away from the stationary point is described by the linearized equation (75), which has the same form in the full renormalization group as in the truncated renormalization group, thereby proving the admissibility of using the truncated renormalization group to calculate the index ν.

7. CHARACTER OF THE SCALING. THE CONDUCTIVITY

The basic renormalization-group equation (97) has the form

$$\mathbf{g}^{(n+1)}=\mathbf{F}(\mathbf{g}^{(n)},\ \{V(\mathbf{q})\}) \tag{98}$$

and differs from the equation postulated in Ref. 11 in the vector character of the Thouless parameter $\mathbf{g}^{(n)}$ and the presence of the random element $\{V(\mathbf{q})\}$. In the zeroth approximation these differences are unimportant. In fact, $g_i^{(n)}$ can be written in the form (see Sec. 5)

$$g_i^{(n)}=(J_i^{(0)}/\beta W_i)^{2^n}g_i^{c(n)},$$

where $g_i^{c(n)}$ has the stationary distribution (76). If g^* is the point of the maximum of $P(g)$ (see Fig. 1), then for the most probable value of $g_i^{(n)}$ we have

$$g_i^{(n)}=g^*(J_i/\beta W_i)^{2^n} \quad \text{or} \quad g_i(bL_n)=(g^*)^{1-b}g_i(L_n)^b,$$

where $L_n=2^n$ is the length scale at the nth step of the renormalization group. From this, for $b\to 1$, we obtain the Gell-Mann–Low equation[11]

$$\partial \ln g/\partial \ln L=\beta_{\mathrm{GL}}(g),\quad \beta_{\mathrm{GL}}(g)=\ln(g/g^*)\quad (g^*\sim\beta) \tag{99}$$

(the index i can be omitted, since this equation is the same for all i). It is evident that by an analogous transformation we can also bring the general equation (98) to an explicitly one-parameter form.

The principal result of the hypothesis of one-parameter scaling is the relation (4) between the indices s and ν. The essentially nontrivial point is the assertion $\nu=\nu'$ (ν and ν' are the correlation-length indices in the insulating and metallic phases, respectively), since the relation $s=\nu'(d-2)$ has a general character (see, e.g., Ref. 10). In the theory described, the equality $\nu=\nu'$ holds; consequently, the relation (4) is also valid. Thus, our results do not contradict the hypothesis of one-parameter scaling.

With regard to the relation (4), one comment must be made. From the analogy with incommensurate systems[19] we must expect that for sufficiently small β the conductivity in the metallic phase will be infinite and the index s indeterminate; a finite conductivity arises for a certain β_c. If the region of existence of a finite conductivity and the region of one-parameter scaling overlap, the index ν does not have a singularity at the point β_c and the index s can be calculated from the relation (4) in the framework of the proposed expansion. In fact, the conductivity of a finite system depends analytically on β in view of the absence of phase transitions in finite systems; the Gell-Mann–Low function describes the relationship between the conductivities of two finite systems and so, for finite values of g, depends analytically on β, as is confirmed by the expression (99), which is valid for $g\lesssim 1$ for any β (we note, however, that the limit $\lim_{g\to\infty}\beta_{\mathrm{GL}}(g)$ has a singularity at the point β_c). The index ν is determined by the behavior of $\beta_{\mathrm{GL}}(g)$ at $g\sim 1$ and does not have a singularity at the point β_c. It also follows from the above arguments that in the region of one-parameter scaling on the (β,d) plane (see Fig. 2) there can be no singularities in β: All singular lines of the type AB and CD are simultaneously boundaries of one-parameter scaling.

8. CHARACTER OF THE UNIVERSALITY

As usual in the theory of phase transitions, the renormalization group constructed makes it possible to investigate the nature of the universality of the critical behavior of the physical quantities.

1. As is clear from the derivation in Sec. 5, the result $\nu=1$ for $\beta\to 0$ has complete universality and does not depend on the statistical nature of the potential, e.g., on the existence or absence of variance, long-range correlations, etc. It is equally valid for random and determinate models. Such universality is similar to the universality of mean-field theory.

2. The first correction to the value $\nu=1$ depends on the statistical properties of the potential, although it turns out to be the same for large classes of analogous models. The universality class of the Gaussian model (28), for which the formula (21) was obtained, turns out to be very wide.

Suppose, for example, that the Fourier components of the potential have arbitrary, independent, finite-variance distributions satisfying (28). In view of (36), each Fourier component $V_{ss'}^{(n)}(\mathbf{k},\mathbf{q})$ of the potential of the nth step of the renormalization group is a linear combination of 2^{nd} Fourier components of the initial potential, with weights of order

unity. By virtue of the central limit theorem, the statistical properites of $V_{ss'}^{(n)}(\mathbf{k},\mathbf{q})$ for large n will be indistinguishable from those for the Gaussian model with the same values of the variance. Therefore, the critical behavior, determined by large n, will be the same as in the Gaussian model (28).

We can also weaken the requirement of equal variances for all $\mathbf{q}\in\Omega_n$ by replacing the last condition of (28) by the following condition:

$$\langle|V(\mathbf{q})|^2\rangle=(\beta^n V)^2 B(\mathbf{q}),\quad \mathbf{q}\in\Omega_n, \tag{100}$$

where $B(\mathbf{q})$ is a non-negative 2π-periodic function. For $\beta\to 1$ such a model goes over into the Anderson model in which the site energies $V_{\mathbf{m}}$ are correlated:

$$\langle V_{\mathbf{n}} V_{\mathbf{n}+\mathbf{m}}\rangle=V^2 B_{\mathbf{m}},\quad B_{\mathbf{m}}=\sum_{\mathbf{q}} B(\mathbf{q})\, e^{i\mathbf{q}\mathbf{m}}. \tag{101}$$

Defining the quantity $D_n(\mathbf{q})$ not by (69) but by

$$D_n(\mathbf{q})=\sum_{\mathbf{a}\in\Gamma_n}\left|R_n\left(\frac{\mathbf{q}}{2^n}+\mathbf{a}\right)\right|^2 B\left(\frac{\mathbf{q}}{2^n}+\mathbf{a}\right),$$

we obtain for it the same functional equation (69a), which must now be solved with the initial condition $D_0(\mathbf{q})=B(\mathbf{q})$. Making the Fourier transformation

$$D_n(\mathbf{q})=\sum_{\mathbf{m}} D_n(\mathbf{m})\, e^{i\mathbf{q}\mathbf{m}},$$

we bring (69a) to the form

$$D_{n+1}(\mathbf{m})=2^d\Big\{D_n(2\mathbf{m})+\sum_i 2g_i^{(n)2}\,[D_n(2\mathbf{m}+\mathbf{e}_i)-2D_n(2\mathbf{m}) +D_n(2\mathbf{m}-\mathbf{e}_i)]\Big\}. \tag{102}$$

With the aid of the inequality

$$|D_{n+1}(\mathbf{m})|\leqslant 2^d\max\{|D_n(2\mathbf{m})|,\quad |D_n(2\mathbf{m}+\mathbf{e}_i)|, \quad |D_n(2\mathbf{m}-\mathbf{e}_i)|\}$$

it is not difficult to show that if $|D_0(\mathbf{m})|<|\mathbf{m}|^{-\alpha}$, then $|D_n(\mathbf{m})|\leqslant 2^{(d-\alpha)n}\,\|\mathbf{m}|-1|^{-\alpha}$ for $|\mathbf{m}|>1$. Assuming that the quantities $D_n(\mathbf{m})$ with $|\mathbf{m}|>1$ are known, we obtain for $D_n(\mathbf{m})$ with $\mathbf{m}=\mathbf{0}$, $\pm\mathbf{e}_i$ an inhomogeneous system of difference equations; investigating this system, we find that under the condition

$$\alpha>(4d/\ln 2)\,\tilde\beta^2\ln(1/\tilde\beta) \tag{103}$$

we have $D_n(\mathbf{m})\to D_n\delta_{\mathbf{m}0}$, i.e., $D_n(\mathbf{q})\to\text{const}(\mathbf{q})$, as in the model (28). Thus, the model (100) belongs to the universality class of the model (28), if the correlations of the levels in the corresponding Anderson model ($\beta\to 1$) have a power-law decay with an exponent α satisfying (103).

3. In order to demonstrate the existence of other universality classes, we shall consider models in which the site energies $V_{\mathbf{m}}$ have distributions with an infinite variance. In view of the action of the law of large numbers, it is sufficient to consider the so-called stable distributions, which, like the normal law, preserve their form under summation of identically distributed terms. Such distributions have a characteristic function of the form[27]

$$\chi_{V_{\mathbf{m}}}(\theta)=\exp(-{}^1\!/_2|V\theta|^\gamma), \tag{104}$$

where γ is a parameter varying from 0 to 2: The normal law corresponds to $\gamma=2$, and the Cauchy distribution to $\gamma=1$ (we confine ourselves to symmetric distributions). The quantity V in (104), characterizing the width of the distribution, will be called the modulus of $V_{\mathbf{m}}$ and will be denoted by $\|V_{\mathbf{m}}\|$.

We shall define the hierarchical model as follows: The Fourier components $V_{\mathbf{q}}$ and $V_{\mathbf{q}'}$ are independent for $\mathbf{q}\neq\pm\mathbf{q}'$, $V_{\mathbf{q}}=V_{\mathbf{q}}^{(1)}+iV_{\mathbf{q}}^{(2)}$, $V_{-\mathbf{q}}=V_{\mathbf{q}}^*$, and the quantities $V_{\mathbf{q}}^{(1)}$ and $V_{\mathbf{q}}^{(2)}$ have a joint distribution with characteristic function

$$\chi_{V_{\mathbf{q}}^{(1)},\,V_{\mathbf{q}}^{(2)}}(\theta_1,\theta_2)=\exp[-2^{-\gamma}\|V_{\mathbf{q}}\|^\gamma(\theta_1^2+\theta_-^2)^{\gamma/2}]. \tag{105}$$

For the moduli of real sums of quantities $V_{\mathbf{q}}$ the following formula is valid:

$$S=S^*=\sum_{\mathbf{q}} A_{\mathbf{q}}V_{\mathbf{q}},\qquad \|S\|^\gamma=\sum_{\mathbf{q}}|A_{\mathbf{q}}|^\gamma\|V_{\mathbf{q}}\|^\gamma, \tag{106}$$

and is analogous to the law of composition of variances in the Gaussian case.

We shall note the modifications that it is necessary to introduce into the theory for application to the case under consideration. The definition (69) of the quantity $D_n(\mathbf{q})$ is replaced by

$$D_n(\mathbf{q})=\sum_{\mathbf{a}\in\Gamma_n}\left|R_n\left(\frac{\mathbf{q}}{2^n}+\mathbf{a}\right)\right|^\gamma.$$

This quantity also turns out to be independent of $\mathbf{q}$, and satisfies the recursion relations

$$D_{n+1}=D_n 2^d\left(1-2\gamma\sum_i g_i^{(n)2}\right).$$

The moduli of $W_{\mathbf{m}}^{(n)}$ and $U_{\mathbf{m}}^{(n)}(\mathbf{q})$ can be expressed in terms of D_n:

$$\|U_{\mathbf{m}}^{(n)}(\mathbf{q})\|=\beta^{p+n}VD_n^{1/\gamma},\quad \mathbf{q}\in\Omega_p,$$
$$\|W_{\mathbf{m}}^{(n)}\|=\beta^n VD_n^{1/\gamma}(2^d-1)^{1/\gamma}.$$

Performing the scale transformation

$$U_{\mathbf{m}}^{(n)}(q)\to U_{\mathbf{m}}^{(n)}(\mathbf{q})\beta^n D_n^{1/\gamma},$$
$$W_{\mathbf{m}}^{(n)}\to W_{\mathbf{m}}^{(n)}\beta^n D_n^{1/\gamma},\quad J_i^{(n)}\to J_i^{(n)}\beta^n D_n^{1/\gamma},$$

we arrive at Eq. (73) with $\tilde\beta=2^{d/\gamma}\beta$. The characteristic function $\chi_B\{\theta_{ij}\}$ is calculated in the same way as in the Gaussian case, up to formula (84). The joint distribution function of $Z_{\mathbf{e}_i}$, $Z_{\mathbf{e}_j}$, and $Z_{\mathbf{e}_i+\mathbf{e}_j}$ has the form

$$P(Z_{\mathbf{e}_i},Z_{\mathbf{e}_j},Z_{\mathbf{e}_i+\mathbf{e}_j})=\frac{1}{(2\pi)^3}\int\int\int d\theta_1\,d\theta_2\,d\theta_3$$
$$\times\exp(-i\theta_1 Z_{\mathbf{e}_i}-i\theta_2 Z_{\mathbf{e}_j}-i\theta_3 Z_{\mathbf{e}_i+\mathbf{e}_j})$$
$$\times\exp(-|\theta_1+\theta_2|^\gamma-|\theta_1+\theta_3|^\gamma-|\theta_2+\theta_3|^\gamma).$$

Substituting into (84), we obtain

$$\chi_B\{\theta_{ij}\}=1+i4\tilde\beta^2\ln(1/\tilde\beta)N\bar X\,\frac{\Gamma(1+1/\gamma)}{2^{1/\gamma}\pi} \times\left(-2\sum_i\theta_{ii}+2I_\gamma\sum_i\sum_{j\neq i}\theta_{ij}\right),$$

where I_γ denotes the integral

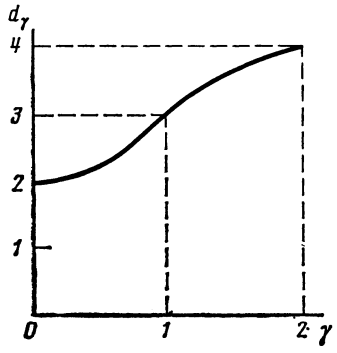

FIG. 5.

$$I_\gamma = 2^{1/\gamma}\int_0^\infty \frac{dx}{[(1+x)^\gamma+1+x^\gamma]^{1+1/\gamma}} = \begin{cases} 1/3, & \gamma=2 \\ 1/2, & \gamma=1. \\ 1, & \gamma\to 0 \end{cases}$$

As a result we arrive at the following expression for the index ν:

$$\nu = 1 - C_\gamma 2^{2d/\gamma}(d-d_\gamma)\beta^2 \ln(1/\beta) + O(\beta^2), \qquad (107)$$

where

$$d_\gamma = 1+\frac{1}{I_\gamma}, \quad C_\gamma = \frac{4I_\gamma}{\ln 2}\frac{\Gamma(1+1/\gamma)}{\Gamma(1/2)}\prod_{m=1}^{\infty}\frac{\Gamma(1-1/\gamma 2^m)}{\Gamma(1-1/2^{m+1})}.$$

The dependence of d_γ on γ is shown in Fig. 5; the existence of this dependence can imply a dependence on γ of the upper and lower critical dimensionalities (see Sec. 2).

4. The result $\nu = 1$, obtained in the limit $\beta \to 0$, is exact for incommensurate systems whose potential is a sum of two periodic potentials with an irrational ratio β_{incom} of periods[18-21] (this result was obtained for the one-dimensional case, but is valid for all d; see the Appendix). The results of this section make it possible to understand the reason for this coincidence.

In incommensurate systems the strong-hierarchy limit can be created by a special choice of the irrational number β_{incom}—namely, in its expansion as a continued fraction

$$\beta_{\text{incom}} = \frac{1}{n_1+\beta_1} = \frac{1}{n_1+\dfrac{1}{n_2+\beta_2}} = \ldots, \qquad (108)$$

all the $n_1, n_2, \ldots$ should be large.[19] Naturally, in the limit

$$\beta = \max(\beta_{\text{incom}}, \beta_1, \beta_2, \ldots) \to 0$$

the result (19) holds. But the correction to it depends on the universality class, and therefore it is not surprising that in the universality class to which incommensurate systems belong it is equal to zero in all orders in β.

APPENDIX

Comment on multidimensional incommensurate systems

The principal results of the theory of localization in incommensurate systems (in particular, $\nu = 1$) were obtained for the equation

$$a_{l+1}+a_{l-1}+V\cos(2\pi\beta_{\text{incom}}l)\,a_l = Ea_l, \qquad (109)$$

which has an Anderson transition at $V = 2$. Its natural multidimensional generalization is the equation

$$\sum_{i=1}^{d}(a_{l+e_i}+a_{l-e_i})+V\sum_{i=1}^{d}\cos(2\pi\beta_{\text{incom}}l_i)\,a_l = Ea_l, \qquad (110)$$

which admits separation of variables; thus, the result $\nu = 1$ is valid for the generalization too.

As shown in Ref. 19, the description of incommensurate systems of general form near the localization edge reduces to Eq. (109), if in the expansion (108) one can find a secondary sequence of sufficiently small β_k (this is true for almost all β_{incom}); i.e., the Hamiltonian (109) with $V = 2$ is the limit Hamiltonian to which the renormalization-group transformation brings an arbitrary system at its critical point. For the same reason, the Hamiltonian (110) with $V = 2$ is the limit Hamiltonian for d-dimensional incommensurate systems possessing cubic symmetry. Thus, the index value $\nu = 1$ turns out to be universal.

The author is grateful to A. F. Andreev and D. E. Khmel'nitskiĭ for discussions.

[1)]To be more precise, we must set $V_q = V_q^{(1)} + iV_q^{(2)}$, $V_q = V^*_{-q}$, and assume that $V_q^{(1)}$ and $V_q^{(2)}$ are independent. Formally, for $\langle V_q^{(1)2}\rangle = \langle V_q^{(2)2}\rangle$, this is the same as (7).

[2)]The sign of J is changed by the replacement $a_l \to a_l(-1)^l$, which does not change the shape of the envelope of the wavefunction.

[3)]In this section, $g_i^{(n)}$ will denote the ratio (59) without the modulus.

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