Quantum chaos and level distribution in the model of two coupled oscillators

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(Submitted 3 July 1986)
Zh. Eksp. Teor. Fiz. **92**, 1564–1573 (May 1987)

The quantum model of two coupled oscillators is investigated under conditions in which the model dynamics in the classical limit is highly stochastic and the stability-island measures are small. For this purpose the ranges of parameter values corresponding to the regular and stochastic dynamics in the classical case and the dependence of the Kolmogorov entropy on the system's energy are determined. In the quantum case histograms of the distribution of the spacings between adjacent energy levels are constructed , and the characteristic level repulsion exponents are determined. It is shown that, even in the region of the system's energies where the influence of the stability islands is clearly negligible, the level repulsion exponents exhibit strong nonmonotonic dependence on the dynamical characteristics of the system.

1. INTRODUCTION

After it became clear that chaos is possible in classical Hamiltonian systems, i.e., that such systems are K systems, there arose the natural question of how this dynamical property is reflected when such systems are quantized. The exact formulation of the resulting problems consists in the determination of the properties of quantum K systems, i.e., those systems that are K systems in the classical limit (see Ref. 1 for a review). The specific nature of the quantum problems leads in the steady-state cases to the necessity of the computation of the energy levels. Therefore, one of the problems of K-system quantization is connected with the determination of the nature of the energy spectrum at those values of the parameters of the problem to which corresponds chaotic dynamics in the classical limit. In Ref. 1 a large number of physical examples are cited which amount to the quantization of K systems. In Refs. 2 and 3 the view is expressed that the spectrum of quantum K systems possesses the property of energy level repulsion, a conjecture which is advanced in Ref. 4 for the spectrum of the excited states of heavy nuclei. By this we mean the following.

We shall consider a set consisting of a large number of levels as an ensemble. Let us introduce for it the probability density $P(\Delta E)$ for the spacing between adjacent levels (the nearest-neighbor spacing) to be equal to ΔE. Then the level-repulsion property consists in $P(\Delta E)\to 0$ as $\Delta E\to 0$. In Ref. 2 the small-ΔE asymptotic form

$$P(\Delta E)\propto(\Delta E)^{\mu} \tag{1.1}$$

is obtained for surface electrons in a magnetic field. Here the repulsion exponent μ depends on the Kolmogorov entropy h, i.e., on the dynamical properties of the system. The idea of repulsion has been confirmed by a numerical spectrum analysis carried out in a billiard quantization model of the "stadium" type.[5] But the determination of the nature of the law of repulsion and the computation of the exponent μ turned out to be a complicated problem even for a numerical analysis. It is shown in Refs. 6 and 7 that a distribution of the type (1.1) should be a characteristic property of stationary quantum K systems under sufficiently general assumptions. Another point of view is expressed in Ref. 8. It consists in the assertion that μ is always equal to unity, as obtains in an orthogonal Gaussian ensemble. The most important point of this assertion is the nondependence of the exponent μ on the dynamical properties of the system.

Below the level distribution law is investigated numerically, largely for two types of systems: 1) for different billiard models in which the particle dynamics is chaotic[9,10]; 2) for the two-particle Hamiltonians

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+U(x,\ y) \tag{1.2}$$

with some nonlinear particle-particle interaction law[11,12] $U(x,y)$. Apparently, the occurrence of level repulsion in K systems whose quantum numbers have been destroyed can be regarded as having been reliably established on the basis of the data reported in these papers. But these data do not allow us to draw any conclusions about the law of repulsion. There are many reasons for this, and it will be useful to dwell on them.

1. The main difficulty encountered in the determination of $P(\Delta E)$ is connected with the necessity to compute a large number of eigenvalues to a high degree of accuracy. Analysis shows that models of the coupled-oscillator type are preferable to the billiard model, for they ensure a higher degree of accuracy in the determination of the spectrum.

2. The second difficulty is connected with the fact that all the models that have thus far been investigated each has in its region of analysis not a very large Kolmogorov entropy h. This implies, for example, for Hamiltonians of the type (1.2), that the phase space of the system contains stability islands of quite large measure. As a result the regular level sequences stemming from the motion on invariant tori in stability islands make a substantial contribution to the level distribution.[13,14] Thus, we lose at not very large h values the possibility of determining the $P(\Delta E)$ for quantum K systems in the pure form. In particular, special attention is given to the control of the influence of the stability islands in the computations carried out below.

3. The dynamical properties and entropy h of the system (1.2) depend on the system's energy E. Therefore, there occurs in the analysis of the distribution of a large number of levels occupying a large energy interval an intermixing of the statistical ensembles with different distribution parameters. The narrower the energy interval containing the levels in question is, the more homogeneous the ensemble of levels will be. On the other hand, a small energy interval will con-

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tain a few levels. Therefore, a reasonable compromise is required in the choice of its dimensions.

The present paper is devoted to the study of the regular and stochastic regimes in the quantum and classical cases for the Hamiltonian (1.2) with the potential

$$U(x,y)=\frac{x^2+y^2}{2}+\frac{\alpha}{4}(x^4+y^4)+\frac{\beta}{2}x^2y^2. \tag{1.3}$$

The potential (1.3) with $\alpha=0$ is derived in Ref. 15 in the Yang-Mills quantum field model. In Ref. 11 this potential is introduced as a model of a system with chaotic motion. It is investigated in Ref. 16 with the aid of the criterion for the overlap of the resonances. The study of quantum chaos in this model is carried out in Refs. 11, 17, and 18.

The potential (1.3) guarantees the finiteness of the motion in the case when $\alpha\geqslant 0$ and $\beta\geqslant -\alpha$. The chosen form of the potential energy is the most general under the following assumptions: 1) $U(x,y)$ is a polynomial of order not higher than four; 2) $U(x,y)$ possesses the C_{4v} symmetry, i.e.,

$$U(x,y)=U(y,x)=U(-x,y).$$

In the $\alpha=0$ case the parameter β can be eliminated from the Hamiltonian with the aid of a scaling transformation. In our paper we study the nature of the classical motion and the properties of the quantum spectra of the model at different α values, which leads to the appearance of an additional parameter in the problem. It turns out, in particular, that the motion is always regular in some range of values of the parameters α and β (see also Ref. 19).

The main result of the investigation is connected with the determination of the level-spacing distribution function under conditions in which we can eliminate the influence of the stability islands. A level repulsion law of the type (1.1) was computed in different intervals of the energy spectrum, i.e., at different values of the Kolmogorov entropy. The data obtained enable us to conclude that a universal $\mu=1$ exponent does not exist. It is possible that the repulsion exponent depends nonmonotonically on the energy corresponding to the spectral interval in question.

2. PROPERTIES OF THE CLASSICAL MODEL

The potential (1.3) does not change its form when the coordinate axes are rotated through $\pi/4$. In this case the parameters α and β transform linearly:

$$\alpha'=(\alpha+\beta)/2,\quad \beta'=(3\alpha-\beta)/2. \tag{2.1}$$

Notice that the Hamiltonian (1.2) with the potential (1.3) allows the separation of the variables, and is integrable in the following obvious cases:

$$\alpha=\beta \tag{2.2a}$$

(cylindrical coordinates) and

$$\beta=0 \tag{2.2b}$$

(Cartesian coordinates). Under the transformation (2.1) the case (2.2b) goes over into

$$\alpha=\beta/3. \tag{2.2c}$$

Thus, the rays (2.2a)–(2.2c) divide the domain of the parameters α and β into four parts:

I. $-\alpha\leqslant\beta<0$
II. $0\leqslant\beta<\alpha$
III. $\alpha\leqslant\beta\leqslant 3\alpha$
IV. $0\leqslant 3\alpha<\beta$.

Under the transformation (2.1) the regions I and IV, II and III go over into each other, so that the indicated parameter domains are physically equivalent.

Analytic investigation of the motion of the system is possible in the region of small energy values:

$$E\max(\alpha,\beta)\ll 1. \tag{2.3}$$

To do this, let us go over in the Hamiltonian (1.2) to the action-angle variables:

$$H=J_x+J_y+\alpha(J_x^2\cos^4\theta_x+J_y^2\cos^4\theta_y)+2\beta J_xJ_y\cos^2\theta_x\cos^2\theta_y. \tag{2.4}$$

When the nonlinear terms are discarded, the natural frequencies for the two degrees of freedom are equal, i.e., the system is degenerate. Let us, for the purpose of investigating the motion go over to the variables, J_1, θ_1, J_2, and θ_2 with the aid of the generating function

$$F=(J_x+J_y)\theta_1+(J_x-J_y)\theta_2.$$

The expression (2.4) goes over into

$$H=J_1+H_1(J_1,\theta_1,J_2,\theta_2),$$

where, on account of (2.3), the term H_1 is small.

Averaging H_1 over the fast phase θ_1, we obtain an averaged Hamiltonian H_1 in which we can, in accordance with (2.3), set $J_1=E$:

$$\bar{H}_1=\frac{3\alpha+2\beta}{16}J_2^2+\frac{\beta}{16}(E^2-J_2^2)\cos 4\theta_2. \tag{2.5}$$

In the $3\alpha<\beta$ case there exists a separatrix passing through the hyperbolic points $J_2=0$ and $\theta_2=\pm\pi/4$. There occurs in the vicinity of the separatrix a stochastic layer that goes over into chaos at $E\gtrsim 1$, a fact which is confirmed by both our computational data and the results obtained in Refs. 14, 15, 17, and 18.

In the $3\alpha>\beta$ case all the trajectories are closed around the elliptic points $J_2=0$, $\theta_2=\pm\pi/4$, 0. The data obtained in the numerical modeling show that the motion is regular in this parameter domain. This agrees also with the results obtained in Ref. 19.

The Lyapunov exponent σ was computed in the following manner. The system's state vector $\mathbf{z}(t)=\{x,p_x,y,p_y\}$ was determined by means of a numerical solution of the equations of motion. Concurrently, the vector $\mathbf{z}_1(t)$ was computed. The initial conditions for it was chosen in the form

$$x_1=x,\ p_{x_1}=p_x+\delta_x,\ y_1=y,\ p_{y_1}=p_y+\delta_y,$$

where x,y,p_x, and p_y are the initial values of the components of the vector $\mathbf{z}$, and δ_x and δ_y are small quantities that are chosen with the aid of the conditions

$$\|\mathbf{z}-\mathbf{z}_1\|=d_0, \tag{2.6a}$$

$$p_x^2+p_y^2=p_{x_1}^2+p_{y_1}^2. \tag{2.6b}$$

The quantity d_0 determines the initial distance between the vectors $\mathbf{z}$ and $\mathbf{z}_1$ along the Euclidean norm. In the computations it was taken to be equal to $d_0=10^{-5}$. The condition

(2.2b) guarantees the equality of the initial energies for the vectors $\mathbf{z}$ and $\mathbf{z}_1$. This circumstance distinguishes our method from the one proposed in Ref. 20.

The above-described procedure was repeated after the quantity

$$d(t)=\|\mathbf{z}(t)-\mathbf{z}_1(t)\|$$

has attained a prescribed value $d_{\max}$ (we chose $d_{\max}=1$), or when $t>t_{\max}$ ($t_{\max}=50$). In this way we computed the sequence of values $\{d_i\}$ and the exponents

$$\sigma_i=(1/t_i)\ln(d_i/d_0),$$

after which we determined the trajectory-averaged Lyapunov exponent σ. The total motion time along the trajectory was chosen from the range $(1\text{–}3)\times 10^4$. This guaranteed a high degree of accuracy in the computation of σ, since this quantity ceases to vary in the region of short times.

Concurrently, we constructed the Poincaré cross section by the method described above. We divided the phase-space cross section in the plane (x,p_x) into $N_0=4450$ cells, and computed the number N_c of cells through which the trajectory passed. Then we determined the fraction

$$R_c=N_c/N_0,$$

of volume occupied by the trajectory and the Kolmogorov entropy $h=\sigma R_c$. The data obained for $\alpha=0,\beta=1$ and for $\alpha=1/6,\beta=1$ are presented in Fig. 1.

Notice that R_c is very close to unity when $E\gtrsim 50$. In fact only $\sim 1\%$ of the region around the elliptic point $x=0$, $p_x=0$ turns out to be unfilled. But the correct computation of R_c in this region is difficult because the numerical determination of N_c in the regions close to the phase space boundary is a difficult task.

Thus, our computations show that, as the energy increases, the stability islands in the phase space of the classical problem virtually vanish. This circumstance is important for the investigation of the quantum problem, since the characteristics of the behavior of the level distribution function cannot be explained by the influence of the islands.

3. THE QUANTUM PROBLEM. STATISTICS OF THE ENERGY LEVELS

In the quantum case to the model (1.2) with the potential (1.3) corresponds the Schrödinger equation

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+\frac{x^2+y^2}{2}\Psi+\frac{\alpha}{4}(x^4+y^4)\Psi+\frac{\beta}{2}x^2y^2\Psi=E\Psi. \tag{3.1}$$

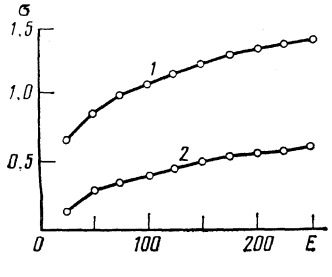

FIG. 1. Dependence of the trajectory- and phase-space-averaged Lyapunov exponent σ on the total energy E: 1) $\alpha=0,\beta=1$; 2) $\alpha=\frac{1}{6},\beta=1$.

We solved Eq. (3.1) through the expansion of Ψ in terms of the wave functions of a harmonic oscillator with frequency equal to unity, taking account of the symmetry of the Hamiltonian. We computed only the eigenvalues corresponding to the completely symmetric representation of the C_{4v} group.

In the computations the infinite Hamiltonian matrix was replaced by a finite matrix of dimension N. The majority of the computations were carried out for $N=2550$.

The limitation of the number of basis functions leads to errors in the computation of the upper eigenvalues. To estimate the number of accurately computed eigenvalues, we compared the results of the computations carried out with $N=2550$ and $N=3660$ for the parameters $\alpha=0$ and $\beta=1$. The comparison shows that only the first 500 eigenvalues in the $N=2550$ case can be computed with error smaller than 0.01%; subsequently, the error increases rapidly. Thus, only 20% of the total number of computed eigenvalues corresponds to energy levels of the original Hamiltonian. The remaining levels are affected by the "boundary effect." In our opinion the accurate estimation of the number of correctly computed energy levels is given insufficient attention in investigations of quantum chaos.

An advantageous characteristic of polynomial potentials of the form (1.3) is the negligibly small errors made in the computation of the Hamiltonian matrix elements. At the same time, in the billiard models, for example, the computation of the matrix elements is complicated, and can serve as a source of additional errors (see, for example, Refs. 9 and 10).

It is well known that energy-level repulsion occurs in quantum systems that are chaotic in the classical limit. For a quantitative description of the effect (or of its absence), a histogram of the level-spacing (ΔE) distribution is constructed which is then approximated by a smooth function $P(s)$, where $s=\Delta E/\langle\Delta E\rangle$.

The most generally used distributions are the distributions

$$P(s;q)=\left[F\left(\frac{\pi^{1/2}}{2}(1-q)s\right)q^2+\frac{\pi}{2}(1-q)^3 s+2q(1-q)\right]\exp\left[-qs-\frac{\pi}{4}(1-q)^2s^2\right],$$

$$F(z)=\frac{2}{\pi^{1/2}}\exp(z^2)\int_z^\infty \exp(-t^2)\,dt, \tag{3.2}$$

where $0\leqslant q\leqslant 1$ (see Refs. 13 and 14), and the distribution

$$P(s;\mu)=\bar{A}s^\mu\exp(-\bar{\gamma}s^{\mu+1}),$$

$$\bar{\gamma}=\{\Gamma[(\mu+2)/(\mu+1)]\}^{\mu+1},\quad \bar{A}=(\mu+1)\bar{\gamma}, \tag{3.3}$$

where $\mu\geqslant 0$ (see Ref. 17). The expressions (3.2) and (3.3) permit the passage to the Poisson-distribution limit

$$P(s)=e^{-s} \tag{3.4}$$

at $q=1,\mu=0$ and the orthogonal-Gaussian-ensemble distribution limit

$$P(s)=\frac{\pi}{2}s\exp\left(-\frac{\pi}{4}s^2\right) \tag{3.5}$$

at $q=0,\mu=1$.

In the distribution (3.2) the parameter q plays the role of the phase-space region occupied by the regular trajectories. But in our case the solution of the classical problem shows that, in the energy region $E \gtrsim 50$, the stability-island portion does not exceed it. Therefore, in the energy region $20 \lesssim E \lesssim 200$ of interest to us, the distribution (3.2) practically reduces to (3.5), a fact which does not correspond with our results.

The expression (3.3) better describes the distributions obtained by us in the region of fairly high energies. But the calculations showed that the variation of one parameter μ does not allow us to approximate the computed distributions with a high degree of accuracy. Therefore, it is convenient to introduce, for example, the following two-parameter distribution:

$$P(s;\ \mu,\ \nu)=As^{\mu}\exp(-\gamma s^{\mu+\nu}),$$

$$\gamma=(\Gamma_2/\Gamma_1)^{\mu+\nu},\quad A=(\mu+\nu)\Gamma_1^{\mu+1}/\Gamma_1^{\mu+2},\qquad (3.6)$$

$$\Gamma_1=\Gamma[(\mu+1)/(\mu+\nu)],\quad \Gamma_2=\Gamma[(\mu+2)/(\mu+\nu)],$$

with $\mu \geqslant 0$ and $\nu > -\mu$. The distribution (3.6) is normalized by the conditions

$$\int_0^\infty P(s)\,ds=1,\quad \int_0^\infty sP(s)\,ds=1.$$

In the course of the computations we constructed level-spacing distribution histograms, which we then approximated by the distribution (3.6) with the aid of the method of least squares.

The computations were carried out according to the following scheme. We computed the eigenvalues of Eq. (3.1) with the parameters $\beta = 1$ and $\alpha = 0$, $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{2}{3}$. To increase the number of events for each of these pairs of parameters, we diagonalized an ensemble of 13 matrices, the parameters α and β of which differed by 10^{-3}. Such a difference in the parameters changes substantially the eigenvalues of the Hamiltonian, but has virtually no effect on the distribution $P(s)$. Thus, we had at our disposal 6500 eigenvalues for each of the four pairs of α- and β-parameter values, which allowed us to decrease the $P(s)$ fluctuations.

Further, following Dyson,[21] we divided the entire energy interval into subintervals, each containing a sufficiently large number of levels. All the events within one subinterval, with different small changes in α and β near their prescribed values (e.g., $\alpha = 0, \beta = 1$), constitute a representative of the ensemble of nearest-neighbor spacings. The various subintervals are characterized by different average values of the energy ΔE, i.e., by different dynamical characteristics (in particular, different values of the entropy h). Of course, we could have done without the division into subintervals, as was done in the first numerical level-statistics analyses. But the narrower a subinterval is, the less important is the role of the fluctuations in the level distribution. This is due to the strong inhomogeneity of the properties of the system of coupled oscillators (1.3) as functions of the energy E. This inhomogeneity is, in particular, indicated by the plot of the function $\sigma(E)$ in Fig. 1. It can be seen from this plot that the exponent σ changes roughly by a factor of two over the entire energy $20 \lesssim E \lesssim 200$ considered by us.

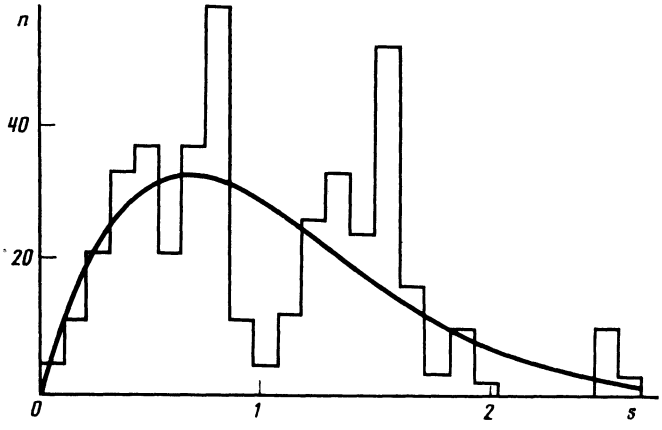

FIG. 2. Histograms of the nearest-neighbor spacing distribution for the parameters $\alpha = 0, \beta = 1$; n is the number of levels and $s = \Delta E/\langle \Delta E \rangle$. The continuous curve is an approximation constructed with the aid of the formula (3.6). The energy region is $20 < E < 40$.

Figure 2 shows an example of the histogram of the level-spacing distribution in some region of the spectrum for $\alpha = 0, \beta = 1$. It can be seen from it that the behavior of the histograms for some subintervals in the region of small s can have a nonmonotonic character. This is corroborated also by the dependences of the parameters of the distribution (3.6)

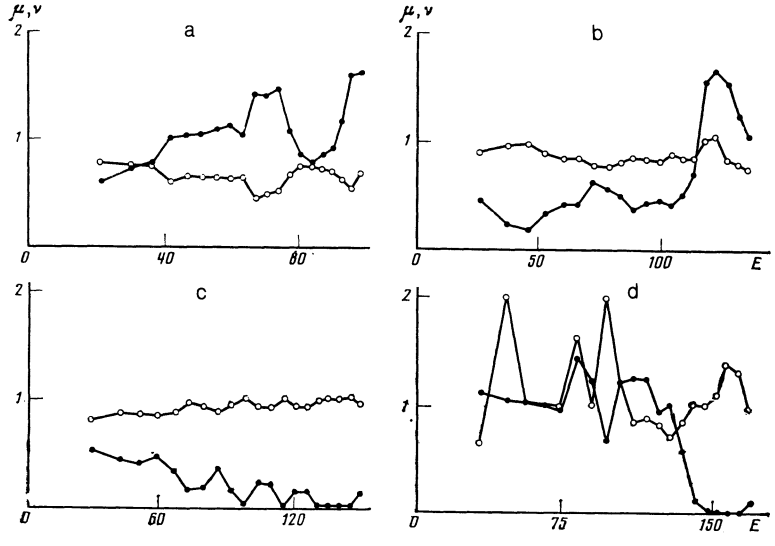

FIG. 3. Values of the distribution parameters μ(●) and ν(○) for different values of the energy E and different Hamiltonian parameters: a) $\alpha = 0$, $\beta = 1$; b) $\alpha = \frac{1}{6}$, $\beta = 1$; c) $\alpha = \frac{1}{3}, \beta = 1$; and d) $\alpha = \frac{2}{3}, \beta = 1$.

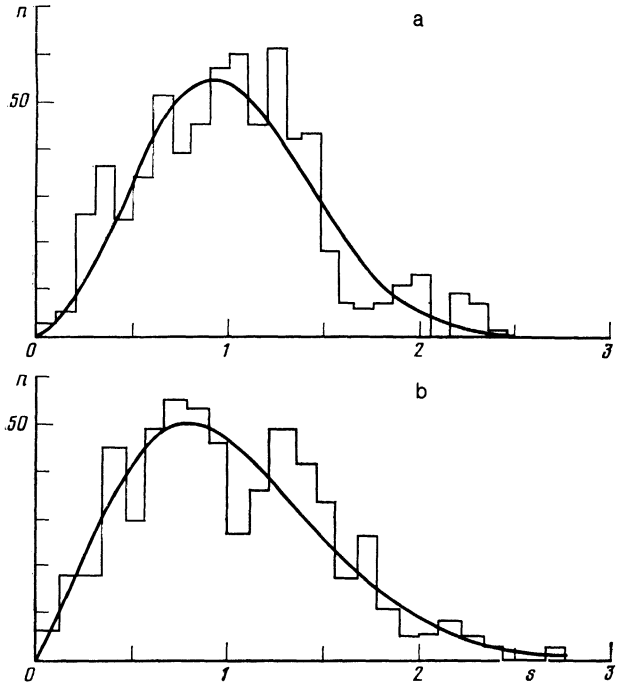

FIG. 4. Histograms of the distribution of the relative nearest-neighbor spacings (s) for $\alpha = 0$ and $\beta = 1$ in two energy intervals (n is the number of levels). Each histogram contains 650 levels. The continuous curve is an approximation constructed with the aid of the formula (3.6): a) $94 < E < 109$, $\mu = 1.6$, $\nu = 0.7$; b) $144 < E < 157$, $\mu = 1.2$, $\nu = 0.7$.

on the energy E corresponding to the middle of the subinterval. The parameters μ and ν depend nonmonotonically on the energy (Fig. 3a), the μ values being fairly far from the value $\mu = 1$, which corresponds to the orthogonal-Gaussian-ensemble distribution (3.5). And what is more, the level-spacing distribution exhibits a dip (Fig. 2) in the region of low energies, which is not described by formulas of the type (3.2)–(3.3).

Figure 3b shows the results of computations carried out for the potential parameters $\alpha = \frac{1}{6}, \beta = 1$. They lie closer to the boundary $\alpha = \beta/3$ of the region of regular motion in this case than in the preceding case. It can be seen that the parameter μ, as a function of the energy, varies nonmonotonically from 0.2 to 1.8. We also carried out computations for a potential with separable variables, i.e., a potential with $\alpha = \frac{1}{3}, \beta = 1$. The results obtained (Fig. 3c) show that the level distribution, to which would have corresponded the parameter values $\mu = 0$ and $\nu = 1$. We can also see here an essential dependence of the parameters on the energy.

Finally, Fig. 3d shows the results of computations carried out for the parameters $\alpha = \frac{2}{3}, \beta = 1$, which correspond to the region of regular dynamics. The level distribution in this case exhibits strong irregularity in the dependence on the energy subinterval, and the parameter μ does not tend to zero.

Since the increase of the parameter $\alpha \neq 0$ leads to the integrable case, it is useful to discuss in greater detail the case $\alpha = 0$, in which the entropy increases with increasing energy. Two corresponding histograms are shown together with the fitting curves in Fig. 4. Each of them was constructed with 650 levels, the accuracy of determination of which allowed us to eliminate the influence of the stability islands. In all, we obtained nine histograms encompassing the energy range $40 < E < 167$, over which the entropy changes by a considerable factor. The respresentative histograms in Fig. 4 exhibit a fairly substantial discrepancy between the level-spacing and Dyson distributions. This shows that, to obtain a function $P(s)$ that varies more appreciably, we should consider a system with a large Kolmogorov entropy. One more comment on these histograms. At small entropy values the procedure of data processing with the aid of formulas of the type (3.5) or (3.6) may turn out to be too crude and incapable of distinguishing the dynamics of level repulsion.

4. CONCLUSION

The above-presented numerical results show that the property of level repulsion in quantum K systems is not the only characteristic of the level-spacing distribution. The parameters obtained for the distribution $P(s)$ vary nonmonotonically with the energy region. This property of nonmonotony is not connected with the influence of the stability islands, since the distribution $P(s)$ was determined in that energy region where chaos in the classical limit was so highly developed that the stability-island measure was negligibly small. Therefore, we can expect the complicated form of the distribution $P(s)$, as a function of the energy region, to be determined by the dynamical properties of the corresponding classical system.

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Translated by A. K. Agyei