Edge plasmons in a two-dimensional bounded plasma

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(Submitted 2 September 1986)
Zh. Eksp. Teor. Fiz. **92**, 1648–1657 (May 1987)

The asymptotics of the edge-plasmon spectrum of a two-dimensional semi-infinite plasma (the electrons occupy a half-plane) in the presence of a screening electrode is obtained in the long-wave limit $qd \ll 1$ (q is the plasmon wave vector and d is the distance to the screen) in the absence of a magnetic field. The results differ qualitatively in cases when the screening is by one or two electrodes. The damping length of the boundary plasmons in the interior of the plasma is found to be anomalously large. For the more realistic circular geometry (the electrons occupy a disk of radius R) only modes with sufficiently large azimuthal indices $n > N \sim (R/d)^{3/4}$ can be interpreted as boundary plasmons.

A number of recent papers are devoted to the experimental and theoretical study of the spectrum of bounded-plasma oscillations localized near its boundary. What are involved in fact are two effects. The first manifests itself distinctly in experiment and has been quite well interpreted quantitatively, viz., the splitting of the natural plasma oscillations ω_n of a bounded $2D$ system by a magnetic field H normal to the plasma layer. The mode for which $\partial\omega_n/\partial H < 0$ tends to become localized, with increase of H and of the mode number, near the free edge of the $2D$ system, and can therefore be called an edge magnetoplasmon.

In the second effect, an edge plasmon in a $2D$ semi-bounded system (the electrons occupy a half-plane) can exist also without a magnetic field. This phenomenon is as yet not quite clear. Thus, it is indicated in Ref. 1 that an edge plasmon cannot exist without a magnetic field in the absence of a screening electrode (a metal surface parallel to the plasma layer). Later papers,[2,4] in which the spectrum of the edge plasmons was calculated approximately, attest to the existence in the same situation of an edge plasmon with a spectrum

$$\omega_p(q) = \text{const}\ \omega_l(q), \quad \text{const} < 1,$$

where $\omega_l(q)$ is the spectrum of two-dimensional plasmons in an unbounded $2D$ plasma, and q is the wave vector of the plasmon along the plasma boundary. Volkov and Mikhailov,[5] reporting a more accurate solution of the problem of an edge plasmon in a screened plasma, obtained its spectrum in a number of ranges of q in the general case $H \neq 0$. However, in the limiting case $qd \ll 1$, where d is the distance to the screening electrode at $H = 0$, was not discussed in their paper. Finally, in Fetter's paper[6] in which contains a numerical solution of the edge-plasmon problem, it is stated that in the limit as $qd \to 0$ and $H = 0$ there is no edge plasmon if the screening is effected by two electrodes placed at equal distances d from the plasma layer. In view of the foregoing, we wish to present here our own arguments in favor of the existence of edge plasmons in the absence of a magnetic field in a screened plasma in the limit as $qd \to 0$, and discuss the possibility of observing them.

1. EDGE PLASMON FOR A SEMI-INFINITE 2D PLASMA

Consider, for the sake of argument, the cell shown schematically in Fig. 1. A parallel-plate capacitor is partially (up to a heighth d above the lower plate) filled with liquid helium. An equilibrium $2d$ electron system was produced on the helium surface. The density $n_0(x)$ of this system is shaped by potentials applied to the capacitor plates and to a guarding electrode represented in the figure by a vertical segment. The dielectric constant of helium is $\varepsilon \approx 1.06$, the potential of the guard electrode is designated V. The capacitor plates occupy the planes $z = 0$ and $z = d' > d$. The point $x = 0$ coincides with the boundary of the $2d$ system. The coordinate of the guard electrode, $x = -L$, is determined in this case from the solution of the electrostatic-equilibrium equation. The same equation yields also the equilibrium distribution of the electron density $n_0(x)$. It turns out that in this case, when the two screening electrodes are at equal distances from the helium surface, $n_0(x)$ has a square-root dependence as $x \to +0$ and rapidly reaches $n_0(\infty)$ at $x \gtrsim d$.[3] It is natural to assume that this $n_0(x)$ dependence is preserved also in a situation with one screening electrode:

$$n_0(x) \sim x^{1/2} \quad \text{as} \quad x \to +0, \quad \partial n_0/\partial x \approx 0 \quad \text{at} \quad x \gtrsim d. \tag{1}$$

A. Consider first a situation with one screening plane $z = 0$. The set of equations that describe the edge plasmon in the linear approximation contains the connection between the electrostatic potential $\varphi(x,y,t) = \varphi(x)\exp(iqy - i\omega t)$ and the oscillating density $n(x,y,t) = n(x)\exp(iqy - i\omega t)$:

$$\varphi(x,y,t) = -|e| \int_0^\infty dx' \int_{-\infty}^{+\infty} dy' K(x-x', y-y') n(x', y', t), \tag{2}$$

where e is the electron charge, y the coordinate along the half-plane boundary, $K(x,y)$ is the Green's function of the Poisson equation of the corresponding electrostatic problem; it includes also the continuity equation

$$\frac{\partial n}{\partial t} + \operatorname{div}[n_0(x)\mathbf{v}] = 0. \tag{3}$$

Here $n_0(x)$ the equilibrium density of the electrons, and v is

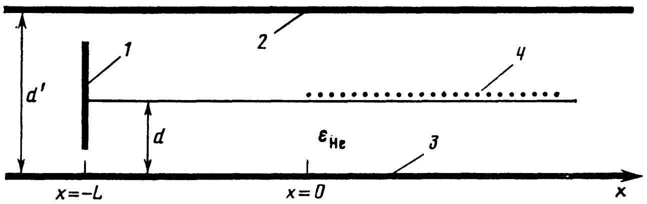

FIG. 1. 1—Guard electrode, 2—upper screening electrode, 3—lower screening electrode, 4—electrons above the helium.

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the two-dimensional field of their hydrodynamic velocity. The value of v is obtained from the equation of motion

$$m\frac{\partial \mathbf{v}}{\partial t}=|e|\nabla_2\varphi, \quad (4)$$

where m is the electron mass and ∇_2 is the two-dimensional gradient in the plane of the electron layer.

One can obtain for $K(x,y)$ the Fourier representation

$$K(x,y)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\exp\{-i(k_x x+k_y y)\}K(k_x,k_y)\,dk_x\,dk_y, \quad (5)$$

where

$$K(k_x,k_y)=\frac{4\pi}{(1+\varepsilon)|k|}\left\{1-\frac{2\varepsilon}{1+\varepsilon}\sum_{n=1}^{\infty}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n-1}\times\exp\{-2n|k|d\}\right\},$$

$$|k|=(k_x^2+k_y^2)^{1/2}. \quad (5a)$$

Substituting φ and n in (2) in a form that contains the factor $\exp(iqy-i\omega t)$, we obtain the equation

$$\varphi(x)=-|e|\int_0^{\infty}\mathcal{L}(x-x')n(x')\,dx', \quad (6)$$

where

$$\mathcal{L}(x)=\frac{4}{1+\varepsilon}\left\{K_0(q|x|)-\frac{2\varepsilon}{1+\varepsilon}\sum_{n=1}^{\infty}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n-1}\times K_0(q[x^2+(nD)^2]^{1/2})\right\}, \quad (7)$$

$D=2d$, and K_0 is a modified Bessel function of the second kind. In the derivation of (5a), the presence of the guard electrode was not taken into account, since it is possible to meet the inequality $L\gg d$ by choosing a high enough guard-electrode potential.

Since $(\varepsilon-1)/(\varepsilon+1)\ll 1$ for helium, we can put $\varepsilon=1$ in (7), thereby simplifying greatly the expression for the kernel:

$$\mathcal{L}(x)\approx L(x)=2\{K_0(q|x|)-K_0(q[x^2+D^2]^{1/2})\}. \quad (8)$$

It was assumed in Refs. 1, 2, and 4–6 that the equilibrium profile of the electron density $n_0(x)$ is the step function $\theta(x)$. In fact, as mentioned above, $n_0(x)$ has a square-root dependence as $x\to+0$ and tends to a constant $n_0(\infty)$ at $x\gtrsim d$. This behavior of $n_0(x)$ as $x\to 0$ imposes certain restrictions on $\varphi(x)$ if it is desired to remain within the framework of the linear approximation. Indeed, from the continuity equation written in the form

$$n(x)=\frac{|e|}{m\omega^2}\operatorname{div}(n_0\nabla_2\varphi)$$

$$=\frac{|e|}{m\omega^2}\left\{\frac{d}{dx}\left[n_0(x)\frac{d\varphi}{dx}\right]-n_0(x)q^2\varphi(x)\right\}, \quad (9)$$

it can be seen that $n(x)$ contains a contribution $\sim n_0'\varphi'$, and the condition $n(x)/n_0(x)\sim n_0'\varphi'/n_0\ll 1$ that permits linearization calls for satisfaction of the relation $\varphi'(x)\sim x^{1+\alpha}$, $\alpha\geqslant 0$ as $x\to+0$, since $n_0'(x)/n_0(x)\sim x^{-1}$ as $x\to+0$. For a step-function distribution $n_0(x)=\theta(x)$, the condition $\varphi'(x)\to 0$ as $x\to+0$ is equivalent (by virtue of the proportionality $v_x\sim\varphi_x'$) to vanishing of the normal current on the plasma boundary, i.e., to absence of a displacement of the boundary or to the fact that no singular charges of type $Q\delta(x)$ are produced on it.

If the boundary has a finite rigidity and its equilibrium position is determined by the electrostatic-equilibrium equation, the fluxes to the boundary can in principle shift the position of the latter. If, however, it is assumed that the oscillation amplitude in the edge plasmon is such that the corresponding displacement of the boundaries of the $2d$ system is small compared with the width $n_0(x)$ of the transition density region, the increments that appear when account is taken of the displacements of the boundary turn out to be of second order in the oscillation amplitude, and should be disregarded in the linear theory.

Substituting (8) and (9) in (6) and introducing the dimensionless coordinate x/D, which will henceforth be designated by the same letter x, we obtain the following equation for $\varphi(x)$:

$$\Omega^2\varphi(x)=-\int_0^{\infty}dx'L_q(x-x')\left((f_0\varphi')'-(qD)^2f_0(x')\varphi(x')\right),$$

$$\varphi'(x)|_{x=0}=0, \quad (10)$$

where $\Omega^2=\omega^2mD/(2\pi n_0(\infty)e^2)$ is the dimensionless frequency,

$$f_0(x)=n_0(x)/n_0(\infty),\quad f_0(x)\sim x^{1/2}\ \text{as}\ x\to+0,$$

$$f_0(x)\to 1\ \text{as}\ x\to+\infty\ \text{и}\ f_0(x)=0\ \text{at}\ x\leqslant 0,$$

$$L_q(x)=\pi^{-1}\{K_0(qD|x|)-K_0[qD(1+x^2)^{1/2}]\}. \quad (11)$$

The primes in the expression $(f_0\varphi')'$ denote differentiation with respect to x'.

The kernel $L_q(x)$ is given by

$$L_q(x)\approx\begin{cases}\pi^{-1}\ln|x|, & x\to 0,\\ 1/2\pi x^2, & 1\leqslant x\leqslant(qD)^{-1},\\ (qD/8\pi|x|^3)^{1/2}e^{-qD|x|}, & |x|\geqslant(qD)^{-1}.\end{cases} \quad (12)$$

We note in addition that

$$\int_{-\infty}^{+\infty}L_q(x)\,dx=\frac{1}{qD}[1-\exp(-qD)]=1+o(qD). \quad (13)$$

It is impossible to obtain an exact solution of the integral equation (10). We solve it approximately by recognizing that the sought function can be taken outside the integral sign if the characteristic interval of its substantial variation exceeds the distance over which the kernel falls off. The calculation results will confirm that the function $\varphi(x)$ varies little over distances $\sim(qD)^{-1}$, so that the following chain of equations is valid (recall that $f_0(x)$ becomes equal to unity at $x\sim 1$):

$$\int_0^{\infty}dx'L_q(x-x')f_0(x')\varphi(x')\approx\varphi(x)\int_0^{\infty}dx'L_q(x-x')f_0(x')$$

$$=\varphi(x)\int_{-\infty}^{+\infty}dx'L_q(x-x')f_0(x')=\varphi(x)\left\{\int_{-\infty}^{+\infty}L_q(x-x')\,dx'\right.$$

$$\left.-\int_{-\infty}^{+\infty}L_q(x-x')(1-f_0(x'))\,dx'\right\}\approx\varphi(x)(1-h(x)),$$

$$h(x)\approx\int_{-\infty}^{0} L_q(x-x')\,dx'=\int_{x}^{\infty} L_q(s)\,ds \quad \text{at} \quad x\gg1. \tag{14}$$

It follows from (12) that

$$h(x)\approx 1/2\pi x \quad \text{at} \quad 1\leqslant x\leqslant (qD)^{-1} \tag{14a}$$

and $h(x)$ decreases exponentially at $x\gtrsim (qD)^{-1}$

$$h(x)\approx(8\pi qDx^3)^{-1/2}\exp(-qDx), \qquad x\geqslant (qD)^{-1}. \tag{14b}$$

As to that part of the integral in (10) which contains $(f_0\varphi')'$, it is convenient to represent it in the form

$$\int_0^{\infty} dx' L_q(x-x')\,(f_0\varphi')'\,dx'=G(x)\,(f_0\varphi')'(x), \tag{15}$$

where $G(x)$ is some indeterminate function, which is in fact defined by (15). As to $G(x)$, it can be stated that $G(x)\approx 1$ in the region of those x in which $(f_0\varphi')'$ varies slowly over distances of the order of unity (since the kernel $L_q(x-x')$ behaves at $1\lesssim x\lesssim (qD)^{-1}$ like $(x-x')^{-2}$, i.e., it is concentrated mainly in the region $|x-x'|\sim 1$). Next, it follows from the fact that as $x\to 0$ the left-hand side of (15) tends to a certain constant, while $\varphi'(x)\sim x^{1+\alpha}$, $\alpha\geqslant 0$ as $x\to +0$ (see the arguments that follow Eq. (9)), it follows that $G(x)$ varies like $x^{-(\alpha+1/2)}$ as $x\to 0$.

Using the approximations (14) and (15), we rewrite Eq. (10) for $\varphi(x)$:

$$d[f_0(x)d\varphi/dx]/dx=-G^{-1}(x)\{\Omega^2-(qD)^2+(qD)^2h(x)\}\varphi(x). \tag{16}$$

At large $x\gg 1$, the tendency of $G(x)$ to unity and the slow variation of $(f_0\varphi')'$, used to assess the properties of the function $G(x)$, actually take place. By the same token it follows from (16) that $G(x)\approx 1$ we have $(f_0\varphi')'\approx C_1+C_2h(x)$, i.e., the assumption $G(x)\approx 1$ at $x\gg 1$ is justified, since it is obtained in this case that $(f_0\varphi')'$ changes to the extent that $h(x)$ of (14a) and (14b) changes, i.e., quite slowly.

Equation (16) recalls the problem of determining the energy level E in a shallow well,[7] where the role of E is played by $\Omega^2-(qD)^2$, the potential energy takes the form $-(qD)^2h(x)$, while shallowness of the well is ensured by the inequality $(qD)^2\ll 1$. In exact analogy with the quantum-mechanical problem, we shall assume that the sought energy level $\Omega^2-(qD)^2$ is the quantity $o[(qD)^2]$, so that in the region $x\lesssim (qD)^{-1}$ we can neglect the term $\Omega^2-(qD)^2$ compared with $(qD)^2h(x)$.

As a result, we have in this region the equation

$$d(f_0 d\varphi/dx)/dx=-G^{-1}(x)(qD)^2h(x)\varphi(x). \tag{17}$$

In the region $x\gtrsim (qD)^{-1}$, where $h(x)$ decreases exponentially and $f_0(x)=G(x)=1$, we obtain

$$d^2\varphi/dx^2=-[\Omega^2-(qD)^2]\varphi(x). \tag{18}$$

The problem now is to match in the intermediate region the logarithmic derivatives of the solution (17) which satisfies the condition $\varphi'(0)=0$, and of that solution of Eq. (18) which decreases exponentially as $x\to +\infty$. From (7) we have

$$f_0\,d\varphi/dx=-(qD)^2\int_0^x G^{-1}(x')\,h(x')\,\varphi(x')\,dx'. \tag{19}$$

Assuming that in the integral of (19) we can replace $\varphi(x')$ by $\varphi(0)$ in the entire integration region (estimates that demonstrate the validity of this approximation will be given below), we obtain at $1\lesssim x\lesssim (qD)^{-1}$

$$\varphi'(x)=-(qD)^2\varphi(0)\,(A+(2\pi)^{-1}\ln x), \tag{20}$$

where the constant $A\sim 1$ results from integration over the region $x\sim 1$, and the logarithmic contribution comes from the region $x\gtrsim 1$, where $G(x)=1$ and $h(x)\approx(2\pi x)^{-1}$.

Having now Eq. (20), we can now justify replacement of $\varphi(x')$ in (19) by $\varphi(0)$. Indeed, substituting in place of $\varphi(0)$ the more accurate value

$$\varphi(x)=\varphi(0)+\int_0^x\varphi'(s)\,ds\approx\varphi(0)-(qD)^2\varphi(0)\left[Ax+\frac{x\ln x}{2\pi}\right] \tag{21}$$

we obtain for $\varphi'(x)$ a correction of order of

$$-(qD)^2\frac{(qD)^2\varphi(0)}{2\pi}\int_0^x\left(As+\frac{s\ln s}{2\pi}\right)\frac{ds}{s}$$

$$\approx-(qD)^2\frac{\varphi(0)\,(qD)^2}{2\pi}\left(Ax+\frac{x\ln x}{2\pi}\right),$$

i.e., at $x\sim (qD)^{-1}$, where it is necessary to carry out the matching (which, incidentally, is not very critical to the location of the point at which it is effected, since $\varphi'(x)$ has a logarithmic dependence on x), the addition is of order of smallness $(qD)\ll 1$ compared with the zeroth approximation (20). Moreover, it follows from the very same Eq. (20) that the value of $\varphi(x)$ at $x\sim (qD)^{-1}$ differs from $\varphi(0)$ by $\sim (qD)\ln(qD)^{-1}\varphi(0)\ll\varphi(0)$, i.e., the function $\varphi(x)$ indeed varies slowly over a distance not only ~ 1, but also over a distance $\sim (qD)^{-1}$, which makes it sensible to retain in (14) the term proportional to $h(x)$.

Finally, equating the value $\varphi'(x)/\varphi(x)$ obtained from (20) at the point $x\sim (qD)^{-1}$ [it is possible, as follows from the foregoing to replace $\varphi(x)$ by $\varphi(0)$] to the logarithmic derivative of the solution of Eq. (18), which vanishes at infinity, we obtain

$$\frac{\varphi'}{\varphi}=-(qD)^2\left(A+\frac{\ln(qD)^{-1}}{2\pi}\right)=-((qD)^2-\Omega^2)^{1/2}, \tag{22}$$

whence

$$\Omega^2=(qD)^2\left[1-(qD)^2\left(A+\frac{\ln(qD)^{-1}}{2\pi}\right)^2\right], \tag{23}$$

i.e., at q so small that $(\ln(qD)^{-1})/2\pi$ exceeds the indeterminate constant A, we have (after changing to dimensional quantities) the following dispersion equation for an edge plasmon as $q\to 0$:

$$\omega=c_0q\left\{1-\frac{1}{2}\left[\frac{qD\ln(qD)^{-1}}{2\pi}\right]^2\right\}, \tag{24}$$

where $c_0=[4\pi n_0(\infty)e^2d/m]^{1/2}$ is the velocity of the long-wave plasmon in an unbounded two-dimensional plasma. It follows from (23) that, in accordance with the assumption made above, the difference $\Omega^2-(qD)^2$ turns out to be a quantity of higher order of smallness in qD than $(qD)^2$. In the quantum-mechanical problem concerning the energy level in a shallow well, this corresponds to the statement that the sought level is of second order of smallness in the well depth.

Thus, in the situation considered, the edge plasmon, whose dispersion law is determined in the $q\to 0$ limit by Eq.

(24), lies lower than the corresponding plasmon of an unbounded plasma, and its damping length, determined by the quantity φ/φ', turns out to be anomalously large: in dimensional units it is equal not to q^{-1}, as might be assumed, but to

$$\Lambda=2\pi q^{-1}/qD \ln(1/qD). \tag{25}$$

B. Particularly noteworthy is the case of screening by two planes. We confine ourselves to the simplest and therefore usually considered variant, in which the metallic electrodes are at equal distances d from the electron sheet. In this case we have in place of (5a) for the Fourier transform of the function $K(x,y)$

$$K(k_x, k_y) = \frac{2}{1+\varepsilon}\frac{2\pi}{|k|} \text{th} |k|d, \tag{26}$$

while Ω^2 and $L_q(x)$ take in Eq. (10) the form

$$\Omega^2=\omega^2(1+\varepsilon)md/(2\pi n_0(\infty)e^2), \tag{27}$$

$$L_q(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} K(k_x, q)\, e^{ik_x x}\, dk_x, \tag{28}$$

where x denotes the dimensionless coordinate $\tilde{x} = x/d$.

To analyze the long-wave approximation it suffices to replace $L_q(x)$ by

$$L_0(x) = -\frac{1}{\pi}\ln \text{th}\frac{\pi x}{4}. \tag{29}$$

In $x \gtrsim 1$

$$L_0(x)=2\pi^{-1}\exp(-\pi x/2), \tag{30}$$

i.e., the kernel falls off exponentially at $x \gtrsim 1$. The function $h(x)$ defined by (14) therefore decreases already exponentially at $x \gtrsim 1$ for such a kernel. The logarithmic derivatives should now be matched at $x \gtrsim 1$, i.e., Eq. (17), in which D must be replaced by d, should be integrated from 0 to a certain value $x \sim 1$. The main contribution to the derivative builds up here in that region of x where the function $G(x)$ is not known with any degree of accuracy (i.e., $(f_0\varphi')'$ varies noticeably over distances ~ 1 in the region considered). It can therefore only be stated that at $x \gtrsim 1$

$$d\varphi/dx=-B(qd)^2\varphi(0), \tag{31}$$

where $B \sim 1$ is a certain constant. At $x \gg 1$ we are left with Eq. (18) (with D replaced by d), so that the matching yields finally for the edge-plasmon spectrum

$$\omega(q)=cq(1-B^2(qd)^2/2), \tag{32}$$

where $c = 2^{-1/2}c_0$ is the velocity of the long-wave plasmons in an unbounded $2d$ plasma screened by two planes. In this screening variant, the plasmon damping length is found to be

$$\Lambda=\frac{1}{q}\frac{1}{Bqd}\sim\frac{1}{q^2}, \tag{33}$$

i.e., in the limit as $q \to 0$ it is larger than in the case of screening by one plane.

Note that in Ref. 6 (pp. 3718–3719) it is stated that there is no edge plasmon at $H = 0$ in the limit $qd \to 0$, if the screening is by two electrodes at equal distances from a two-dimensional plasma layer. This conclusion was drawn after the kernel $L_q(x)$ (designated $K(x - x')$ in Ref. 6) of Eq. (10) was replaced by $\delta(x)$. As a result, the integral equation is reduced to a differential one, from which it is seen that in the presence of a magnetic field, $H \neq 0$, the plasmon damping length is $\Lambda \sim H^{-1}$, i.e., Λ becomes infinite at $H = 0$. From the arguments advanced above it follows that the approximation of $L_q(x)$ by a δ-function is too crude: an edge plasmon exists also at $H = 0$, but the length of its damping into the plasma interior turns out to be anomalously large. It makes sense also to formulate an answer to the qualitative question of the role of the $f_0(x)$ profile in the formation of the spectrum of edge plasmons. The detailed behavior of the function f_0 (which describes the equilibrium electron density) in the region $0 < x \lesssim 1$, where it increases from zero to unity, affects the values of the constants A and B in (23) and (32). This means that if the screening is effected by one plane, the asymptotic dispersion law for the plasmon as $q \to 0$, given by Eq. (24) (which does not contain A), is the same for any function f_0 that reaches unity rapidly enough, including $f_0(x) = \theta(x)$ used in the preceding papers.[1,2,4–6] At the same time, in the case of screening by two plane the explicit form of the function $f_0(x)$ affects the numerical coefficient B in the asymptotic form of the dispersion law (32). The qualitative character of the spectrum on variation of $f_0(x)$, however, remains unchanged in this case.

3. EDGE PLASMON IN THE CASE OF CIRCULAR GEOMETRY

Since real experiments are always performed in finite geometry, with the electron sheet occupying a circle or a rectangle, a discussion of edge plasmons in bounded geometry is in order.

Let the electron be located in a circle of radius R and let the screening be effected by two planes at a distance $d \ll R$ from the electrons. The plasma oscillations are then described by the equation

$$\varphi(r,t)=-|e|\int_{|\mathbf{r}'|\leq R} K(\mathbf{r}-\mathbf{r}')\,n(\mathbf{r}',t)\,d^2\mathbf{r}', \tag{34}$$

where K is given by Eqs. (4) and (26), and n is connected with φ by the continuity equation [first equality in (9)]. We use a cylindrical coordinate system (φ,θ) with origin at the center of the circle occupied by the electrons, and with $\varphi(\mathbf{r},t)$ in the form $\varphi(\rho)\exp(in\theta - i\omega t)$. Far from the perimeter, Eq. (34), assuming slow variation of $\varphi(\mathbf{r},t)$ in space, reduces to a differential equation for $\varphi(r)$, where $r = \rho/d$:

$$r^2\varphi''+r\varphi'+\left[\left(\frac{\omega d}{c}\right)^2 r^2-n^2\right]\varphi=0. \tag{35}$$

The solution of this equation is

$$\varphi=J_n\left(\frac{\omega d}{c}r\right), \tag{36}$$

where J_n is a Bessel function of order n.

In the vicinity of the boundary of the circle, $\varphi(\mathbf{r},t)$ is a certain wave that travels along the perimeter and has a wave vector $q = n/R$. This can be used to obtain data boundary condition for (35), which follows from (31)

$$(\varphi'/\varphi)_{r=R/d}=B(qd)^2=B(nd/R)^2. \tag{37}$$

Introducing the variable $v = \omega/(cq) = \omega R/(nc)$, we rewrite (37) in the form

$$vJ_n'(nv)/J_n(nv)=Bn\alpha,\quad \alpha=d/R\ll 1. \tag{38}$$

Note that a dispersion equation for $2d$ plasmons is given in Ref. 3 for the case of circular geometry, with account taken of the magnetic field H. At $H = 0$ it reduces to an equation such as (38), but its solutions in the absence of a magnetic field were not investigated specifically in Ref. 3. It is expedient therefore to consider this particular case in detail.

For each n, Eq. (38) has an infinite number of solutions $v_{n,k}$, where k is the radical index. The edge plasmon corresponds to values $v<1$, for in this case we obtain a wave traveling along the circle and having a frequency lower than that of a $2d$ plasmon with a wave vector $q=n/R$ in an unbounded plasma (this is precisely the property possessed by the edge plasmons considered in Sec. A. If α is small enough, the right-hand side of (38) can be set equal to zero for small n. This means that at these n the solution of (38) is the quantity nv, which differs very little from the root of the equation $J_n'(x)=0$ that describes ordinary $2d$ plasmons under circular-geometry conditions, subject to the additional requirement that there be no normal component of the current on the circle boundary. It is known (Ref. 8, Sec. 9.5.2) that the smallest root of this equation $x_{\min}>n$, i.e., $v>1$. Values of v smaller than unity appear therefore among the solutions of (38) only for sufficiently large numbers $n>N$. The critical value of N can be obtained from (38) by putting in it $v=1$. Using the known (Ref. 8, Secs. 9.3.5 and 9.3.31) asymptotic expressions for $J_\nu'(\nu)$ and $J_\nu(\nu)$ at large ν, we easily obtain

$$N\approx(B\alpha)^{-3/4}, \tag{39}$$

i.e., only oscillations having a sufficiently large azimuthal index $n>N$ can be regarded as edge plasmons under condition of circular geometry.

To determine the behavior of $v<1$ in the region $n>N$ we can use the asymptotes of Bessel functions at large ν (Ref. 8, Secs. 9.3.7 and 9.3.11):

$$J_\nu'(\nu\,\mathrm{ch}^{-1}\gamma)\sim(\mathrm{sh}\,2\gamma/4\pi\nu)^{1/2}\exp[\nu(\mathrm{th}\,\gamma-\gamma)],$$

$$J_\nu(\nu\,\mathrm{ch}^{-1}\gamma)\sim(2\pi\nu\,\mathrm{th}\,\gamma)^{-1/2}\exp[\nu(\mathrm{th}\,\gamma-\gamma)],$$

from which follows the approximate equality

$$J_\nu'(\nu v)/J_\nu(\nu v)\approx(1-v^2)^{1/2}/v. \tag{40}$$

Substitution of (40) in (37) leads to the equation $(1-v^2)^{1/2}=nB\alpha$ from which we obtain for ω the expression

$$\omega\approx\frac{cn}{R}\left[1-\frac{B^2}{2}\left(\frac{nd}{R}\right)^2\right], \tag{41}$$

which coincides with (32) if the substitution $n/R=q$ is made. Thus, the dispersion law (41) does indeed describe an edge plasmon that propagates along the boundary of an electron disk.

It must be noted that (41) is valid only in a very narrow range: the edge plasmon described by this equation has a lower bound on n $(n>N)$ and an upper one $(qd\ll1)$ $\leftrightarrow(n\ll R/d)$. These two conditions can be met simultaneously only if the inequality $(d/R)^{1/4}\ll1$ holds. Noteworthy among the experimental studies of the spectrum of $2d$ plasmons in bounded systems, which have a bearing on the questions touched upon here, is Ref. 3, where the long-wave part of the $2d$-plasmon spectrum was investigated in detail for circular geometry in the absence and presence of a magnetic field. It was found there that the observed natural frequencies $\omega_{n,k}$ of an electron plasma with small indices $(n,k\leqslant4)$ are well described by a relation in the form (38) with a constant $\alpha\lesssim10^{-1}$. Thus, the nonvanishing of the parameter α in (38), which is the necessary condition for the existence of edge plasmons, is confirmed by experiment. To go into the region $n>N$, where the edge plasmon would be sufficiently distinctly defined at $H=0$, calls only for creation of conditions under which modes with sufficiently large n can be excited.

The authors thank V. A. Volkov and S. A. Mikhaĭlov for a discussion of the results of the work and for helpful remarks.

APPENDIX

The integral in the left-hand side of (15) can be rewritten, after a single integration by parts, in the form

$$I=-\int_0^\infty dx' f_0(x')\varphi'(x')\frac{\partial}{\partial x'}L_q(x-x'). \tag{A.1}$$

The contribution from the boundaries of the integration interval yields zero, since $(f_0\varphi')\to0$ as $x\to0,+\infty$.

In the region $x\lesssim(qD)^{-1}$ we have

$$\partial L_q(x-x')/\partial x'\approx-\{\pi(x'-x)[(x'-x)^2+1]\}^{-1}.$$

Assuming $\varphi'(x)$ to change slowly over distances ~1, we can expand $\varphi'(x')$ in (A.1) in a Taylor series in $(x'-x)$ and confine ourselves to the first two terms of this series:

$$\varphi'(x')\approx\varphi'(x)+\varphi''(x)(x'-x).$$

We then obtain

$$I\approx g_1(x)\varphi''(x)+g_2(x)\varphi'(x),$$

where

$$g_1(x)=\frac{1}{\pi}\int_0^\infty\frac{f_0(x')dx'}{(x'-x)^2+1},$$

$$g_2(x)=\frac{1}{\pi}\int_0^\infty\frac{f_0(x')dx'}{(x'-x)[(x'-x)^2+1]}.$$

Recognizing that $f_0(x)$ tends to unity if $x\gtrsim1$, we find that in the region $1\lesssim x\lesssim(qD)^{-1}$

$$g_1(x)=1+o(1/x),$$

i.e., for large x the integral in the right-hand side of (15) can be approximately replaced by $\varphi''(x)+g_2(x)\varphi'(x)$. This replacement, obviously, reduces (10) to a differential equation that has at $1\lesssim x\lesssim(qD)^{-1}$ the same form as (16).

[1)]An alternative for the transformation of the integral in (15) by expanding the function $\varphi(x')$ in a Taylor series $\varphi'(x')\approx\varphi'(x)+\varphi''(x)(x'-x)+\ldots$ is discussed in the Appendix. The final conclusions are similar to those of the main text: in the limit $qD\ll1$ Eq. (10) can be reduced to a differential one in the form (16).

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Translated by J. G. Adashko