Electrostatic oscillations in bounded superlattices in a strong magnetic field

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The quasistationary approximation is used in an analysis of low-frequency ($\omega \ll \omega_c = eH/mc$) electrical properties of bounded quantum superlattices in a strong magnetic field. It is assumed that superlattices can be described by the average permittivity tensor and the Hall currents can exceed greatly the ohmic currents. These assumptions are used to show that natural oscillations of the Hall current can exist in bounded superlattices. The spectrum of natural oscillations is calculated for samples of simple shape (sphere, ellipsoid of revolution). The main distinguishing feature of these oscillations is that resonance frequencies depend only on the shape but not on the absolute dimensions of a sample.

1. INTRODUCTION

The discovery of the quantum Hall effect has stimulated theoretical and experimental studies of two-dimensional (2D) conducting channels in strong magnetic fields. In addition to investigations of microscopic properties of 2D channels, there have been several studies of the electro-dynamics of ideal Hall conductors (i.e., 2D channels in which the number of filled Landau levels is an integer). One can include in this group the work on the distribution of the current in an ideal Hall conductor[1,2] and at the boundary of such a conductor with a metal contact,[2] and also studies of the screening of the charge and eddy currents in 2D channels.[3,4] In work of this type it is usual to study an inhomogeneous distribution of the charge in a 2D channel. Such a distribution is the source of an electric field and of the associated Hall currents (i.e., of currents perpendicular to the electric field). In the case of an ideal Hall conductor the ohmic currents (parallel to the field) can be ignored compared with the Hall currents. We shall consider low-frequency electrical properties of bounded superlattices, for example those based on GaAs–AlGaAs, in the approximation when a superlattice can be approximated by a homogeneous anisotropic medium. Superlattices consist of alternate layers of semiconductors of two types and conducting 2D channels appear at the interfaces between them. The electrical conductivity of a single 2D channel in a magnetic field is described by the tensor

$$\hat{\bar{\sigma}}=\begin{pmatrix}\bar{\sigma}_{xx} & -\bar{\sigma}_{xy} & 0\\ \bar{\sigma}_{xy} & \bar{\sigma}_{xx} & 0\\ 0 & 0 & 0\end{pmatrix}. \quad (1)$$

In Eq. (1) we are using a system of Cartesian coordinates with the x and y axes lying in the plane of a 2D layer; the z axis is perpendicular to this layer. An external static magnetic field is directed along the z axis. When the magnetic field is sufficiently strong, we find that $\bar{\sigma}_{xx} \ll \bar{\sigma}_{xy}$, whereas under the conditions of the quantum Hall effect when carriers in a 2D channel fill an integer (n) number of Landau levels, we find that $\bar{\sigma}_{xx} = 0$ and $\bar{\sigma}_{xy} = e^2n/h$.

The low-frequency oscillations are of interest because when the frequencies of these oscillations are sufficiently low, we find that 2D channels can retain their unique properties which are demonstrated under dc conditions. Therefore, we can expect to observe oscillatory processes under the conditions of the quantum Hall effect. In view of this, we shall study low-frequency ($\omega \ll \omega_c$) oscillations which can appear in media with zero diagonal components of the electrical conductivity tensor, i.e., oscillations of the Hall current. We shall show that if the delay effects are ignored, such oscillations can appear only in the presence of an inhomogeneity in the system. In particular, the boundaries of a sample can act as such an inhomogeneity. We are therefore facing the problem of natural oscillations in bounded quantum superlattices.

It is formally simpler to discuss a superlattice than a bounded single 2D layer if the superlattice can be modeled by a continuous anisotropic medium. The natural oscillations in bounded superlattices discussed below are closely related to surface magnetoplasmons, which seems to be natural in view of the important role played by boundaries in the oscillation mechanism. Natural oscillations of the Hall current in a superlattice will be shown to be formally very similar to oscillations of the magnetization in bounded ferromagnets (Walker modes[5]). The similarity between oscillations of the Hall current and the Walker modes is due to the symmetry of the Maxwell equations when a magnetic field is replaced with an electric one, and also due to the circumstance that oscillations of the magnetization in the Walker modes, like oscillations of the Hall current are two-dimensional, i.e., the corresponding modes depend on coordinates in planes perpendicular to a static magnetic field. We shall show that the distribution of the electric field in oscillations in superlattices is simply identical in form with the distribution of the magnetic field in the corresponding Walker modes. We can therefore say that oscillations in a superlattice are an "electric" analog of magnetostatic oscillations in ferromagnets. We shall stress the similarity by referring to oscillations of the Hall current in a superlattice as electrostatic oscillations.

We shall now formulate the main assumptions used in the present study.

1. The distance d between the 2D channels in a superlattice is much less than the characteristic dimensions of a change in the field (and, in particular, the dimensions of a sample). In this case we can describe electrical properties of a superlattice by means of the average values of the electrical conductivity ($\hat{\sigma}$) and permittivity ($\hat{\varepsilon}$) tensors:

$$\hat{\sigma}=d^{-1}\hat{\bar{\sigma}}, \quad \hat{\varepsilon}=\varepsilon_0-4\pi i\hat{\sigma}/\omega. \quad (2)$$

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Here, ε_0 is the average permittivity of the semiconductors composing the superlattice; ω is the frequency of the field; the time dependence is selected to be $\exp(i\omega t)$. Therefore, a superlattice can be modeled by an anisotropic ($\sigma_{zz} = 0$) homogeneous medium.

2. We shall also assume that $\sigma_{xx} = 0$. We shall show that this assumption does not distort the results significantly if σ_{xx} is much smaller than the frequencies of natural oscillations of a superlattice found below. Since these natural frequencies are often of the order of σ_{xy} ($\sigma_{xy} = d^1 \bar{\sigma}_{xy}$), we then find that $\sigma_{xx} \ll \sigma_{xy}$. Clearly, the values of the external magnetic field most favorable for satisfying this condition correspond to the filling of an integral number of the Landau levels in the 2D channels of the superlattice.

3. We shall consider sufficiently low frequencies (in particular, we shall assume that $\omega \ll \omega_c = eH/m^*c$), so that we can ignore the frequency dependence of σ_{xy}.

4. We shall employ a quasistationary approximation, i.e., we shall ignore the delay effects. This approximation is known to be valid if the frequency is sufficiently low. It implies that we can ignore the vortex electric field ($\mathbf{E}_1$) compared with the potential field ($\mathbf{E}$). If the characteristic dimension in the change of the field in an oscillation is a, it follows from the Maxwell equations that

$$\pi a H \sim \left(\frac{\omega\varepsilon_0}{c} + \frac{4\pi}{c}\sigma_{xy}\right) a^2 E, \quad \pi a E_1 \sim \frac{\omega}{c} H a^2.$$

Hence, we find that

$$E_1/E \sim 4(a/\lambda)^2(1+4\pi\sigma_{xy}/\varepsilon_0\omega), \tag{3}$$

where $\lambda = 2\pi c\varepsilon_0^{-1/2}\omega^{-1}$ is the electromagnetic wavelength in a medium with a permittivity ε_0.

We shall show below that in cases of practical interest the quantity a should be regarded as of the order of the size of a sample. If the right-hand side of Eq. (3) is much less than unity, then the quasistationary approximation is valid. We shall derive formulas relating the oscillation frequency ω to the value of σ_{xy} and thus use Eq. (3) to determine (set the upper limit to) the permissible values of σ_{xy}.

We shall first consider qualitatively the mechanism of the appearance of electrostatic oscillations. First of all, we note that in the quasistationary approximation and when the condition $\sigma_{xx} = 0$ is met an electric charge cannot accumulate (or change) within the superlattice. The quasistationary approximation means that the electric field in the superlattice can be represented by $\mathbf{E} = \nabla\psi$ and the equation of continuity

$$\partial\rho/\partial t + \operatorname{div} \mathbf{j} = 0$$

yields

$$\partial\rho/\partial t = \sigma_{xy}(-\partial^2\psi/\partial x\partial y + \partial^2\psi/\partial y\partial x) = 0.$$

However, at the boundary of the superlattice an electric charge may accumulate and this circumstance is decisive for the appearance of low-frequency oscillations ($\omega \ll \omega_c$) in bounded superlattices. We shall assume that on the surface of a sample there is a certain charge distribution (and the total charge of each 2D layer is zero). The surface charge creates an electric field inside the sample and this induces a Hall current (we are ignoring the ohmic current). The Hall current carries charges to the surface of the sample and thus alters the initial distribution of the surface charge.

There is a set of distributions of the surface charge (this will be shown later) for which this change simply reduces to rotation about the direction of a static external magnetic field. In other words, the surface charge induces the Hall current which seems to rotate the initial charge and, therefore, the electric field in the sample. The rotated field rotates the Hall current, and so on. This allows us to estimate the frequency of such oscillations. We shall assume that the sample is spherical (with a diameter a) and it consists of a superlattice. On one part of the surface of the sample there is a charge $+q$ and on the other the charge is $-q$. Since in the quasistationary approximation the field E is determined entirely by the electric charge, it follows that $E \propto q/a^2\varepsilon_0$. The density of the Hall current is $j \propto \sigma_{xy}E \propto \sigma_{xy}q/a^2\varepsilon_0$ and the total Hall current is of the order of $I \propto ja^2 \propto q\sigma_{xy}/\varepsilon_0$. We shall find the oscillation period from the condition $IT \propto q$ which gives the period $T \propto \varepsilon_0\sigma_{xy}^{-1}$ and the oscillation frequency $\omega \propto \sigma_{xy}\varepsilon_0^{-1}$.

2. QUANTITATIVE ANALYSIS

In the quasistationary approximation the Maxwell equations reduce to

$$\operatorname{curl} \mathbf{E}=0 \quad \text{or} \quad \mathbf{E}=\nabla\psi, \tag{4}$$

$$\operatorname{div} \mathbf{D}=0, \tag{5}$$

where

$$\mathbf{D}=\hat{\varepsilon}\mathbf{E} \tag{6}$$

and $\hat{\varepsilon}$ is defined by Eqs. (1) and (2). Adopting the Cartesian coordinates (we recall that the x and y axes are parallel to the 2D layers and the z axis is perpendicular to these layers), we obtain

$$\left(1-\frac{4\pi i}{\varepsilon_0\omega}\sigma_{xx}\right)\left(\frac{\partial^2\psi^{(i)}}{\partial x^2}+\frac{\partial^2\psi^{(i)}}{\partial y^2}\right)+\frac{\partial^2\psi^{(i)}}{\partial z^2}=0, \tag{7}$$

$$\Delta\psi^{(e)}=0. \tag{8}$$

In Eqs. (7) and (8) the quantities $\psi^{(i)}$ and $\psi^{(e)}$ represent the values of the function ψ inside and outside the sample, respectively. Equations (4) and (5) yield the boundary conditions of continuity of the potential function ψ and of the normal (to the surface) component of the electric induction vector $\mathbf{D}$:

$$\psi^{(i)}|_s=\psi^{(e)}|_s, \tag{9}$$

$$\mathbf{n}_0[\varepsilon_0\nabla\psi^{(i)}-(4\pi i/\omega)\hat{\sigma}\nabla\psi^{(i)}]_s=\mathbf{n}_0\nabla\psi^{(e)}|_s. \tag{10}$$

Here, $\mathbf{n}_0$ is a unit vector along the normal to the surface of the sample. As already mentioned, we shall use the approximation $\sigma_{xx} = 0$. It follows from Eq. (7) that this is permissible if

$$4\pi\sigma_{xx}/\varepsilon_0\omega \ll 1. \tag{11}$$

The real part of σ_{xx} describes the damping, whereas the imaginary contributes to the frequency of electrostatic oscillations. Since $\varepsilon_0 \sim 10$ for superlattices of the GaAs–AlGaAs type, Eq. (11) indicates that we can ignore the damping if $\operatorname{Re} \sigma_{xx} \ll \omega$. This is one of the principal approximations we shall make in this study. It should be pointed out that oscillations of this nature have been observed experimentally in a single 2D channel of a GaAs–AlGaAs heterostructure in the frequency range 10^8–10^9 Hz (Ref. 6). This supports the hypothesis that weakly damped low-frequency oscillations exist in a superlattice. We can estimate the correction to the frequency of electrostatic oscillations associated with the imaginary part of σ_{xx} on the basis of the Drude model. In this model we find that if $\omega \ll \omega_c$, then

$$\operatorname{Im} \sigma_{xx} \approx \omega_p^2 \omega / 4\pi\omega_c^2, \quad \omega_p^2 = 4\pi^2 e n / m^*.$$

We can see that the correction to the frequency of electrostatic oscillations $\operatorname{Im} \sigma_{xx}$ can be ignored if

$$\omega_p^2/\varepsilon_0\omega_c^2 \ll 1.$$

This condition is easily satisfied. For example, if the distance between the 2D layers in a superlattice is $d = 1\,\mu$, the density of electrons in a layer is $\bar{n} \sim 10^{11}$ cm^{-2}, the effective mass is $m^* \simeq 0.06 m_0$, and the applied magnetic field is $B \sim 10$ T, we find that $\varepsilon_0^{-1}\omega_p^2\omega_c^{-2} \approx 10^{-4} \ll 1$. If $\sigma_{xx} = 0$, Eq. (7) for $\Psi^{(i)}$, known in the theory of magnetism as the Walker equation,[5] reduces to the Laplace equation

$$\Delta \psi^{(i)} = 0. \tag{12}$$

We therefore have to solve Eqs. (8) and (12) subject to the boundary conditions of Eqs. (9) and (10).

We shall first consider a spherical sample of radius R. We shall introduce spherical coordinates r, θ, and φ:

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta.$$

We can easily show that the boundary condition of Eq. (10) is of the following form in terms of these spherical coordinates:

$$\varepsilon_0 \frac{\partial \psi^{(i)}}{\partial r} + \frac{4\pi i}{\omega R} \sigma_{xy} \frac{\partial \psi^{(i)}}{\partial \varphi} = \frac{\partial \psi^{(e)}}{\partial r}. \tag{13}$$

These solutions of Eqs. (8) and (12), which are finite everywhere and satisfy the boundary condition (9), can be written in the form[7]

$$\psi_{n,m}^{(i)} = R^{-2n-1} r^n P_n{}^m (\cos\theta) e^{im\varphi}, \tag{4}$$

$$\psi_{n,m}^{(e)} = r^{-n-1} P_n{}^m (\cos\theta) e^{im\varphi}, \tag{15}$$

where P_n^m are the associated Legendre polynomials of the first kind. Substituting the solutions (14) and (15) into the boundary condition (13), we obtain the dispersion equation for the frequencies of natural oscillations of a sphere:

$$\omega_{n,m} = 4\pi\sigma_{xy} m / [n(\varepsilon_0 + 1) + 1]. \tag{16}$$

The case of electrostatic oscillations with $m = n = 1$ was discussed in Ref. 8 by a different method. In the present case the numbers n and m are integers such that $0 < m \leqslant n$. The solution with $m = 0$ should be considered separately, because Eq. (2) loses its physical meaning for $m = 0$. An analysis shows that this solution describes a situation when constant annular currents flow in a sample, but the total charge on the boundary of a single layer is no longer zero (although the total charge on the surface of a sample is still zero). Since in a superlattice the charge cannot leak from one 2D layer to another, the solution with $m = 0$ should be rejected. We can therefore see that a spherical sample can support natural electrostatic oscillations and that their frequency is independent of the size of the sample.

Since the electric field of electrostatic oscillations exists also outside a sample (the sample is then called an "open" resonator), the ambient medium may affect the resonance frequencies. We shall consider the case which is simple to tackle by calculation when a spherical sample is surrounded by a concentric metal sphere of radius R_1. Then, the solutions of Eqs. (8) and (12), which are finite inside the metal sphere, can be written in the form

$$\psi_{n,m}^{(i)} = C r^n P_n{}^m (\cos\theta) e^{im\varphi}, \tag{17}$$

$$\psi_{n,m}^{(e)} = r^{-n-1} P_n{}^m (\cos\theta) e^{im\varphi} - A r^n P_n{}^m (\cos\theta) e^{im\varphi}. \tag{18}$$

We shall regard the metal sphere as perfectly conducting. Then the following boundary condition is obeyed:

$$\psi_{n,m}^{(e)}(R_1) = 0. \tag{19}$$

The boundary condition of Eq. (8) reduces to

$$\psi_{n,m}^{(i)}(R) = \psi_{n,m}^{(e)}(R). \tag{20}$$

The conditions (19) and (20) allow us to determine the constants C and A in Eqs. (17) and (18). Substituting then Eqs. (17) and (18) into the boundary condition (10), we obtain the following expression for the frequencies of natural oscillations of the system under discussion:

$$\omega_{n,m} = 4\pi\sigma_{xy} m (1-x) [\varepsilon_0 n (1-x) + n(1+x) + 1]^{-1}, \tag{21}$$

where $x = (R/R_1)^{2n+1}$. It follows from Eq. (21) that $\omega_{n,m} \to 0$ if $R_1 \to R$. Therefore, a metal sphere can reduce considerably the natural frequencies of electrostatic oscillations. The reduction in $\omega_{n,m}$ occurs because charges appear on the metal surface and the sign of these is opposite to the sign on the nearby surface of the sample (superlattice). This reduces the electric field and the Hall currents in the sample and, consequently, lowers the natural frequency. Clearly, this conclusion applies to samples of any shape.

We shall now consider samples in the form of an ellipsoid of revolution (spheroid). The equation for a spheroid in Cartesian coordinates is

$$(x^2 + y^2)/a^2 + z^2/b^2 = 1. \tag{22}$$

Here, a and b are the semiaxes of the spheroid, where b is the rotation axis directed along the magnetic field. We shall introduce spheroidal coordinates ξ, η, and φ (Fig. 1):

$$x^2 = c^2(1+\xi^2)(1-\eta^2)\cos^2\varphi,$$

$$y^2 = c^2(1+\xi^2)(1-\eta^2)\sin^2\varphi, \quad z = c\eta\xi. \tag{23}$$

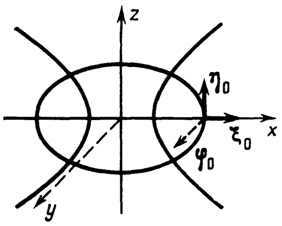

FIG. 1. System of coordinates of a oblate spheroid (showing transverse section). The coordinate surfaces are: oblate spheroids, ξ = const; 2) uniaxial hyperboloids, ξ = const; 3) half-planes φ = const, passing through the z axis.

In these formulas we have $c^2 = a^2 - b^2$. We shall now assume that $a > b$, i.e., we shall consider an oblate (flattened) spheroid. The surface of a sample is described by the following equation in terms of the spheroidal coordinates:

$$\xi = \xi_0 = b/c. \tag{24}$$

It is known from Ref. 7 that the solutions of the Laplace equations (8) and (12), finite everywhere and satisfying the boundary condition (9), can be written in the form

$$\psi_{n,m}^{(e)} = Q_n{}^m(i\xi)P_n{}^m(\eta)e^{im\varphi}, \tag{25}$$

$$\psi_{n,m}^{(i)} = \frac{Q_n{}^m(i\xi_0)}{P_n{}^m(i\xi_0)}P_n{}^m(i\xi)P_n{}^m(\eta)e^{im\varphi}, \tag{26}$$

where Q_n^m are the associated Legendre polynomials of the second kind. We shall now write down the boundary condition (10) in spheroidal coordinates. We shall consider separately the term $\mathbf{n}_0\hat{\sigma}\nabla\psi^{(i)}$, which stands on the right-hand side of the condition (10). We shall expand the electric field at the boundary of the sample into the components along the coordinate lines ξ, η, and φ (Fig. 1):

$$\nabla\psi^{(i)} = \mathbf{E}_\varphi^{(i)} + \mathbf{E}_\eta^{(i)} + \mathbf{E}_\xi^{(i)}.$$

If $\sigma_{xx} = 0$, we can readily see that the Hall currents due to the components $\mathbf{E}_\eta^{(i)}$ and $\mathbf{E}_\xi^{(i)}$ flow along the surface and, moreover,

$$\mathbf{n}_0\hat{\sigma}\mathbf{E}_\eta^{(i)} = \mathbf{n}_0\hat{\sigma}\mathbf{E}_\xi^{(i)} = 0.$$

It now remains to consider the expression $\mathbf{n}_0\hat{\sigma}\mathbf{E}_\varphi^{(i)}$. We have

$$\mathbf{n}_0\hat{\sigma}\mathbf{E}_\varphi^{(i)} = j_x^{(i)}\cos(\widehat{\mathbf{n}_0\mathbf{x}_0}) + j_y^{(i)}\cos(\widehat{\mathbf{n}_0\mathbf{y}_0}). \tag{27}$$

Here, $\mathbf{x}_0$ and $\mathbf{y}_0$ are unit vectors directed along the indicated Cartesian axes; $j_x^{(i)}$ and $j_y^{(i)}$ are the components of the Hall current created by the field $\mathbf{E}_\varphi^{(i)}$:

$$j_x^{(i)} = -\sigma_{xy}E_\varphi^{(i)}\cos\varphi, \quad j_y^{(i)} = \sigma_{xy}E_\varphi^{(i)}\sin\varphi. \tag{28}$$

The direction cosines of the normal to the boundary of a sample ($\mathbf{n}_0$ is directed along the coordinate line ξ) can be obtained from the system (23):

$$\begin{aligned}\cos(\widehat{\mathbf{n}_0\mathbf{x}_0}) &= \left(\frac{1-\eta^2}{a^2} + \frac{\eta^2}{b^2}\right)^{-1/2}\frac{(1-\eta^2)^{1/2}}{a}\cos\varphi,\\ \cos(\widehat{\mathbf{n}_0\mathbf{y}_0}) &= \left(\frac{1-\eta^2}{a^2} + \frac{\eta^2}{b^2}\right)^{-1/2}\frac{(1-\eta^2)^{1/2}}{a}\sin\varphi.\end{aligned} \tag{29}$$

Substituting Eqs. (28) and (29) into Eq. (27), we find that

$$\mathbf{n}_0\hat{\sigma}\mathbf{E}_\varphi^{(i)} = -\sigma_{xy}E_\varphi^{(i)}[(1-\eta^2)/a^2 + \eta^2/b^2]^{-1/2}(1-\eta^2)^{1/2}/a. \tag{30}$$

In the case of $E_\varphi^{(1)}$, we have

$$E_\varphi^{(i)} = h_\varphi^{-1}\partial\psi^{(i)}/\partial\varphi = c^{-1}(1+\xi^2)^{-1/2}(1-\eta^2)^{-1/2}\partial\psi^{(i)}/\partial\varphi. \tag{31}$$

Here, h_φ is the corresponding Lamé coefficient.

At the boundary of a sample we have $\xi = \xi_0 = b/c$ and

$$E_\varphi^{(i)} = a^{-1}(1-\eta^2)^{-1/2}\partial\psi^{(i)}/\partial\varphi. \tag{32}$$

Using Eqs. (30) and (32), we obtain the boundary condition (10) in the form

$$\frac{\varepsilon_0}{b(a^2-b^2)^{1/2}}\frac{\partial\psi^{(i)}}{\partial\xi} + \frac{4\pi i}{\omega}\frac{\sigma_{xy}}{a^2}\frac{\partial\psi^{(i)}}{\partial\varphi} = \frac{1}{b(a^2-b^2)^{1/2}}\frac{\partial\psi^{(e)}}{\partial\xi}. \tag{33}$$

Substituting into Eq. (33) the solutions (25) and (26) we get the dispersion equation for natural frequencies of a spheroid:

$$i\xi_0\left\{\varepsilon_0\frac{P_n{}^{m'}(i\xi_0)}{P_n{}^m(i\xi_0)} - \frac{Q_n{}^{m'}(i\xi_0)}{Q_n{}^m(i\xi_0)}\right\} = \frac{4\pi}{\omega_{n,m}}\sigma_{xy}\frac{b^2}{a^2}m, \tag{34}$$

where

$$\begin{aligned}P_n{}^{m'}(i\xi_0) &= \frac{d}{dz}P_n{}^m(z)\big|_{z=i\xi_0} = \frac{1}{i}\frac{d}{d\xi_0}P_n{}^m(i\xi_0),\\ Q_n{}^{m'}(i\xi_0) &= \frac{1}{i}\frac{d}{d\xi_0}Q_n{}^m(i\xi_0).\end{aligned} \tag{35}$$

We shall consider the natural frequencies $\omega_{n,m}$ of some of the simplest oscillation modes in the case of a strongly oblate ($b \ll a$) ellipsoid of revolution. The expressions for the Legendre polynomials in the case of low values of the argument reduce to

$$\begin{aligned}&P_1{}^1(ix) = 1 + O(x^2), && Q_1{}^1(ix) = 2x - {}^1/_2\pi + O(x^3),\\ &P_2{}^1(ix) = 3ix + O(x^3), && Q_2{}^1(ix) = 2i - {}^3/_2\pi ix + O(x^2),\\ &P_2{}^2(ix) = 3 + O(x^2), && Q_2{}^2(ix) = {}^3/_2\pi i - 8ix + O(x^2),\\ &P_3{}^1(ix) = -{}^3/_2 + O(x^2), && Q_3{}^1(ix) = {}^3/_4\pi - 8x + O(x^2),\\ &P_3{}^2(ix) = 15ix + O(x^3), && Q_3{}^2(ix) = 8 - {}^{15}/_2\pi x + O(x^2).\end{aligned}$$

Using these formulas, we find from Eqs. (34) and (35) that

$$\begin{aligned}&\omega_{1,1} \approx \pi^2\sigma_{xy}\frac{b}{a}, \quad \omega_{2,1} \approx \frac{4\pi\sigma_{xy}}{\varepsilon_0}\frac{b^2}{a^2},\\ &\omega_{2,2} \approx \frac{3}{2}\pi^2\sigma_{xy}\frac{b}{a}, \quad \omega_{3,1} \approx \frac{3\pi^2}{8}\sigma_{xy}\frac{b}{a}, \quad \omega_{3,2} \approx \frac{8\pi\sigma_{xy}}{\varepsilon_0}\frac{b^2}{a^2}.\end{aligned} \tag{36}$$

In these expressions we retain the terms with the lowest power of the parameter b/a.

We shall now consider the expressions for the potential function $\psi_{n,m}$. In the case of $\psi_{1,1}$ we have

$$\psi_{1,1} = P_1{}^1(i\xi)P_1{}^1(\eta)e^{i(\varphi+\omega t)} = [(1+\xi^2)(1-\eta^2)]^{1/2}e^{i(\varphi+\omega t)}. \tag{37}$$

Using the transformation of Eq. (23), we obtain

$$\psi_{1,1} \sim (x+iy)e^{i\omega t}. \tag{38}$$

Similarly, in the case of $\psi_{2,2}$ we have

$$\psi_{2,2} = (1+\xi^2)(1-\eta^2)e^{i(2\varphi+\omega t)} \sim (x+iy)^2e^{i\omega t}, \tag{39}$$

and we can generally show that

$$\psi_{n,n} \sim (x+iy)^ne^{i\omega t}. \tag{40}$$

If $m = n - 1$, then

$$\psi_{m+1,m} \sim z(x+iy)^me^{i\omega t}. \tag{41}$$

In the remaining cases the expressions for $\psi_{n,m}$ are more complex. For example, we have

$$\psi_{3,1} = -\left(\frac{3}{2}\right)^2\frac{x+iy}{c}\frac{20z^2-5(x^2+y^2)+4c^2}{c^2}e^{i\omega t}. \tag{42}$$

In contrast to the expressions in the system (36), Eqs. (37)–(42) are valid for any (and not just small) values of b/a. In particular, they are valid for a spherical sample. The structure of Eqs. (38)–(42) shows that in a coordinate system rotating about the axis z in the counterclockwise direction at an angular frequency ω/m, the function $\psi_{n,m}$ is independent of time. The functions $\psi_{n,m}$ and $\psi_{n,n-1}$ are identical with the functions of the magnetic potentials $\psi_{n,n,0}$ and $\psi_{n,n-1,0}$ obtained by Walker.[5] The expressions for the other functions $\psi_{n,m}$ also show a strong formal similarity to the functions of the magnetic potential.[9] It therefore follows that a superlattice may exhibit oscillations in which the electric field is distributed exactly in the same way as the magnetic field in the corresponding magnetostatic oscillations.

The expressions for the natural frequencies given by Eq. (36) for the case of a strongly flattened spheroid are similar to the corresponding expressions for the Walker modes. It is the frequencies of different (even or odd relative to the coordinate z) oscillations that approach in a different way the limiting value (zero in the case of electrostatic oscillations and a finite value in the magnetic case) as b/a approaches zero. For example, $\omega_{n,n} \sim b/a$, whereas $\omega_{n,n-1} \sim (b/a)^2$. This result is important in the experiments because it provides a technique for determining very low natural frequencies of electrostatic oscillations. Obviously, we need to use a sample with a sufficiently small ratio b/a and excite oscillations which are odd relative to the coordinate z. The reason why oscillations with $m = n$ and $m = n - 1$ behave differently on approach of b/a to zero can be demonstrated by considering oscillations with $m = n = 1$ and $m = 1$, $n = 2$. In the former cases the oscillations of the field in all the 2D layers of the investigated spheroid are in phase, whereas in the latter case the oscillations in the 2D layers with $z > 0$ are shifted in phase by 180° relative to the oscillations in layers characterized by $z < 0$. In other words, parts of the spheroid surface with the same coordinates x and y and with identical magnitudes but different signs of the coordinate z carry opposite charges. Therefore, the electric field inside the spheroid is weaker because of mutual compensation of the charges at the boundaries with $z > 0$ and $z < 0$. The degree of decompensation increases on reduction in the ratio b/a. A reduction of the internal field results in an additional (by a factor of b/a) reduction in the natural frequency of electromagnetic oscillations. An analogy with the Walker modes is supported also by experimental observations of electrostatic oscillations. The oscillations with $m = n = 1$ can be excited by a homogeneous external alternating electric field. The other types of electrostatic oscillations appear in a sample subjected to an inhomogeneous external field of suitable symmetry. In experiments carried out in the microwave range it is desirable to use the methods of Ref. 10 (naturally, replacing magnetic fields in the resonator with electric fields).

An increase in the indices m and n drives the electric field and current associated with electrostatic oscillations toward the surface of a sample [this follows from Eqs. (17) and (25)]; electrostatic oscillations in this case are of small scale compared with the dimensions of an excited sample. Since the influence of the shape of a sample on such perturbations is fairly weak, small-scale excitations can clearly be regarded as due to a surface magnetoplasma.[11] Such a description has the advantage that it allows us to classify excitations by means of the quasimomentum. However, large-scale excitations are described satisfactorily as electrostatic oscillations.

Clearly, the distinguishing property of electrostatic oscillations [which follows from Eq. (16) and (34)] is independence of natural frequencies of the absolute dimensions of a body when its shape is maintained. This property does not allow us to interpret electrostatic oscillations as resonances which occur when a certain relationship between the wavelength of a surface magnetoplasmon and the size of the sample is obeyed. From the logic point of view the relationship between electrostatic oscillations and surface magnetoplasmons is analogous to the relationship between the Walker modes and spin waves.[5]

We shall now obtain numerical estimates. A characteristic conductivity of a single 2D layer is $\bar{\sigma}_{xy} \sim (10^4\ \Omega)^{-1} \sim 10^8$ cm/sec. At a distance of $d \sim 10^{-3}$ cm we obtain $\sigma_{xy} = d^{-1}\bar{\sigma}_{xy} \sim 10^{11}$ sec^{-1} and in the case of a spherical sample we find that $\omega_{1,1} \sim 10^{11}$ sec^{-1} ($f_{1,1} \sim 15$ GHz). For these values of σ_{xy} the quasistationary approximation of Eq. (3) limits the size of the sample to $a \lesssim 1$ mm. Therefore, the quasistationary approximation imposes quite stringent conditions on the value of σ_{xy} and, consequently, on the distance between the 2D channels. When this distance is reduced, σ_{xy} and $\omega_{n,m}$ increase and the influence of the delay effects becomes stronger. However, in the case of oscillations which are odd relative to $z (m = n - 1)$ the delay effects in a strongly flattened spheroid are unimportant even for $d \ll 10^{-3}$ cm. This is due to the low natural frequency of such oscillations. If $d \sim 1000$ Å and $\bar{\sigma}_{xy} \sim 10^8$ cm/sec, we obtain $\sigma_{xy} \sim 10^{13}$ sec^{-1} and for $b/a \sim 10^{-2}$ Eq. (36) yields $\omega_{2,1} \sim 10^9$ sec$^{-1} \sim 150$ MHz. Equation (3) imposes the limit $a \leqslant 1$ mm on the spheroid diameter. However, oscillations with $m \neq n - 1$ clearly will not be observed in such a sample.

Our calculation shows that electrostatic oscillations are possible in any bounded body the conductivity of which is described by a tensor of the Eq. (1) type. A superlattice is an actual (and possibly one of the most interesting, because of the quantum Hall effect) realization of such a body. Important properties of a superlattice necessary for the observation of electrostatic oscillations is the anisotropy ($\sigma_{zz} = 0$, and in fact σ_{zz} can be subjected to the same limitations as σ_{xx}) and fairly low values of σ_{xy} (obtained because of large values of d). The latter property is essential in order to satisfy the quasistationary approximation. Oscillations with $m = n$ can exist also in media with an arbitrary value of σ_z, because for these oscillations we have $E_z = 0$. Clearly, electrostatic oscillations should also be observed in a superlattice with two types of carrier.

3. CONCLUSIONS

We investigated natural oscillations of a bounded anisotropic ($\sigma_{zz} = 0$) and gyrotropic ($\sigma_{xy} = -\sigma_{yx}, \sigma_{xx} = \sigma_{yy}$) conducting sphere. A practical realization of such a sphere is a quantum superlattice in a strong magnetic field applied at right-angles to 2D layers. It is shown above that natural oscillations of the Hall current can occur in such a medium and they represent the limiting case of surface magnetoplasma oscillations under conditions when the spatial scale of the change in the fields and currents is comparable with the dimensions of a sample. A distinguishing feature of these oscillations, which are called electrostatic, is the fact that their natural frequencies depend only on the shape and not on the absolute dimensions of a sample. There is a formal analogy between electrostatic oscillations in a superlattice and magnetostatic oscillations (Walker modes) in bounded ferromagnets; the relationship between electrostatic oscillations and surface magnetoplasmons is similar to the relationship between the Walker modes and spin waves.

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[1]A. H. MacDonald, T. M. Rice, and W. F. Brinkman, Phys. Rev. B **28**, 3648 (1983).
[2]D. J. Thouless, J. Phys. C **18**, 6211 (1985).
[3]J. B. Sokoloff, J. Phys. C **18**, L997 (1985).
[4]J. B. Sokoloff, Phys. Rev. B **31**, 1924 (1985).
[5]L. R. Walker, Phys. Rev. **105**, 390 (1957).
[6]S. A. Govorkov, M. I. Reznikov, A. P. Senichkin, and V. I. Tal'yanskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 380 (1986) [JETP Lett. **44**, 487 (1986)].
[7]P. M. Morse and H. Feshbach, Methods of Theoretical Physics, 2 vols., McGraw-Hill, New York (1953).
[8]V. I. Tal'yanskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 96 (1986) [JETP Lett. **43**, 127 (1986)].
[9]P. C. Fletcher and R. O. Bell, J. Appl. Phys. **30**, 687 (1959).
[10]J. F. Dillon Jr., Phys. Rev. **112**, 59 (1958).
[11]Ji-Wei Wu, P. Hawrylak, G. Eliasson, and J. J. Quinn, Phys. Rev. B **33**, 7091 (1986).

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