Red shift, horizon potential, and solitary wave in the electrodynamics of a uniformly accelerated charge

V. I. Ritus

P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow
(Submitted 26 November 1986)
Zh. Eksp. Teor. Fiz. **92**, 1936–1945 (June 1987)

The global properties of the field of a uniformly accelerated charge due to its mass shift are considered. Relativistic invariance and causality are used to establish a relationship between the integrated characteristics of the fields moving with the charge and the radiation fields. The mass shift is the red shift of the energy of the dragged field. It is also equal to the momentum transported by the radiation field in the direction of acceleration. The shift can be expressed in terms of the nonzero horizon potential, and can be related to the change in the topology of the manifold occupied by the field, by expressing the self-energy as half the product of the charge and the change in the self-field potential. It is shown that the radiation field has a discrete dynamic radius-time symmetry and forms a solitary light wave carrying unit electric field flux. Arguments are presented in favor of the mass shift being proportional to the spin of the self-field.

I. INTRODUCTION

It was found in Ref. 1 that the real part of the electron mass shift in a constant uniform electric field ε, where $\varepsilon \ll m^2c^3/e\hbar$, has the classical limit[1]

$$\operatorname{Re}\Delta m^{\text{кл}} = -\frac{1}{2}\alpha\frac{|e\varepsilon|}{m}. \tag{1}$$

This shift is not predicted by the classical Abraham-Lorentz-Dirac equation that takes into account the local self-interaction effects of the charge, which suggests that the shift is of nonlocal origin. Another distinctive property of Re Δm^{cl} is that it is linear in the modulus of e, i.e., it is nonanalytic and, consequently, nonperturbative in the external field. Although it does not change the classical equation of motion, the shift does alter the expression for the action,[1,2] and should therefore manifest itself in quantum-mechanical processes.

Together with its imaginary part, which, in the same limit, is given by

$$\operatorname{Im}\Delta m \approx -\frac{\alpha}{2\pi}m\beta\left(2\ln\frac{2\beta m}{\gamma\mu}-1\right),\quad \beta=\frac{|e\varepsilon|}{m^2}\ll 1, \tag{2}$$

the mass shift determines the probability amplitude exp $(-i\Delta m\tau)$ of finding the electron in the electric field in a state with zero transverse momentum $p_\perp = 0$ during the proper time τ. The mass μ of the photon, introduced to remove the infrared divergence, can be replaced with the minimum wave number $k_{\perp\min} = \mu c/\hbar$ of the radiated quanta, which also determines the precision with which the transverse momentum $p_\perp$ of the electron can be measured. It follows that $-2\operatorname{Im}\Delta m$ is the photon emission probability per unit proper time, or the frequency of transition of the electron from the state with $p_\perp = 0$ to a state with $p_\perp \neq 0$. In principle, such inverse transitions are observable. The differential probability that they will occur (i.e., the $p_\perp \to p'_\perp$ transition probability) was deduced by Nikoshov.[3] We have shown[4] that the real part of the shift enhances pair production by the electric field.

In the present paper, we examine the global properties of the self-field of a uniformly accelerated electron that are in some way related to the mass shift. In Section 2, we use relativistic invariance and causality to derive the relation, given by (4), between the integrals of the Lagrange function of the self-field in the invariance regions of Rindler and Milne, which is valid for any time $t > 0$. In other words, we establish the relationship between the global dynamic characteristics of the attached fields and the fields radiated by the charge. This enables us to express the mass shift in terms of the field in the Rindler region, in which it constitutes the red shift of the energy of the attached field, or in terms of the field in the Milne region, in which it is equal to the momentum transported with the field in the direction of acceleration through a surface perpendicular to the acceleration (Sections 3 and 4). In the same Section, we give the covariant generalization of the well-known formula of classical electrodynamics for the self-energy (half the product of the charge and the change in the self-field potential; see Ref. 5, Section 37), in which the mass shift is represented by a finite term describing the change in the topology of the manifold occupied by the field: because of the nonzero acceleration of the charge, a horizon with nonzero potential, which essentially determines the shift, appears at a finite distance from the charge. In Section 5, we examine the dynamic radius-time symmetry of the field due to the uniformly accelerated charge in the Milne system, which explains the absence of a mass shift in the case of the uniformly accelerated scalar charge and, when conservation of the electric field flux is taken into account, the magnitude of the change in the self-field potential, due to the uniformly accelerated electric charge. This symmetry emerges naturally in the course of the analytic continuation of the field from the Minkowski space R^4_1 to the space R^4_2 with zero signature, the result of which is the compactification of the second cyclic coordinate. Hence, if, in the Milne sector of the Minkowski space, the electromagnetic field evolves into a soliton in the course of time, then, in the space R^4_2, it takes the form of an instanton. Finally, in the concluding Section, we collect together arguments in favor of the conclusion that the mass shift of the uniformly accelerated charge is proportional to the spin of the self-field.

As in Refs. 1 and 2, we use the following expressions for the radiative correction to the action of a charge moving in

 0038-5646/87/061087-06$04.00

an external field:

$$\Delta W=\int_0^t dt\left[\int d^3x\frac{E^2-H^2}{2}\right]_0^F=\int_0^t dt\left[\int d^3x\,\partial_\alpha(F^{\alpha\beta}A_\beta)\right]_0^F, \quad (3)$$

where A_α, **E**, **H** are, respectively, the potential and the self-field intensities, and the indices F, 0 denote the difference between the values of the quantity in brackets for a charge in the external field and in vacuum for the same positions and velocities, at a given instant of time.

II. RELATIONSHIP BETWEEN GLOBAL CHARACTERISTICS OF THE ATTACHED AND RADIATED FIELDS DUE TO A UNIFORMLY ACCELERATED CHARGE

The field due to a uniformly accelerated charge traveling along the z-axis in accordance with the expression $z(t)=(w_0^{-2}+t^2)^{1/2}$ with proper acceleration w_0 occupies two relativistically invariant regions of space time, namely, the Rindler region $z>|t|$ and the Milne region $t>|z|$, and has a definite symmetry. The relativistic invariance of the regions R and M, and the symmetry of the field, ensure that these regions and the field transform into themselves under all Lorentz transformations for which the trajectory $z^2-t^2=w_0^{-2}$ of the uniformly accelerated charge transforms into itself (rotations around the z-axis and boosts along the z-axis). The significant difference between the Rindler and Milne regions is that the field in region R is entrained by the charge, i.e., it continues to interact with it, whereas the field in region M propagates independently, forming a solitary wave, and perturbing it does not affect the motion of the charge. It is precisely for this reason that the radiation due to the uniformly accelerated charge if found in region M.

If we use the field due to a uniformly accelerated charge given, for example, by Fulton and Rohrlich[6] or in our previous paper,[2] we can show by direct calculation that, for given $t>0$, the integrals of the Lagrange function of the field over spacelike projections of regions M and R are related by

$$\frac{1}{2}\int_{-t<z<t} d^3x\frac{E^2-H^2}{2}=-\int_{t<z<\infty} d^3x\frac{E^2-H^2}{2}\bigg|_0^F. \quad (4)$$

This is a consequence of relativistic invariance and causality. Actually, the integral of the Lagrange function of the field due to a uniformly accelerated charge, evaluated over the x, y plane,

$$\int dx\,dy\frac{E^2-H^2}{2}=\alpha w_0^2 g_1(u^2)=\frac{\alpha w_0^2}{2}\left(\frac{1}{1-u^2}+\frac{\operatorname{Arth}u}{u}\right)' \quad (5)$$

is an analytic function on the complex plane of u^2 with the cut $1\leqslant u^2<\infty$, where the prime represents the derivative with respect to $u^2=w_0^2(z^2-t^2)$. The function $g_1(u^2)$ assumes complex conjugate values at points that are symmetric relative to the real axis of u^2 and, in particular, on the edges of the cut. Hence the integal

$$\operatorname{Re}\int_0^\infty dz g_1(u^2)=\frac{1}{2}\int_{-\infty}^\infty dz g_1(u^2) \quad (6)$$

evaluated over the contour lying just above the real z-axis must be zero because $g_1(u^2)$ is analytic in the upper half-plane of z, and falls off rapidly at infinity. Hence, for any $t>0$,

$$\int_0^t dz g_1(u^2)=-\operatorname{Re}\int_t^\infty dz g_1(u^2). \quad (7)$$

This is actually equivalent to (4) because, by virtue of the above properties of $g_1(u^2)$, the left-hand integral is equal to

$$\frac{1}{2}\int_{-t}^t dz g_1(u^2),$$

and the integration with respect to z over a path above the pole $z=(w_0^{-2}+t^2)^{1/2}$ in the right-hand integral is equivalent to taking the difference $|_0^F$.

To prove (4) or (7), we have used: (1) relativistic invariance, according to which g_1 depends only on z^2-t^2 and the integrals are evaluated over the regions R and M, and (2) the analyticity of g_1 in the z^2-t^2 plane, except for the cut $w_0^{-2}\leqslant z^2-t^2<\infty$, which is a consequence of causality.

III. RINDLER SECTOR: THE RED SHIFT OF THE ENERGY OF THE ENTRAINED FIELD, AND THE HORIZON POTENTIAL

By virtue of (4), the change in the action for the field due to a uniformly accelerated charge can be expressed in terms of the field in the Rindler region. Here, it is natural to use the accelerated (Rindler) frame x', y', z', t' (Ref. 7), related to the inertial x,y,z,t frame by

$$x=x',\quad y=y',\quad z=z'\operatorname{ch}w_0t',\quad t=z'\operatorname{sh}w_0t' \quad (8)$$

and having the metric $g'_{\alpha\beta}=\operatorname{diag}(1,1,1,-w_0^2z'^2)$. On the Minkowski z,t diagram, the hypersurfaces of the accelerated system with constant z' are represented by hyperbolas, whereas hypersurfaces with constant t' (simultaneous events) are represented by straight lines passing through the coordinate origin. Planes in the Rindler system with fixed z' travel relative to the inertial system in the z direction with accelerations $1/z'$, so that the system is characterized by constant distances between its spatial points. The uniformly accelerated charge is at rest at the point $z'=w_0^{-1}, \rho=0$ in this system, its proper time is equal to t', and the field symmetry consists of the lack of dependence on the cyclic coordinates, i.e., the azimuthal angles φ and time t':

$$E_z'=-\frac{e(w_0^{-2}+\rho^2-z'^2)}{\pi w_0^2\xi'^3},\quad E_\rho'=\frac{2e\rho z'}{\pi w_0^2\xi'^3},$$
$$E_\varphi'=0,\quad \mathbf{H}'=0, \quad (9)$$
$$\xi'^2=(w_0^{-2}-z'^2-\rho^2)^2+4w_0^{-2}\rho^2.$$

Thus, the change in the action for the field due to the uniformly accelerated charge can be written in the form

$$\Delta W=-\int_{R_+t} d^4x\frac{E^2-H^2}{2}\bigg|_0^F=-\int_{R_+'\tau} d^4x'\sqrt{-g'}\frac{E'^2-H'^2}{2}\bigg|_0^F$$
$$=-t'\int_{0<z'<\infty} d^3x'\sqrt{-g'}\frac{E'^2}{2}\bigg|_0^F, \quad (10)$$

where $g'=\det g'_{\alpha\beta}$. In transforming to the Rindler coordinates, we have replaced the region R'_+, bounded by the hyperplanes $t=0$ and $t>0$, with the region R'_+, bounded by the hyperplanes $t'=0$ and $t'=\tau(t)$, where

$$\tau(t)=w_0^{-1}\operatorname{Arth}\frac{t}{(w_0^{-2}+t^2)^{1/2}}, \quad (11)$$

which is valid for $t\gg w_0^{-1}$.

Since $W = \tau \Delta m$, the last integral in (10) is the mass shift of the charge, which has an exceedingly simple interpretation in the accelerated frame, namely, it is the red shift of the energy of the self-field. In point of fact, general relativity (see Ref. 5, Sections 83, 84, and 88) shows that the energy $d^3x'(g'/g'_{00})^{1/2}E'^2/2$ of the field in the geometric volume element $d^3x'(g'/g'_{00})^{1/2}$ transforms into the energy

$$\left[d^3x'(g'/g_{00}')^{1/2}\frac{E'^2}{2}\right]\sqrt{-g_{00}'} \equiv d^3x'\sqrt{-g'}\frac{E'^2}{2}, \quad (12)$$

under displacement toward the charge, which, after summation over all the elements of the volume of the entrained field, forms the field mass of the charge. The difference between this and the field mass of the uniformly moving charge is

$$\Delta m^{cl} = \int\limits_{0<z'<\infty} d^3x'\sqrt{-g'}\frac{E'^2}{2}\Big|_0^F$$

$$= \alpha w_0 \int_0^\infty du u g_1(u^2) = -\frac{1}{2}\alpha w_0. \quad (13)$$

It is necessary to integrate above the singularity of $g_1(u^2)$ at $u^2 = 1$ [see (5)], which is equivalent to taking the difference $|_0^F$.

The expression given by (13) can be rewritten in the form of a conserved invariant (independently of the spacelike hypersurface S):

$$\Delta m^{\kappa\pi} = \int\limits_S dS_\alpha \sqrt{-g}\, T^\alpha{}_\beta \xi^\beta |_0^F, \quad (13')$$

where T^α_β is the energy-momentum tensor of the field due to the charge, ξ^β is the Killing vector field that generates boosts along the z-axis, and dS_α is a "volume" element on S. In the Rindler system, $\xi'^\beta = \delta^\beta_0$. The contribution due to the part of S lying in the Milne region is zero.

Let us now consider the expression for ΔW in terms of the covariant divergence of the vector $R^\alpha = F^{\alpha\beta}A_\beta$ in the Rindler system:

$$\Delta W = \frac{1}{2}\int\limits_{R_+\tau} d^4x'\partial_\alpha'\left(\sqrt{-g'}R'^\alpha\right)\Big|_0^F$$

$$= \frac{1}{2}\tau\int d^3x'\partial_\alpha'\left(\sqrt{-g'}R'^\alpha\right)\Big|_0^F. \quad (14)$$

Since $\sqrt{-g'}R'^\alpha$ is independent of t', integration with respect to t' gives the proper time τ of the charge and the three-dimensional divergence of the 3-vector $\sqrt{-g'}\mathbf{R}' = \sqrt{-g'}A'_t\mathbf{E}'$ remains under the integral sign. Transforming from Cartesian coordinates x, y, z' or cylindrical coordinates z', ρ, φ to the bispherical coordinates ψ, χ, φ (Ref. 8), we have

$$\rho = w_0^{-1}\sin\chi(\operatorname{ch}\psi - \cos\chi)^{-1}, \quad z' = w_0^{-1}\operatorname{sh}\psi(\operatorname{ch}\psi - \cos\chi)^{-1}, \quad (15)$$

in which the electric field has only the ψ-component,

$$E_\psi' = -\frac{ew_0^2}{4\pi}(\operatorname{ch}\psi - \cos\chi)^2, \quad F_{01}' = \frac{ew_0}{4\pi}\operatorname{sh}\psi, \quad (16)$$

and the covariant zero component of the potential depends only on ψ,

$$A_0' = -\sqrt{-g_{00}'}A_t' = -\frac{ew_0}{4\pi}\operatorname{ch}\psi, \quad (17)$$

and hence we find that

$$\int d^3x'\partial_\alpha'(\sqrt{-g'}R'^\alpha) = \int d\psi\, d\chi\, d\varphi\frac{\partial}{\partial\psi}(h_\chi h_\varphi\sqrt{-g_{00}'}R_\psi')$$

$$= -A_0'(\psi)\int d\sigma_\psi E_\psi'\Big|_{\psi_1=0}^{\psi_2=\infty} = eA_0'(\psi)\Big|_{\psi_1=0}^{\psi_2=\infty}. \quad (18)$$

The tensor indices with alphabetic rather than numerical values are always used for components in the locally comoving orthonormal frame, so that $A_x = \sqrt{g_{11}}A_1, \ldots,$ $A_t = \sqrt{-g_{00}}A^0$, and so on, $h_\psi, h_\chi, h_\varphi$ are the Lamé parameters, $ds_\psi = h_\psi d\psi$ is an element of length in the ψ-direction, and $d\sigma_\psi = h_\chi h d\chi d\varphi$ is an element of area on the $\psi =$ const surface in the bispherical system.[8]

Since $-e$ is equal to the flux of the electric field through a $\psi =$ const surface, the triple integral reduces to the difference between the values of the covariant zero component of potential on the boundaries of the Rindler region. The $\psi_1 = 0$ boundary coincides with the $z' = 0$ horizon surface and the $\psi_2 = \infty$ boundary coincides with the spherical surface centered on $\rho = 0$, $z' = w_0^{-1}\coth\psi_2$ having radius $r = (w_0\sinh\psi_2)^{-1} \to 0$, drawn around the charge, so that

$$eA_0'(\infty) = -\frac{e^2}{4\pi r} + O(r), \quad r\to 0, \quad eA_0'(0) = -\frac{e^2 w_0}{4\pi}. \quad (19)$$

In the expression for the mass shift, the first term in (19), which leads to the energy of the self-field of the unaccelerated charge, vanishes when the difference $|_0^F$ is evaluated, and we obtain

$$\Delta m = -\tfrac{1}{2}e[A_0'(\infty) - A_0'(0)]_0^F = \tfrac{1}{2}eA_0'(0) = -\tfrac{1}{2}\alpha w_0. \quad (20)$$

In other words, in the chosen gauge, the potential near the charge is equal to the potential of an unaccelerated charge, so that the mass shift is due to the nonzero horizon potential which appears at a finite distance w_0^{-1} from the charge because of its acceleration. It follows that the mass shift can be related to the change in the topology of the manifold occupied by the field: the space with the distinguished point (i.e., the charge) is replaced during acceleration by a half-space with the distinguished point. Because the field is azimuthally symmetric and vanishes at infinity, these manifolds are homeomorphic to the disk D^2 and the ring $S^1\times I$, and have different Euler parameters[9] $\chi = 1$ and $\chi = 0$.

We note in conclusion that the change in the covariant zero component of the potential, i.e.,

$$eA_0'\Big|_{\psi_1}^{\psi_2} = e\int_{\psi_1}^{\psi_2} E_\psi'\sqrt{-g_{00}'}\,h_\psi\, d\psi \quad (21)$$

can be interpreted as the work done by the electric field along a line of force, taking into account the red or blue shift described by the factor $\sqrt{-g'_{00}}$.

IV. MILNE SECTOR: SOLITARY LIGHT WAVE WITH UNIT ELECTRIC FIELD FLUX

According to (4), the change in the action for a uniformly accelerated charge can be expressed in terms of the field in the Milne region. It is convenient to introduce the Milne coordinate frame x', y', z', t' with the metric $g'_{\alpha\beta}$ (Refs. 10 and 11):

$$x = x', \quad y = y', \quad z = t'\operatorname{sh} w_0 z', \quad t = t'\operatorname{ch} w_0 z',$$
$$g_{\alpha\beta}' = \operatorname{diag}(1, 1, w_0^2 t'^2, -1). \quad (22)$$

On the Minkowski z, t diagram, its hypersurfaces with constant t' are hyperbolas, whereas hypersurfaces with constant z' are straight lines passing through the origin. We then have

$$\Delta W = \frac{1}{2}\int_{M^t} d^4x' \sqrt{-g'}\frac{E'^2-H'^2}{2} \approx \frac{1}{2}\int_{M'^\zeta} d^4x' \sqrt{-g'} \times \frac{E'^2-H'^2}{2} = \zeta\alpha w_0\int_0^\infty dv v g_1(-v^2) = \frac{1}{2}\alpha w_0\zeta. \quad (23)$$

The second equation is written for $t \gg w_0^{-1}$, whereas the region M^t can be replaced with M'^ζ, which is the part of M lying between the hyperplanes $z' = -\zeta$ and $z' = \zeta$, where

$$\zeta=\zeta(t)=w_0^{-1}\,\mathrm{Arth}\frac{(t^2-w_0^{-2})^{1/2}}{t}. \quad (24)$$

The expression for Δm that follows from (23) can be represented by the integral (13′) over an arbitrary time-like hypersurface S; the Rindler region does not contribute to it.

In the Milne frame, the field of the uniformly accelerated charge is independent of the cyclic coordinates φ and z',

$$E_z' = -\frac{e(w_0^{-2}+\rho^2+t'^2)}{\pi w_0^2\xi'^3}, \quad H_\varphi' = \frac{2e\rho t'}{\pi w_0^2\xi'^3}, \quad (25)$$
$$\xi'^2=(w_0^{-2}+t'^2-\rho^2)^2+4w_0^{-2}\rho^2,$$

and is identical with the field of the uniformly accelerated charge on the $z = 0$ plane of the inertial frame. This field and the corresponding set of Maxwell equations

$$\partial(\rho H_\varphi')/\rho\partial\rho=\partial E_z'/\partial t', \quad \partial E_z'/\partial\rho=\partial(t'H_\varphi')/t'dt', \quad (26)$$

have an interesting symmetry under the interchange $\rho \rightleftarrows t'$: the dependence of the field on time t' is the same as its dependence on the radius ρ. This important global property of the field is discussed below.

Let us now examine the expression for ΔW in terms of the covariant divergence in the Milne system:

$$\Delta W = -\frac{1}{4}\int_{M'^\zeta} d^4x'\partial_\alpha'(\sqrt{-g'}R'^\alpha) = -\frac{1}{2}\zeta\left(I_1(t')\Big|_{t'=0}^{t'=\infty} + I_2(\rho)\Big|_{\rho=0}^{\rho=\infty}\right). \quad (27)$$

Since $\sqrt{-g'}R'^\alpha$ is independent of z', the integral with respect to z' yields 2ζ, and the integral of the three-dimensional divergence is equal to the two terms in brackets in (27), which can be explicitly evaluated:

$$I_1(t')=2\pi\int_0^\infty d\rho\rho w_0 t' E_z' A_z' = \frac{\alpha w_0}{2}\left(\frac{\mathrm{arctg}\, v}{v}+\frac{1}{1+v^2}\right),$$
$$v=w_0t', \quad (28)$$

$$I_2(\rho)=-2\pi\int_0^\infty dt' w_0 t'\rho H_\varphi' A_z' = \frac{\alpha w_0}{2}\left(\frac{\mathrm{arctg}\, r}{r}-\frac{1}{1+r^2}\right),$$
$$r=w_0\rho. \quad (29)$$

Since the field $E_z'(\rho,t')$ does not change sign, and the potential $A_3'(\rho,t') \equiv w_0 t' A_z'(\rho,t)$ varies continuously with ρ, t', we can use the mean value theorem and the conservation of the flux of the electric field through the x,y plane

$$\int dx\, dy E_z'(\rho,t')=-e, \quad (30)$$

to write the first integral in the form

$$I_1(t')=A_3'(\bar\rho,t')\int dx\, dy E_z'(\rho,t')=-eA_3'(\bar\rho,t'),$$
$$\bar\rho=\bar\rho(t'), \quad (31)$$

i.e., in the form of the covariant third component of the potential, averaged over the electric field flux. It is clear from (27)–(29) that the mass shift is completely determined by the first integral:

$$\Delta m = \frac{1}{2}I_1(t')\Big|_{t'=0}^{t'=\infty} = -\frac{1}{2}eA_3'(\bar\rho,t')\Big|_{t'=0}^{t'=\infty} = -\frac{1}{2}\alpha w_0 \quad (32)$$

so the total change in time of the third component of the potential averaged over the field flux is well-defined.

If we integrate the relationship between the field and the potential with respect to time,

$$E_z'=-\partial(t'A_z')/t'dt', \quad H_\varphi'=-\partial A_z'/\partial\rho, \quad (33)$$

we find that the change in the potential at any point with fixed ρ is

$$A_3'(\rho,t')\Big|_{t'=0}^{t'=\infty}=-\int_0^\infty dt' w_0 t' E_z'(\rho,t') = \frac{ew_0}{2\pi}. \quad (34)$$

The quantity $ew_0/2\pi$ follows exclusively from the $\rho \rightleftarrows t'$ symmetry of the field and the conservation of the flux (30). On the other hand, it is clear from (32) that, at the point corresponding to the average flux, the result is smaller by a factor of two:

$$A_3'(\bar\rho,t')\Big|_{t'=0}^{t'=\infty}=\frac{ew_0}{4\pi}. \quad (35)$$

The point is that $\bar\rho$ depends significantly on t'. The electric field flux through the x,y plane has a maximum density at the center $\rho = 0$ only for $0 \leqslant t' < 2^{-1/2}w_0^{-1}$. This maximum is reduced in the course of time and eventually becomes a minimum for $t' > 2^{-1/2}w_0^{-1}$, whereas the flux density maximum shifts to $\rho > 0$ and the flux concentrates in the ring of radius $\rho \simeq t'$ and width $\Delta\rho \sim w_0^{-1}$ for $t' \gg w_0^{-1}$, so that $\bar\rho \simeq t'$. In other words, the entire electric field flux is transported by the resulting solitary light wave (which is cylindrical in the Milne system) with a symmetric field distribution relative to $\rho = t'$:

$$E_z'(\rho,t')\approx-H_\varphi'(\rho,t')\approx-\frac{ew_0}{2\pi(\rho+t')}[w_0^2(\rho-t')^2+1]^{-3/2},$$
$$\rho\sim t'\gg w_0^{-1}, \quad (36)$$

and an antisymmetric distribution of potential, so that the change in the potential on the crest of the wave is equal to half the total change in the potential.

The unvarying asymptotic form of (36), the constant velocity of the crest, and the constant electric field give the electromagnetic field (25) in the Milne sector the properties of a soliton. It would be interesting to find other symmetric solutions of (26) with a constant flux, e. g., flux distributed over several crests.

We draw attention to the fact that the appearance of the solitary light wave is accompanied by the vanishing, in a time $\sim w_0^{-1}$, of the negative pressure (tension) in the field along the z-axis, given by

$$T_{zz}'=-{}^1/_2(E_z'^2-H_\varphi'^2)<0,$$

or the force of attraction that decreases with time in accordance with (5) and is analytically continued into the Milne region: $u = iv$, $v = w_0 t'$. Because of the flux conservation described by (30), the relaxation of this tension results not in the vanishing of the potential, but in the appearance of a magnetic field of amplitude equal to the amplitude of the electric field. The flux of the negative z-component of momentum, which decays in time, flows through the $z' = \text{const}$ plane, so that the change in this flux is always equal to the mass shift; see (32). This is the significance of this physical quantity for Milne observers.

V. RADIUS-TIME SYMMETRY IN THE MILNE SYSTEM

We have seen that the Maxwell equations (26), the electric and magnetic fields (25), and the Lagrangian of the field of the uniformly accelerated charge in the Milne system of coordinates exhibit the $\rho \rightleftarrows t'$ symmetry. For the uniformly accelerated scalar charge (source of the scalar field Φ), this symmetry immediately leads to a zero mass shift. Actually, the wave equation for Φ, and Φ itself, are $\rho \rightleftarrows t'$ symmetric in the Milne system,

$$\frac{\partial}{\rho\partial\rho}\left(\rho\frac{\partial\Phi'}{\partial\rho}\right)=\frac{\partial}{t'\partial t'}\left(t'\frac{\partial\Phi'}{\partial t'}\right),\quad \Phi'=\frac{e}{2\pi w_0\xi'},\tag{37}$$

which may be compared with (25) and (26). Hence, the Lagrangian

$$\frac{1}{2}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi=\frac{1}{2}\left[\left(\frac{\partial\Phi'}{\partial\rho}\right)^2-\left(\frac{\partial\Phi'}{\partial t'}\right)^2\right]=\frac{2\alpha(\rho^2-t'^2)}{\pi w_0^2\xi'^4}\tag{38}$$

is antisymmetric, and its integral, which determines the mass shift, is equal to zero because of the $\rho \rightleftarrows t'$ symmetry of the volume element and the region of integration.

All the general relationships for the uniformly accelerated scalar charge can be deduced from the corresponding relations for a uniformly accelerated electric charge by introducing the replacements

$$A_\alpha j_\alpha \to -\Phi j,\quad E^2-H^2\to(\partial_\alpha\Phi)^2,\quad F_{\alpha\beta}A_\beta\to-\Phi\partial_\alpha\Phi,$$
$$g_1(u^2)\to g_0(u^2)=\frac{1}{2}\left(\frac{1}{1-u^2}-\frac{\operatorname{Arth} u}{u}\right)'.\tag{39}$$

The fact that the Lagrangian (38) is antisymmetric ensures that the force

$$\int dx\,dy\,T_{zz}'=-\alpha w_0^2 g_0(-v^2),\qquad v=w_0 t',\tag{40}$$

between portions of the field in the Milne region separated by the plane $z' = \text{const}$ will eventually change sign: attraction will be replaced by repulsion, so that the change in the z-component of the field momentum will always be zero. From the point of view of an accelerated observer, the absence of a mass shift in the case of the uniformly accelerated scalar charge is due to the precise cancellation of the red shift of the energy of the self-field in the region $0 < z' < w_0^{-1}$ by the blue shift in the region $w_0^{-1} < z' < \infty$; see (13) and (39).

The $\rho \rightleftarrows t'$ symmetry in the Milne system is a consequence of the dynamic symmetry of the field equations and the invariance of the field due to the uniformly accelerated charge under rotations around the z-axis and boosts along the z-axis. Thus, the scalar-field dynamics is described by the general covariant wave equation

$$\Box\Phi\equiv\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\,g^{\alpha\beta}\partial_\beta\Phi)=j.\tag{41}$$

Invariance under rotations and boosts indicates that the field is independent of the coordinates z' and φ in the Milne cylindrical frame with the metric given by (22). As a result, the d'Alembertian in (41) becomes $\rho \rightleftarrows t'$-antisymmetric and, since there is no source in the Milne system, (41) reduces to the $\rho \rightleftarrows t'$-symmetric equation (37). For the same reasons, in the Rindler system, the d'Alembertian is $\rho \rightleftarrows z'$-symmetric but, because of the nonsymmetric source $j' = e\delta(\rho)\delta(z' - w_0^{-1})/2\pi\rho$, neither the equation nor the field exhibits this symmetry.

Using the analytic continuation ($z \to iz_u, t \to t_u$) of (41) from the Minkowski space into the 4-space R_2^4 of zero signature and metric given by $g_{\alpha\beta} = \text{diag}(1,1,-1,-1)$, we find that (41) becomes an ultrahyperbolic equation[12] and the symmetry with respect to the boosts along the z-axis becomes a symmetry under rotations of the plane z_u, t_u around the origin. Taking into account the symmetry of the field under rotations in the x,y, and z_u, t_u planes and transforming to polar coordinates, we obtain (37), where $t' = (z_u^2 + t_u^2)^{1/2}$, $\rho = (x^2 + y^2)^{1/2}$. For the same reasons, the Maxwell equations become indentical with (26) after analytic continuation into R_2^4. All these scalar field functions in the Milne region of the Minkowski space are identical with their analytic continuations into the space R_2^2 if the points $z = t'\sinh\psi$, $t = t'\cosh\psi$ on hyperbolas in the Milne region are continued into the points $z_u = t'\sin\theta$, $t_u = t'\cos\theta$ on circles in the z_u, t_u plane in the cyclic coordinate $\psi \to i\theta$, $\theta = 2\arctan\psi$, which is thus compactified: $-\pi < \theta < \pi$. The edge $t = |z|$ of the Milne region thus becomes the center $z_u = t_u = 0$ of the z_u, t_u plane. The fields (37) and (25) of the uniformly accelerated scalar and electric charges have the properties of instantons in the space: R_2^4: they are localized near the origin in a region of size w_0^{-1}, they have no singularities, and they have finite action that vanishes for the scalar field and equals

$$\int_{R_2^4} d^4x_u\left(\frac{E^2-H^2}{2}\right)_u=2\pi\alpha\int_0^\infty\int_0^\infty\frac{dr^2\,dv^2}{[(1+v^2-r^2)^2+4r^2]^2}=\pi\alpha\tag{42}$$

for the Maxwell field, where

$$r=w_0\rho,\quad v=w_0t',\quad d^4x_u=w_0^{-4}r\,dr\,d\varphi\,v\,dv\,d\theta.$$

Similarly, we can perform the analytic continuation $z \to z_E$, $t \to it_E$ of the wave equation from the Minkowski space w_0^{-2} to the Euclidean space R^4, having transformed the symmetry under boosts along the z-axis into the symmetry under rotations of the z_E, t_E plane. Scalar field functions in the Rindler region of the Minkowski space will be indentical with their analytic continuations into the Euclidean space if the point $z = z'\cosh\psi$, $t = z'\sinh\psi$ are continued into the points $z_E = z'\cos\theta$, $t_E = z'\sin\theta$ in the cyclic coordinate $\psi \to i\theta$, $\theta = 2\arctan\psi$, so that the edge $z = |t|$ of the Rindler region becomes the center of the Euclidean plane. The field of the uniformly accelerated scalar and electric charges are localized near the origin of the Euclidean space in a region of size $\sim w_0^{-1}$, but are singular on the source circle $\rho = 0$, $z'^2 = z_E^e + t_E^2 = w_0^{-2}$. Their action changes by a finite amount, given by

$$\int_{R^4} d^4x_E \left(\frac{E^2-H^2}{2}\right)_E \Big|_0^F = 2\pi\alpha \int_0^\infty \int_0^\infty \frac{dr^2\, du^2}{[(1-u^2-r^2)^2+4r^2]^2} = -\pi\alpha, \quad (43)$$

$u = w_0 z'$ for the Maxwell field and equal to zero for the scalar field.

We thus see that analytic continuation in the cyclic variable and its compactification ensure that the action is finite and there is no change in the dynamic equations or in their solutions.

VI. CONCLUSION

Since our basic results have already been adequately summarized in the text and in the Introduction, we need only emphasize that the change in the topology of the manifold occupied by the self-field and the nonzero spin of the field are the most significant qualitative conditions for the existence of the mass shift. In this connection, it would be interesting to deduce the mass shift of a uniformly accelerated field source of spin 2 or more. It is reasonable to suppose that the shift is proportional to the spin of the field. At any rate, after the compactification of the cyclic variable into an angle, the compactified action and angle become conjugate quantities. Hence, $\Delta m w_0^{-1}$ may be looked upon as the radiative correction to the component of the angular momentum of the uniformly accelerated charge in the direction of acceleration, which physical considerations indicate should be of the order of $\alpha \hbar n$, where $\hbar n$ is the spin of the free field.

The author is indebted to A. I. Nikishov for discussions and advice.

[1)] We use the system of units in which $\hbar = c = 1$, $\alpha = e^2/4\pi\hbar c \simeq 1/137$.

[1]V. I. Ritus, Zh. Eksp. Teor. Fiz. **75**, 1560 (1978) [Sov. Phys. JETP **48**, 788 (1978)].
[2]V. I. Ritus, Zh. Eksp. Teor. Fiz. **80**, 1288 (1981) [Sov. Phys. JETP **53**, 659 (1981)].
[3]A. I. Nikishov, Zh. Eksp. Teor. Fiz. **59**, 1262 (1970) [Sov. Phys. JETP **32**, 690 (1971)].
[4]V. I. Ritus, Dokl. Akad. Nauk SSSR **275**, 611 (1984) [Sov. Phys. Dokl. **29**, 227 (1984)].
[5]L. D. Landau and E. M. Lifshitz, *Field Theory*, Pergamon Press, various editions [Russ. original, Nauka, Moscow (1973), p. 504].
[6]T. Fulton and R. Rohrlich, Ann. Phys. **9**, 499 (1960).
[7]W. Rindler, Am. J. Phys. **34**, 1174 (1966).
[8]G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill, 1961 [Russ. transl., Nauka, Moscow (1977), p. 582].
[9]C. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press, London, 1983, p. 311.
[10]E. A. Milne, Nature **130**, 9 (1932).
[11]N. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* [Russ. transl., Nauka, Moscow (1984), p. 356].
[12]R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Wiley, 1953 [Russ. transl. (of a previous edition), Gostekhizdat, Moscow (1945), Vol. 2, p. 620].

Translated by S. Chomet