Loss of equilibrium of elongated magnetic configurations

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(Submitted 31 March 1986)
Zh. Eksp. Teor. Fiz. **92**, 2050–2059 (June 1987)

Elongated magnetic configurations, i.e., configurations whose longitudinal dimension is considerably greater than the transverse dimension, are analyzed. It is shown that in several cases an equilibrium prevails only in the form of a one-dimensional field, i.e., of straight field lines. Such a degeneracy occurs (1) in plane geometry when there is a straight null line, (2) in plane geometry when there are two null lines of arbitrary shape, and (3) for an axisymmetric field with a poloidal component that vanishes on some surface. Restrictions are also found on the form of the equilibrium for a field which has the property that it is weak on some straight line, although not zero (like a coronal streamer or the tail of the earth's magnetosphere). In all cases in which the boundary conditions prevent the attainment of a one-dimensional equilibrium, current sheets will unavoidably arise.

Current sheets, which are of fundamental importance in cosmic electrodynamics as well as in plasmas in toroidal devices, are the subject of active research. Formation of a current sheet can take place because of a loss of equilibrium.

We know that not all initial magnetic configurations have an equivalent equilibrium configuration. In other words, that motion of a plasma with a frozen-in magnetic field which is caused by the loss of an equilibrium of the initial state will not always be capable of bringing the configuration to an equilibrium state. In several cases the corresponding equilibrium absolutely must involve discontinuities of the magnetic field, even if the initial field is smooth. This freezing in occurs in an ideally conducting medium. In such a case, discontinuities of the magnetic field are permissible. In a real, highly conducting plasma, in contrast, current sheets of small but nonzero thickness arise at the points of discontinuity.

A departure of a magnetic field from equilibrium arises either as the result of evolution of the field caused by external forces of some sort (e.g., when a field is generated by motions of a plasma) or because of changes in boundary conditions. In the present paper we are concerned with the second of these possibilities.

We will use a model proposed by Moffatt[1]: The viscosity is high, while the ohmic loss is essentially zero (in terms of the conductivity, we are considering the case $\sigma\to\infty$). In this model a nonequilibrium initial field causes motion, which in turn causes a monotonic—nonoscillatory—decrease in the magnetic-field energy. The viscosity damps the motion, with the result that an equilibrium is reached.

1. GENERAL PROPERTIES OF AN EQUILIBRIUM IN THE PRESENCE OF A NULL LINE

We restrict the analysis to two-dimensional configurations, which depend on two coordinates. In §§ 1, 2, 4, and 5 we deal with planar geometry: all quantities depend on x and y alone. The equilibrium condition is written

$$\Delta A=f(A),\quad f(A)=-4\pi dP/dA,$$
$$P=p+\frac{1}{8\pi}H_y^2,\quad p=p(A),\quad H_y=H_y(A),\tag{1}$$

where $\mathbf{A}$ is the magnetic vector potential, $\mathbf{H}=\operatorname{curl}\mathbf{A}$, $A\equiv A_y$, and p is the pressure. Despite the complexity of Eq. (1) (P is generally a nonlinear function of A_y), many equilibrium solutions are known (Ref. 2, for example). To demonstrate that a given configuration is not an equilibrium configuration, one must prove that the corresponding A does not satisfy (1) for any function $f(A)$.

We will be constructing several examples below, but we would first like to call attention to a remarkable property of Eq. (1) in the case in which there is a null line. By a "null line" we mean a curve in the XY plane on which the transverse field component $\mathbf{H}_\perp=\{H_x,0,H_z\}$ vanishes (since we have $\partial/\partial y=0$; it would be more precise to speak in terms of a surface on which the condition $\mathbf{H}_\perp=0$ holds). We will show that if a null line is a straight line segment then the solution will be degenerate, and the equilibrium configuration will become a one-dimensional configuration throughout space, regardless of the boundary conditions.

Let us examine the more general case in which we have $H_x=0$ and $H_z=\text{const}$ on the line segment $x=0$, $-a\leqslant z\leqslant a$ (one particular case would be $H_z=0$). Near $x=0$ we accordingly have

$$A=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\dots,\tag{2}$$

with $A_0=\text{const}$ (since $H_x=0$) and $A_1=\text{const}$ (since $H_z=\text{const}$). Substituting (2) into (1), we find, for the various powers of x,

$$2A_2=f(A_0),\quad 3\cdot 2A_3=f'(A_0)A_1,$$
$$4\cdot 3A_4=f'(A_0)A_2+{}^1/_2f''(A_0)A_1^2.$$

We then find in succession that A_2, A_3, etc., are independent of z. The entire solution A is thus independent of z in the band $-a\leqslant z\leqslant a$. The field lines are straight lines which run parallel to the z axis.

Let us construct a solution outside this band. We consider the half-space $z>a$ (the entire discussion for $z<-a$ is quite similar). We write the solution in the form

$$A=\tilde{A}_0+\tilde{A}_1(z-a)+\tilde{A}_2(z-a)^2+\tilde{A}_3(z-a)^3+\dots,\tag{3}$$

where $\tilde{A}_n$ is a function of x. By virtue of the condition $H_x(z=a)=0$ we have $\tilde{A}_1=0$. Furthermore, we have

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$\tilde{A}_0'' + 2\tilde{A}_2 = f(\tilde{A}_0)$. Comparing this expression with (1), written for the band, we find $\tilde{A}_0'' = f(\tilde{A}_0)$; since, according to (1), $f(A)$ does not vary along a field line, we find $\tilde{A}_2 = 0$. We then find $\tilde{A}_3 = 0$, etc., in succession. Consequently, A is independent of z throughout space, and the solution is uniform.

In this proof we have leaned heavily on the assumption that the function A is analytic. It is easy to see that the derivation can be repeated if it is assumed that the current [i.e., ΔA and thus $f(A)$, contains an even number of jumps and discontinuities; it is sufficient to join the solutions to the right and left of a jump. If the function $f(A)$ is not differentiable at some point, e.g., if it contains a term of the type $(A - A_0)^\alpha$, $0 < \alpha < 1$, then the expansion (2) becomes more complicated, but not in a fundamental way. We need to add to it terms with nonintegral powers, e.g., $A_{\alpha+2}x^{\alpha+2}$. There is no change in the derivation as a result. If the expansion (2) has a finite convergence radius, this derivation becomes modified in the following way. We assume that the expansion (2) holds up to a certain x_0, i.e., that it holds for $x < x_0$. At the point $x = x_0$ we must then write a new expansion, in powers of $x - x_0$, retaining the same boundary conditions: $H_x = 0$, $H_z =$ const at $x = x_0$. If this expansion also holds up to x_1, then we can expand the field around x_1, etc. The case $H_x = 0$, $H_z = 0$ on a straight line segment is a particular case of this situation.

We can now reformulate and slightly amplify Hahm and Kulsrud's result[3] regarding stimulated reconnection of field lines. They showed that a weak (sinusoidal) perturbation of the boundaries of a uniform magnetic configuration (Fig. 1) leads to the formation of current sheets at $x = 0$, with the result that magnetic islands appear. By virtue of the symmetry of the perturbation, the null line must remain straight. As was shown above, the equilibrium field does not depend on the coordinate z in this case. The field in Fig. 1, in contrast, must depend on z, because of the given shape of the perturbed boundary. An equilibrium is thus not possible in the class of continuously differentiable fields. In an ideally conducting medium, the field lines do not reconnect, so the only equilibrium state which the system can reach for the given perturbed boundary contains jumps in the magnetic field, in this case along the z axis. As a result of the finite conductivity, a current sheet of finite thickness forms at the z axis, and magnetic islands eventually appear. Hahm and Kulsrud's result[3] is thus formulated in terms of a system which is not in magnetic equilibrium. Furthermore, we can state immediately that the formation of current sheets is a more general phenomenon—not restricted to the case of a slight perturbation of the boundary (the case studied in Ref. 3) - for the same reason.

2. ELONGATED CONFIGURATIONS OF PLANE GEOMETRY

We will call "elongated" magnetic field configurations those which have two greatly different length scales. More specifically, we are interested in a field with a length scale l for its horizontal variation (along the x axis) and with a length scale L, $L \gg l$, for its variation in the vertical direction (along the z axis). We can say that the magnetic field is elongated in the z direction. Elongated configurations have the advantage that the nature of the equilibrium can be studied by perturbation theory (as in Ref. 3).

Let us consider fields which have a certain symmetry with respect to the z axis. For example, the vector potential component A_y ($\equiv A$) for the field in Fig. 1 is even under the substitution $x \to -x$. No less interesting is the configuration in Fig. 2, where A_y is an odd function of the coordinate x. It turns out the presence of a null line (with $\mathbf{H}_\perp = 0$) substantially restricts the nature of the equilibrium. In particular, if there are two null lines, as in Fig. 2, an equilibrium is possible only if these lines are straight and, accordingly to §1, if the configuration is one-dimensional.

We are thus concerned with solving Eq. (1) in the band shown in Fig. 2. At the boundaries of this band we have $\nabla A = 0$, which corresponds to imposing the condition $\mathbf{H}_\perp = 0$ at the boundaries.

In order to show that an equilibrium configuration would have to be one-dimensional, we expand the solution of

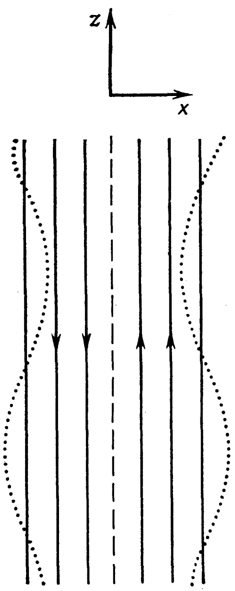

FIG. 1. Perturbation of the plasma boundary (dotted lines) with a magnetic field containing a null line.

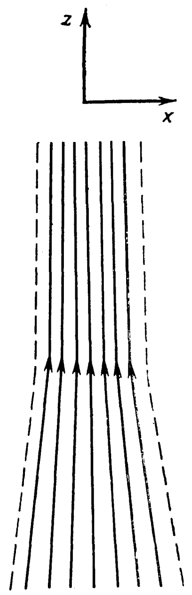

FIG. 2. Magnetic configuration with two null lines (the dashed lines).

Eq. (1) in powers of $z-a$ around some arbitrary point a; i.e., we again use the series (3). Collecting terms $\sim(z-a)^0$ and $\sim(z-a)$, we find

$$\tilde{A}_0''+2\tilde{A}_2=f(\tilde{A}_0), \tag{4}$$

$$\tilde{A}_1''+6\tilde{A}_3=f'(\tilde{A}_0)\tilde{A}_1. \tag{5}$$

A simple estimate of the terms of series (3) yields

$$\tilde{A}_2\approx\tilde{A}_0/L^2,\quad \tilde{A}_3\approx\tilde{A}_1/L^2.$$

Noting that we have $\tilde{A}_1''\approx\tilde{A}_1/l^2$, and making use of the pronounced difference in lengths scales, i.e, the small parameters l/L, we ignore the quantity $\tilde{A}_3$ in (5) [for the same reason, we can discard $\tilde{A}_2$ in (4), but this point is not of consequence for the discussion below]. The resulting equation can be derived from the exact equation,

$$H_z\Delta H_x=H_x\Delta H_z, \tag{6}$$

which is equivalent to Eq. (1) (Ref. 3), by making use of

$$H_x=-\partial_zA,\quad H_z=\partial_xA,\quad |\partial_xA|\approx|\partial_zA|L/l,$$

i.e., $|\partial_xA|\ll|_zA|$, and by retaining in (6) the terms of lowest order in the derivative $\partial/\partial z$:

$$H_x''=(H_z''/H_z)H_x. \tag{7}$$

Noting that we have $H_x=-\partial_zA=-\tilde{A}_1$ at $z=a$, we find the following conclusion: Equation (7) is the same as (5) if we discard $\tilde{A}_3$ and assume $f'(\tilde{A}_0)=H_z''/H_z$. In Eq. (7) we can assume that H_z is given.

Expression (7) is an equation for H_x, which is being sought at $z=a$ (and therefore depends only on x). On the boundary, at some points $x=\pm R$, the field vanishes, $\mathbf{H}_\perp=0$, so that we have $H_x(x=\pm R)=0$. This is the boundary condition.

Equation (7) may be regarded as an eigenfunction problem for the equation

$$H_x''-UH_x=-\lambda H_x,\quad U=H_z''/H_z, \tag{8}$$

where the eigenvalue λ of the unknown function must vanish. The solution $H_x=\alpha H_z$, $\alpha=\text{const}$ vanishes only at the boundaries $x=\pm R$ and is therefore the lowest eigenfunction, with $\lambda=\lambda_0=0$. The solution in which we are interested must be odd in x, i.e., must vanish at least at the point $x=0$ (in addition to the boundary points). It thus corresponds to the first (or higher) eigenvalue $\lambda=\lambda_1>0$ and thus does not satisfy (7). A solution of (7) with this symmetry can be written in the form

$$H_x=\beta H_z(x)\int_0^x\frac{1}{H_z^2(x')}dx',\quad \beta=\text{const}. \tag{9}$$

It does indeed vanish at $x=0$, but it does not satisfy the boundary conditions, as follows from the discussion above. To see this, we assume $\beta H_z(x)>0$ for $x>0$ for definiteness; then H_x is positive everywhere for $x>0$, according to (9). In the limit $x\to R$, we can write H_z in the form $H_z=H_n(R-x)^n$; we would then have

$$H_x\to\frac{\beta}{H_n}\frac{1}{(R-x)^{n-1}}. \tag{10}$$

If $n>1$, there will be a singularity in H_x as $x\to R$. If $n<1$, a singularity will appear in the derivatives of H_x and/or H_z; i.e., the current will become infinite. A unique solution without singularities can occur for $n=1$. In this case we have

$$H_x\to\beta/H_n>0.$$

Equation (7) thus has no nontrivial solutions which satisfy the boundary conditions and have the required symmetry. In other words, we have $H_x\equiv0$ at $z=a$. Since a is arbitrary, we have $H_x\equiv0$ everywhere. The condition div. $\mathbf{H}=0$ means that we have $H_z=H_z(x)$; i.e., the configuration is one-dimensional.

3. AXIAL SYMMETRY

Let us consider equilibrium symmetric configurations. In this case we can assume that Fig. 2 shows the poloidal component of the field. In the cylindrical coordinate system (r,φ,z) an equilibrium is described by the Grad-Shafranov equation[4]

$$\Delta^*\psi+\frac{1}{2}\frac{dF^2}{d\psi}+4\pi\frac{dp}{d\psi}r^2=0,$$

$$\Delta^*=\frac{\partial^2}{\partial z^2}+r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r},\quad \psi=rA_\varphi,$$

$$F=rH_\varphi=F(\psi),\quad p=p(\psi),$$
$$H_r=-\frac{1}{r}\frac{\partial\psi}{\partial z},\quad H_z=\frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{11}$$

Differentiating Eq. (11) with respect to r and z, we can eliminate the pressure; i.e., we can find the analog of Eq. (6):

$$H_z\frac{1}{r}\Delta^*(rH_r)=H_r\Delta^*H_z+\frac{1}{r}\frac{\partial H_r^2}{\partial z}-\frac{1}{r^2}\frac{dF^2}{d\psi}H_r. \tag{12}$$

To find an analog of Eq. (7) which corresponds to an elongated configuration $(\partial/\partial r\gg\partial/\partial z)$, we consider the terms of lowest order in $\partial/\partial z$ in (12):

$$\frac{H_z}{H_r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial(rH_r)}{\partial r}-r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial H_z}{\partial r}=-\frac{1}{r^2}\frac{dF^2}{d\psi}. \tag{13}$$

The vanishing of the poloidal field on the dashed line in Fig. 2 means that at a certain $r=R$ we have $H_z=H_r=0$. We first consider the case $F=0$. The general solution of Eq. (12) (for H_z) is then

$$H_z=\alpha rH_r+\beta\Phi(r),\quad \Phi(r)=rH_r\int_{r_0}^{r}\frac{1}{r_1H_r^2(r_1)}dr_1, \tag{14}$$

where α and β are constants. For this equation, in contrast with (9), the lower limit on the integral in (14) is r_0, where $0<r_0<R$, for otherwise the integral would have no meaning. Since $\Phi(R)\neq0$ [cf. (10)], we have $\beta=0$. The solution $H_z=\alpha rH_r$ satisfies the boundary conditions. It should be kept in mind, however, that solution (14) with $\beta=0$ must be real for arbitrary z [we recall that (13) is being considered for the case of a fixed but arbitrary z]. Expressing H_r and H_z in terms of ψ, and expanding ψ in powers of r, we easily see that a solution $H_z=\alpha rH_r$ could only be a trivial solution, $H_r\equiv0$, $H_z\equiv0$ (we reach the same conclusion if we expand the solution $H_z=\alpha rH_r$ in powers of $R-r$ around $r=R$).

In the general case ($F\neq0$) the solution can be written with the help of a Green's function with the following boundary conditions: $\partial H_z/\partial r = 0$ at $r = 0$ and $H_z = 0$ at $r = R$:

$$H_z = -rH_r \int_0^r \frac{\Phi(r')}{r'} \frac{dF^2}{d\psi} \frac{dr'}{(r')^2} - \Phi(r) \int_r^R H_r(r') \frac{dF^2}{d\psi} \frac{dr'}{(r')^2}. \quad (15)$$

A Green's function exists precisely because the corresponding homogeneous problem ($F = 0$) has only a trivial solution. For the same reason, solution (15) is unique. In particular, it must not depend on the parameter r_0 which appears in the definition of $\Phi(r)$ [see (14)]. Taking the derivative of (15) with respect to r_0, and equating it to zero, we find

$$\int_0^R H_r(r') \frac{dF^2}{d\psi} \frac{dr'}{(r')^2} = 0, \quad (16)$$

$$H_z = rH_r \int_0^r \left(\int_{r'}^r \frac{dr_1}{r_1 H_r^2(r_1)} \right) H_r(r') \frac{dF^2}{d\psi} \frac{dr'}{(r')^2}. \quad (17)$$

The latter equation is indeed independent of r_0. Like (14), however, it can not hold for arbitrary z. Since we must be able to write the solution of the general equation (11) near the origin in the form

$$\psi = \sum_n a_{2n}(z) r^{2n},$$

we find that the lowest power of r for a field $H_z \sim 2ma_{2m} r^{2m-2}$ is lower by at least 2 than the power for a field $rH_r = -r^{2m} da_{2m}/dz$. At the same time, if we substitute $rH_r \sim r^{2m}$ into (17), we find $H_z \sim r^{2m+2}$; i.e., the power of the field H_z is higher than that of rH_r. We can attempt to construct a solution of Eq. (12) [from which Eq. (13), which we are using, follows] in the following way: $H_r = 0$ for $0 < r < a$ and $H_r \neq 0$ at $r > a$. The lower limit on the outer integral in (17) is then replaced by a. We then have

$$\psi = \sum_n a_n(z) (r-a)^n,$$

and the lowest powers are

$$H_z \sim r^{-1}(r-a)^{m-1}, \quad rH_r \sim (r-a)^m,$$

while according to (17) we would have $H_z \sim (r-a)^{m+1}$.

The unique solution of (12) for elongated configuration is therefore trivial for these particular boundary conditions: $H_r \equiv 0$. In this case, ψ depends on r alone; i.e., the configuration degenerates into a one-dimensional configuration.

4. LOSS OF EQUILIBRIUM BECAUSE OF SPECIAL BOUNDARY CONDITIONS

In some models which have been studied previously, the loss of equilibrium is caused by the initial field configuration. Parker[5] called it a "topological equilibrium" (see also Refs. 6–8 and 1). In Hahm and Kulsrud's study,[3] the equilibrium is lost because of special boundary conditions (see the dotted lines in Fig. 1). In this section we construct some examples in which there is no equilibrium because of special initial and boundary conditions; these examples are more interesting for astrophysical applications (in comparison with the problem studied by Hahm and Kulsrud[3]).

To simplify the analysis we consider extended plane configurations as shown in Figs. 1 and 2. The field depends on x and z alone. We assume that the ends of the field lines are frozen in an ideally conducting solid surface. In other words at (say) $z = 0$ the field component H_z is given: $H_z(z=0) = H_z(x)$. These are our boundary conditions. Obviously, for any $H_z(x)$ an equilibrium exists at $z = 0$. The field depends on x alone throughout the region $z > 0$, the field lines are straight, and we have $P(x) + H_z^2/8\pi = p + H^2/8\pi = \text{const}$.

Let us examine the following initial conditions. Again the field depends only on x, and we have

$$P(x) + H_z^2/8\pi = C = \text{const} \quad \text{for } |x| \geqslant x_1 > 0,$$
$$P(x) + H_z^2/8\pi < C \quad \text{for } |x| < x_1. \quad (18)$$

An equilibrium exists only for $|x| \geqslant x_1$; at $|x| < x_1$, a force acts on the plasma and tends to compress toward the z axis.

If there were no boundary conditions at $z = 0$, the initial conditions would compress the plasma toward the z axis; the field would remain one-dimensional, and an equilibrium would be reached (we recall that we are using the model proposed by Moffatt[1]). The boundary conditions at $z = 0$ prevent the plasma from undergoing a displacement at $z = 0$, and the field lines have the behavior shown in Fig. 2. The configuration tends toward such a state because of the effect of the forces (18). As was shown above, when there are null lines, as in Figs. 1 and 2, a multidimensional state of this sort is not an equilibrium state. The initial deviation from equilibrium, (18), is due to a deficiency of the pressure $P(x) = p + H_y^2/8\pi$, in the region $|x| < x_1$. As the system reverts to a one-dimensional configuration associated with "straightening" of the field lines, the quantity $H_y^2/8\pi$ returns to its initial value, while the pressure p may in principle change because of the gas heating at $|x| < x_1$. If energy may not be supplied to the region $|x| < x_1$ (across the $z = 0$ plane, for example), and if the gas is not heated, then the deficiency of the pressure $P(x)$ in this region returns to its initial level; relations (18) remain in force; and there is no equilibrium.

An initial one-dimensional field of exceedingly simple geometry, but with an unbalanced pressure as in (18), thus cannot reach an equilibrium state. In other words, there is no equilibrium in the class of continuous fields. In a viscous but ideally conducting fluid,[1] the initial conditions (18) lead to the formation of discontinuities. For the field shown in Fig. 1, the discontinuity obviously appears at the z axis (as in §1). Correspondingly, a discontinuity is formed on the neutral line in Fig. 2 if the line $\mathbf{H}_\perp = 0$ (the dashed line) separates fields of different signs. An interesting situation arises in a different case: that in which we have $\mathbf{H}_\perp \neq 0$ only within a certain band. On one side of the $\mathbf{H}_\perp = 0$ line we will have a field $\mathbf{H}_\perp \neq 0$, while on the other we will have $\mathbf{H}_\perp \equiv 0$ (just as shown in Fig. 2). In this case a discontinuity generally arises somewhere other than at the null line. The condition at the discontinuity is[9]

$$\{H_\perp^2/8\pi + P\} = 0.$$

where the braces specify the difference between the values on the two sides of the discontinuity (in the case at hand, two discontinuities appear, because of the symmetry). When the conductivity is only finite, the discontinuity corresponds to a current sheet of small but nonzero thickness. If this sheet forms on a neutral line, as in Fig. 1 ohmic diffusion will lead to reconnection of the field lines and to the formation of magnetic islands. If the current sheet instead arises on a field line, with $\mathbf{H}_\perp \neq 0$, the field will penetrate through the matter. More precisely, the matter will penetrate through the field in the direction toward the z axis, leading to an increase in the total pressure between two field lines. As a result, the deficiency of the pressure $P(x)$ at $|x| < x_1$ is eliminated, and an equilibrium in the form of a one-dimensional configuration is reached.

5. ELONGATED CONFIGURATION IN THE MORE GENERAL CASE

A stricter limitation on the form of the elongated configuration can be found in a case of importance for applications. Up to this point, the equilibrium has necessarily been of a one-dimensional nature if the field $\mathbf{H}_\perp$ had a null line in the XZ plane, either a single null line as in Fig. 1 or two as in Fig. 2. Let us construct a more general example. Figure 3 shows a field which is reminiscent of a coronal streamer,[10] on the one hand, and the field of the tail of the earth's magnetosphere, on the other. In the region of field lines which are closed onto the $z = 0$ plane, the field may be electrostatic, so that the horizontal and vertical length scales of the field are identical. This is not an elongated configuration. The extension results from an increase in the flux of the external field, shown in Fig. 3 in the form of the unclosed field lines. An increase of the external field is equivalent to an increase in the external pressure, with the result that the elongated part becomes compressed in the horizonatal direction and extended in the vertical direction (Ref. 11, for example). After the vertical dimension becomes greater than the horizontal dimension, the configuration can be classified as elongated.

Since this compression occurs slowly, the configuration may be regarded as a quasiequilibrium configuration.[11] At any rate, it is treated in this way in searches for instabilities (in particular, the tearing-mode instability) which lead to reconnection of the field lines. In contrast with the field in Fig. 1, the configuration in Fig. 3 has no null lines. We cannot make the *a priori* assumption that we have $\mathbf{H}_\perp = 0$ on the z axis. The so-called transverse component, $H_x \neq 0$, remains here.

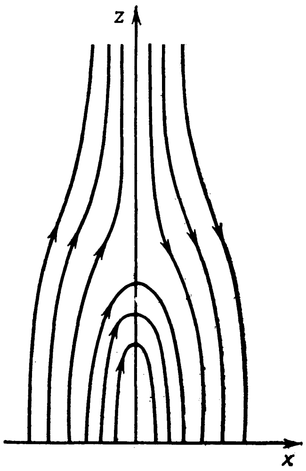

FIG. 3. Elongated configuration without a null line.

To explain the properties of an equilibrium which satisfies general equation (1), we return to expansion (4), (5) and the corresponding relation (7). One solution of relation (7), $H_x = aH_z$, has odd parity along the x axis, while the H_x component of the field in Fig. 3 has even parity. The second solution cannot be written in form (9), since the integral has no meaning in this case. We write it in the following way:

$$H_x=\beta H_z(x) \int \frac{1}{H_z^2}\, dx, \tag{19}$$

where the arbitrary constant in this indefinite integral is chosen in such a way that the integral itself becomes an odd function (this can be done, since the integrand is an even function). For example, we could write

$$\int \frac{dx}{H_z^2}=\frac{1}{2}\left[\Phi(x)-\Phi(-x)\right], \quad \Phi(x)=\int_a^x \frac{dx_1}{H_z^2(x_1)},$$

with $a \neq 0$. In particular, for Harris's solution[12]

$$H_z(x)=H_0 \,\mathrm{th}(x/l),$$

which describes a kinetic steady state (i.e., a state more general than the MHD equilibrium which we are considering here; see Ref. 12) in the absence of a transverse component H_x, expression (19) can be written explicitly as follows:

$$H_x=\frac{\beta}{H_0}\left(x\,\mathrm{th}\frac{x}{l}-l\right). \tag{20}$$

The solution (19), (20) has the symmetry we need: It is even in x. It describes the H_x component of the configuration shown in Fig. 3 wherever the condition $H_x \ll H_z$ holds, i.e., in the band $0 \leqslant z < L$, where L is the vertical dimension of the closed region (Fig. 3). There is the important point, however, that for $|x| > l$ the field H_x increases linearly. This can be seen directly from (20), and it is of course a general property of solution (19), since H_z asymptotically approaches a constant at $|x|: H_z \to H_z(\infty)$. The integral increases linearly with increasing $|x|$.

It might seem that we could construct a solution H_x in a slightly different way. Let us assume tht H_x is given by (20) for $|x| < x_1$, that the solution (20) vanishes at the point $x = x_1$, and that we have $H_x \equiv 0$ at $|x| > x_1$. It should be kept in mind, however, that (20) was derived from (7) or from Eq. (8), which is equivalent. The field corresponding to the vector potential $\tilde{A} + \tilde{A}_1(z-a)$ must be continuous.For a solution constructed in this way, the component $H_x = -\partial_z A = -\tilde{A}_1$ would indeed be continuous, while $H_z = \partial_x \tilde{A}_0 + (z-a)\partial_x \tilde{A}_1 = \partial_x \tilde{A}_0 - (z-a)\partial_x H_x$ would have a discontinuity. A solution without field discontinuities (i.e., without prespecified current sheets) should coincide with (19), (20) for all x. An increase in the field at large x imposes a severe restriction on the ratio $b = |H_x(x=0)|/|H_z(\infty)|$, which plays an extremely important role in the development of the tearing-mode instability.[13] In a medium which is unbounded along the coordinate x, for example, H_x increases without bound; this is a physically meaningless result. We thus have $H_x \equiv 0$, and the configuration is strictly one-dimensional.

Let us assume that the field occupies a bounded region $|x| < R$. In cases of practical interest we would have $R \gg l$. This is the case if the plasma pressure is low in comparison with the magnetic pressure. The thickness l of the layer in which the field changes sign (the plasma layer) is small in comparison with the length scale of the field, R. We then have

$$|H_x(0)/H_x(R)|=l/R, \tag{21}$$

and the maximum value of $H_x = H_x(R)$ must still be much smaller than $H_z(\infty)$. For example, if we have $H_x(R) \approx H_z(\infty)$, then the configuration should expand greatly with increasing z, in contrast with the case in Fig. 3. In specific cases, the quantity $\varepsilon = |H_x(R)/H_z(\infty)|$ can be found experimentally. The quantity b in which we are interested is then smaller by a factor of R/l, according to (21):

$$b=|H_x(x=0)/H_z(\infty)|=\varepsilon l/R. \tag{22}$$

In particular, for the tail of the earth's magnetosphere we would have $b \leqslant 0.01$, which is considerably smaller than the value usually adopted, $b = 0.1$. We might note in this connection that the quantity b is not crucial for the onset of an anisotropic tearing-mode instability[14]; the theory of Ref. 14 apparently would not change substantially if a small value of b were to be taken in account.

Let us review the contents of this section of the paper. First, we have another length scale here, R, in additon to l and L. We should therefore give a more general definition of an elongated configuration: For it, the relation $|H_x| \ll |H_z|$ or $b \ll 1$ must hold. If we abandon the assumption $\mathbf{H}_\perp = 0$ on a straight line (as in §1) but assume the existence of a weak field on this line (the H_x component on the z axis in Fig. 3), then the equilibrium configuration will generally not be one-dimensional. If, however, the configuration in the main volume is a quasiuniform field (under the conditions $l < |x| < R$, $R \gg l$), while in a small region $|x| < l$ the field varies over a short distance l (Fig. 3), the configuration will be nearly one-dimensional in this small region. More specifically, in addition to the small parameter ε, which is a measure of the elongation of the configuration, the expression for b acquires the additional small parameter l/R [see (22)]. As a result, we find $b \ll l/L$. In graphical terms, the field lines run nearly parallel to the z axis. In a region in which the field lines are closed, they intersect the z axis essentially nowhere in the entire region $0 < z < L - l_1$, $l_1 \approx l$. Only near the "vertex" of the closed field lines, at $L - l_1 < z < L$, do they intersect the z axis. In this region we have $H_x \approx H_z$; the field is not extended; and the equilibrium theory derived above does not apply. By way of comparison, under the condition $b \approx l/L$ the field lines intersect the z axis uniformly between $z = 0$ and $z = L$ (as in Fig. 3). Consequently, the observation that there is a transverse field and that it is not weak can be explained in terms of continuously existing magnetic islands.

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Translated by Dave Parsons