Movement of the interface between the *A* and *B* phases in superfluid helium-3: linear theory

N. B. Kopnin

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

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A microscopic theory is developed for calculating the frictional force at a moving interface between the *A* and *B* phases in superfluid helium-3 to first order in the velocity.

1. INTRODUCTION

The superfluid *A* and *B* phases in helium-3 can coexist over a comparatively wide range of temperature and pressure, because it is possible to supercool the *A* phase well below the equilibrium temperature T_{AB} (P) when the helium is cooled from the *A* to the *B* phase. This supercooling is made possible by virtue of the extraordinarily high energy barrier for critical nucleation of *B*-phase in the bulk of the *A*-phase.[1] This has the consequence that under certain conditions, a boundary (*AB* interface) may form between *A* and *B* phases occupying different regions of the containing vessel. If the *AB* interface is not pinned at inhomogeneities on the vessel wall or immobilized by some other means, it will move and its motion will be accompanied by a transition between the *A* and *B* phases. The motion of the *AB* interface has the following important property. The thickness of the interface is of the order of the coherence length ξ and is much less than the mean free path of the quasiparticles. The quasiparticle distribution function is therefore essentially constant over distances comparable to the thickness of the interface (at least if the latter moves slowly enough), and the quasiparticles do not collide with one another. Since this situation is analogous to an object moving in a rarefied gas, the interface can be pictured as a moving structure (a relief pattern superimposed on the order parameter) with a stable topology, while the resistance to the motion is caused by scattering of excitations by the moving structure.

Studies of the motion of the *AB* interface moves may give valuable insight into the kinetic processes in superfluid helium-3. Experimental studies along these lines were begun in Ref. 2. The qualitative model proposed in Ref. 3 corresponds to the general scenario outlined above. In the present paper we develop a more rigorous microscopic theory for the motion of the *AB* interface. As in Ref. 3, the key idea is to regard the interface as a moving relief pattern superimposed on the order parameter $\Delta(\mathbf{r})$; at large distances to either side of the interface, $\Delta(\mathbf{r})$ approaches the equilibrium values Δ_A and Δ_B for the *A* and *B* phases at the given temperature. The normal excitations are assumed to be in equilibrium with the walls of the vessel, and collisions between quasiparticles over distances comparable to the interface thickness are neglected. As might be expected, these starting assumptions lead to a moving interface of the type described above; the motion is driven by the difference in the thermodynamic potentials for the *A* and *B* phases, while the frictional force results from the scattering of quasiparticles by the moving hills and valleys in the potential. The final result of this paper is an expression for the viscous friction coefficient. It is similar in form to the result in Ref. 3 and gives the same order of magnitude for the friction coefficient. Unlike the result in Ref. 3, however, it does not involve any unknown quantities (such as the quasiparticle transmission coefficient across the *AB* interface, which appears in Ref. 3) and can be used to calculate the viscosity for arbitrary values of the parameters that describe the state of the system. The calculations in this paper are valid to first order in the velocity *u* of the interface.

The above formulation obviously requires that all hydrodynamic fluxes vanish at large distances from the *AB* interface. Problems related to the presence of such fluxes are considered in Ref. 4.

2. GENERAL EQUATIONS

We will derive an equation which describes the uniform motion of a two-dimensional relief pattern in the order parameter $\hat{\Delta}_{\mathbf{p}}(\mathbf{r})$. We first examine the steady-state case. The thermodynamic potential of the system is equal to[5]

$$\Omega=-T\ln\langle S\rangle, \tag{1}$$

where

$$S=T_\tau\exp\left[-\int_0^{1/T}H_{int}(\tau)\,d\tau\right], \tag{2}$$

T_τ is the time-ordering operator, τ is the "time," $\langle\cdots\rangle$ denotes a statistical average, and the Hamiltonian is

$$\begin{aligned}H_{int}=\mathrm{Tr}\Big\{&\int d^3\mathbf{r}\Big[\psi_\alpha^+(\mathbf{r})\Big(-\frac{\nabla^2}{2m}-E_F\Big)\psi_\beta(\mathbf{r})\\&\qquad+\frac{\Delta_{\alpha\gamma}^*(\mathbf{r})\Delta_{\gamma\beta}(\mathbf{r})}{2|\lambda|}\Big]\\&-\frac{1}{2}\int\int d^3\mathbf{r}\,d^3\mathbf{r}'[\psi_\alpha^+(\mathbf{r})\psi_\gamma^+(\mathbf{r})V(\mathbf{r},\mathbf{r}')\Delta_{\gamma\beta}(\mathbf{r}')\\&\qquad+\Delta_{\alpha\gamma}^*(\mathbf{r})V^+(\mathbf{r},\mathbf{r}')\psi_\gamma(\mathbf{r}')\psi_\beta(\mathbf{r}')]\Big\}.\end{aligned} \tag{3}$$

Here $\psi_\alpha(\mathbf{r})$ are the one-particle operators, $\lambda<0$ is the interaction constant, $V(\mathbf{r},\mathbf{r}')$ is the normalized interaction operator for the *p*-pairing interaction (see below), Tr denotes a trace over the spin indices, and $\Delta_{\alpha\beta}$ is the order parameter. The latter is regarded in Eq. (3) as an external field satisfying the following self-consistency equation, which results from equating the variation $\delta H_{\rm int}/\delta\Delta^*_{\alpha\beta}$ to zero:

$$\frac{\hat{\Delta}_{\mathbf{p}}(\mathbf{k})}{|\lambda|}=\int\hat{F}_\varepsilon\Big(\mathbf{p}'+\frac{\mathbf{k}}{2},\mathbf{p}'-\frac{\mathbf{k}}{2}\Big)V(\mathbf{p},\mathbf{p}')\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{d\varepsilon}{4\pi i}. \tag{4}$$

The temperature Green's function in (4)

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$$F_{\alpha\beta}(x,\ x')=\langle T_\tau\psi_\alpha(x)\psi_\beta(x')S\rangle/\langle S\rangle \tag{5}$$

is expressed in the momentum and frequency representations relative to the spatial coordinates and the time $\tau-\tau'$; in this representation we have $V(\mathbf{p},\mathbf{p}')=3(\mathbf{pp}')/p_F^2$. The function $\hat{F}_\varepsilon$ in Eq. (4) has been analytically continued along the real frequency axis ε, and the hat ($\hat{}$) denotes a matrix in spin space. Writing

$$\delta\Omega=-T\langle\delta S\rangle/\langle S\rangle$$

for the variation of the thermodynamic potential (1) with respect to the external fields $\delta\hat{\Delta}$, $\delta\hat{\Delta}^*$ and using Eqs. (2)–(5), one can show that

$$\int\delta\Omega\, d^3\mathbf{r}=-\frac{1}{2}\operatorname{Tr}\operatorname{Sp}\int\delta\hat{\mathscr{H}}\hat{\mathscr{G}}_\varepsilon(\mathbf{p},\mathbf{r})\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d\varepsilon}{4\pi i}d^3\mathbf{r}$$
$$+\frac{1}{2}\operatorname{Tr}\int\delta\frac{|\hat{\Delta}_\mathbf{p}|^2}{|\lambda|}\frac{d\Omega_\mathbf{p}}{4\pi}d^3\mathbf{r}. \tag{6}$$

Here we have introduced the matrices

$$\hat{\mathscr{G}}=\begin{pmatrix}\hat{G} & \hat{F}\\ -\hat{F}^+ & \hat{\bar{G}}\end{pmatrix},\qquad \hat{\mathscr{H}}=\begin{pmatrix}0 & -\hat{\Delta}_\mathbf{p}\\ \hat{\Delta}_\mathbf{p}^* & 0\end{pmatrix}$$

in function space, and Sp denotes a trace over the matrix indices. Equation (6) makes use of the mixed coordinate-momentum representation

$$\hat{\mathscr{G}}(\mathbf{p},\mathbf{r})=\int\hat{\mathscr{G}}\left(\mathbf{p}+\frac{\mathbf{k}}{2},\mathbf{p}-\frac{\mathbf{k}}{2}\right)e^{i\mathbf{kr}}\frac{d^3\mathbf{k}}{(2\pi)^3}.$$

In a few cases we will also use

$$\hat{\mathscr{G}}(\mathbf{r},\mathbf{r}')=\int\hat{\mathscr{G}}(\mathbf{p}_1,\mathbf{p}_2)\exp(i\mathbf{p}_1\mathbf{r}-i\mathbf{p}_2\mathbf{r}')\frac{d^3\mathbf{p}_1\,d^3\mathbf{p}_2}{(2\pi)^6},$$

i.e., the coordinate representation with respect to both sets of spatial variables.

In the time-dependent case the concept of thermodynamic potential is meaningful only if the system is nearly steady-state. In this case one must take $\hat{\mathscr{G}}_\varepsilon$ in (6) to be the stationary part $\hat{\mathscr{G}}_\varepsilon^{\mathrm{st}}$ of the complete Green's function

$$\hat{\mathscr{G}}^{tot}_{\varepsilon_+,\varepsilon_-}=\hat{\mathscr{G}}_\varepsilon^{st}2\pi\,\delta(\omega)+\hat{\mathscr{G}}^{nst}_{\varepsilon_+,\varepsilon_-},$$

where $\varepsilon_\pm=\varepsilon\pm\omega/2$. Substituting $\hat{\mathscr{G}}^{\mathrm{st}}=\hat{\mathscr{G}}^{\mathrm{tot}}-\hat{\mathscr{G}}^{\mathrm{nst}}$ in (6) and recalling the equation

$$\frac{\hat{\Delta}_\mathbf{p}(\mathbf{k},\omega)}{|\lambda|}=\int\hat{F}^{tot}_{\varepsilon_+,\varepsilon_-}\left(\mathbf{p}'+\frac{\mathbf{k}}{2},\mathbf{p}'-\frac{\mathbf{k}}{2}\right)V(\mathbf{p}',\mathbf{p})\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{d\varepsilon}{4\pi i} \tag{7}$$

for the order parameter in the time-dependent case, we find the following expression for the change in the thermodynamic potential due to time-dependent processes:

$$\int\delta\Omega\,d^3\mathbf{r}=\frac{1}{2}\operatorname{Tr}\operatorname{Sp}\int\delta\hat{\mathscr{H}}\hat{\mathscr{G}}^{nst}_{\varepsilon_+,\varepsilon_-}(\mathbf{p},\mathbf{r})\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d\varepsilon}{4\pi i}\frac{d\omega}{2\pi}d^3\mathbf{r}. \tag{8}$$

Equation (8) is similar to the one used in Ref. 6 for moving vortices in a type-II superconductor.

The method in Ref. 7 for calculating the motion of vortices in a pure type-II superconductor can also be used to calculate the nonstationary part $\hat{\mathscr{G}}^{\mathrm{nst}}$ of the Green's function. We will assume that to first order the interface moves as a whole and set

$$\hat{\Delta}(\mathbf{r},\ t)=\hat{\Delta}_0(\mathbf{r}-\mathbf{u}t)\exp(2im\mathbf{ur})+\hat{\Delta}_1,$$
$$\hat{G}^{R(A)}_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)=\hat{G}^{R(A)}_{(0)\varepsilon-\mathbf{pu}}(\mathbf{p}_+-m\mathbf{u},\mathbf{p}_--m\mathbf{u})\delta(\omega-\mathbf{ku})+\hat{G}^{R(A)}_{(1)},$$
$$\hat{F}^{+R(A)}_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)$$
$$=\hat{F}^{+R(A)}_{(0)\varepsilon-\mathbf{pu}}(\mathbf{p}_+-m\mathbf{u},\mathbf{p}_--m\mathbf{u})\delta(\omega-\mathbf{ku})+\hat{F}^{+R(A)}_{(1)}. \tag{9}$$

Similar definitions, with $\mathbf{p}-m\mathbf{u}$ replaced by $\mathbf{p}+m\mathbf{u}$, hold for $\overline{G}_{\varepsilon+,\varepsilon-}(\mathbf{p}_+,\mathbf{p}_-)$ and $\overline{F}_{\varepsilon+,\varepsilon-}(\mathbf{p}_+,\mathbf{p}_-)$. Here $\mathbf{p}_\pm=\mathbf{p}\pm\mathbf{k}/2$, and the unperturbed functions $\hat{G}^{R(A)}_{(0)\varepsilon}(\mathbf{p}_+,\mathbf{p}_-)$ and $\hat{F}^{R(A)}_{(0)\varepsilon}(\mathbf{p}_+,\mathbf{p}_-)$ satisfy the stationary equations, in which the unperturbed order parameter $\hat{\Delta}=\Delta_0(\mathbf{k})$ appears. Formulas (9) have the virtue that the corrections to the regular Green's functions $\hat{G}^{R(A)}_{(1)}$, $\hat{F}^{R(A)}_{(1)}$, etc., are expressed in terms of $\hat{\Delta}_1$ and $\hat{\Delta}_1^*$ only and do not involve any time derivatives of the unperturbed function $\hat{\Delta}_0(\mathbf{r}-\mathbf{u}t)$. This is easily seen by using the equations for the regular Green's functions. We stress that this result is valid only for pure materials (superfluid helium with negligible wall effects, pure superconductors[7,8]).

The complete Green's function is of the form[9]

$$\hat{\mathscr{G}}^{tot}_{\varepsilon_+,\varepsilon_-}=(\hat{\mathscr{G}}^R_{\varepsilon_+,\varepsilon_-}-\hat{\mathscr{G}}^A_{\varepsilon_+,\varepsilon_-})\operatorname{th}\left(\frac{\varepsilon}{2T}\right)$$
$$-\frac{\omega}{4T}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)(\hat{\mathscr{G}}^R_{\varepsilon_+,\varepsilon_-}+\hat{\mathscr{G}}^A_{\varepsilon_+,\varepsilon_-})+\hat{\mathscr{G}}^{(a)}_{\varepsilon_+,\varepsilon_-}. \tag{10}$$

This assumes that the normal excitations are in equilibrium (at rest) with the walls of the vessel. In the coordinate representation, the second term in (10) can be written in the form

$$-\frac{\omega}{4T}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)\hat{\mathscr{G}}^{R(A)}_{\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}')=\frac{\omega}{4T}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)\int\hat{\mathscr{G}}^{R(A)}_{(0)\varepsilon}(\mathbf{r},\mathbf{r}_1)$$
$$\times\hat{\mathscr{H}}(\omega,\mathbf{r}_1)\hat{\mathscr{G}}^{R(A)}_{(0)\varepsilon}(\mathbf{r}_1,\mathbf{r}')d^3\mathbf{r}_1. \tag{11}$$

If we neglect inelastic collisions between the quasi-particles, the anomalous part of the Green's function has the form[9]

$$\hat{\mathscr{G}}^{(a)}_{\varepsilon_+,\varepsilon_-}(\mathbf{r},\mathbf{r}')=-\frac{\omega}{2T}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)\int\hat{\mathscr{G}}^R_{(0)\varepsilon}(\mathbf{r},\mathbf{r}_1)\hat{\mathscr{H}}(\omega,\mathbf{r}_1)$$
$$\times\hat{\mathscr{G}}^A_{(0)\varepsilon}(\mathbf{r}_1,\mathbf{r}')d^3\mathbf{r}_1. \tag{12}$$

In Eqs. (11) and (12), $\hat{\mathscr{G}}^{R(A)}_{(0)\varepsilon}$ are regular stationary Green's functions. In what follows we will drop the subscript (0) from regular functions depending only on a single frequency. Since the interface moves as a whole, the quantity $\omega\hat{\mathscr{H}}(\omega,\mathbf{r})$ in (11) and (12) should be understood as $-i\mathbf{u}\nabla\hat{\mathscr{H}}_0(\mathbf{r})$. We now use the equations

$$\hat{\mathscr{G}}_\varepsilon^{-1}(\mathbf{r})\hat{\mathscr{G}}^{R(A)}_\varepsilon(\mathbf{r},\mathbf{r}')=\hat{1}\delta(\mathbf{r}-\mathbf{r}') \tag{13}$$

for the regular Green's functions, where the inverse matrix is

$$\hat{\mathscr{G}}_\varepsilon^{-1}(\mathbf{r})=\left(-\frac{\nabla^2}{2m}-E_F\right)-\varepsilon\sigma_3+\hat{\mathscr{H}}_0(\mathbf{r})$$

and σ_3 is the Pauli matrix in function space. Recalling the relation

$$\mathbf{u}\nabla\hat{\mathscr{H}}_0(\mathbf{r})=u\nabla\hat{\mathscr{G}}_\varepsilon^{-1}(\mathbf{r}) \tag{14}$$

we transform expressions (10)–(12) into

$$\int \hat{\mathcal{G}}^{tot}_{\varepsilon_+,\varepsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\frac{d\varepsilon}{4\pi i}\frac{d\omega}{2\pi}=\int \mathrm{th}\left(\frac{\varepsilon}{2T}\right)[\hat{\mathcal{G}}_\varepsilon{}^R(\tilde{\mathbf{p}}_+,\tilde{\mathbf{p}}_-)+\hat{\mathcal{G}}^R_{(1)}-\hat{\mathcal{G}}_\varepsilon{}^A(\tilde{\mathbf{p}}_+,\tilde{\mathbf{p}}_-)-\hat{\mathcal{G}}^A_{(1)}]\frac{d\varepsilon}{4\pi i}+\int\hat{\mathcal{G}}_\varepsilon{}^{sc}\frac{d\varepsilon}{4\pi i}, \tag{15}$$

where $\tilde{\mathbf{p}}=\mathbf{p}\pm m\mathbf{u}$. The function $\hat{\mathcal{G}}^{\text{sc}}_\varepsilon$ is more conveniently expressed using the coordinate representation:

$$\begin{aligned}\hat{\mathcal{G}}_\varepsilon{}^{sc}(\mathbf{r},\mathbf{r}')=\frac{i}{4T}\mathrm{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)\int d\mathbf{S}_1\frac{1}{2m}\{[\hat{\mathcal{G}}_\varepsilon{}^R(\mathbf{r},\mathbf{r}_1)\\ -\hat{\mathcal{G}}_\varepsilon{}^A(\mathbf{r},\mathbf{r}_1)]\nabla_1(\mathbf{u}\nabla_1)\hat{\mathcal{G}}_\varepsilon{}^A(\mathbf{r}_1,\mathbf{r}')-\nabla_1[\hat{\mathcal{G}}_\varepsilon{}^R(\mathbf{r},\mathbf{r}_1)\\ -\hat{\mathcal{G}}_\varepsilon{}^A(\mathbf{r},\mathbf{r}_1)](\mathbf{u}\nabla_1)\hat{\mathcal{G}}_\varepsilon{}^A(\mathbf{r}_1,\mathbf{r}')+\hat{\mathcal{G}}_\varepsilon{}^R(\mathbf{r},\mathbf{r}_1)\nabla_1(\mathbf{u}\nabla_1)[\hat{\mathcal{G}}_\varepsilon{}^A(\mathbf{r}_1,\mathbf{r}')\\ -\hat{\mathcal{G}}_\varepsilon{}^R(\mathbf{r}_1,\mathbf{r}')]-\nabla_1\hat{\mathcal{G}}_\varepsilon{}^R(\mathbf{r},\mathbf{r}_1)(\mathbf{u}\nabla_1)[\hat{\mathcal{G}}_\varepsilon{}^A(\mathbf{r}_1,\mathbf{r}')-\hat{\mathcal{G}}_\varepsilon{}^R(\mathbf{r}_1,\mathbf{r}')]\}.\end{aligned} \tag{16}$$

Here the integration is over a surface at infinity (cf. Ref. 7).

We now consider the expression

$$\begin{aligned}&\mathrm{Sp\,Tr}\int\delta\hat{\mathcal{H}}(\mathbf{r})\hat{\mathcal{G}}^{tot}_{\varepsilon_+,\varepsilon_-}(\mathbf{p},\mathbf{r})\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d\varepsilon}{4\pi i}\frac{d\omega}{2\pi}d^3\mathbf{r}\\ &=\mathrm{Sp\,Tr}\int\mathrm{th}\left(\frac{\varepsilon}{2T}\right)\delta\hat{\mathcal{H}}(-\mathbf{k})[\hat{\mathcal{G}}_\varepsilon{}^R(\mathbf{p}_+,\mathbf{p}_-)+\hat{\mathcal{G}}^R_{(1)}-\hat{\mathcal{G}}_\varepsilon{}^A(\mathbf{p}_+,\mathbf{p}_-)\\ &-\hat{\mathcal{G}}^A_{(1)}]\frac{d\varepsilon}{4\pi i}\frac{d^3\mathbf{p}\,d^3\mathbf{k}}{(2\pi)^6}+\mathrm{Sp\,Tr}\int[\delta\hat{\mathcal{H}}(\mathbf{r})\hat{\mathcal{G}}_\varepsilon{}^{sc}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}}\frac{d\varepsilon}{4\pi i}d^3\mathbf{r}.\end{aligned} \tag{17}$$

In the integration over $d\mathbf{p}^3$ in the first term we have made a translation $\tilde{\mathbf{p}}\to\mathbf{p}$ in momentum space and used the definition of the unperturbed functions. The first term in (17) gives the stationary contribution, since it does not contain the velocity $\mathbf{u}$ or any nonlocal operations involving the time coordinate. To use Eq. (8) one must thus take $\hat{\mathcal{G}}^{\text{nst}}=\hat{\mathcal{G}}^{\text{sc}}$ as the nonstationary part, where $\hat{\mathcal{G}}^{\text{sc}}$ is given by Eq. (16).

We assume that the Hamiltonian is perturbed by an infinitesimal displacement $\mathbf{d}$ of the system along the normal $\mathbf{n}$ to the AB interface; then $\delta\mathcal{H}=\mathbf{d}\nabla\mathcal{H}$ and (8) becomes

$$S(\Omega_+-\Omega_-)=\frac{1}{2}\mathrm{Tr\,Sp}\int[\mathbf{n}\nabla\hat{\mathcal{H}}(\mathbf{r})\hat{\mathcal{G}}_\varepsilon{}^{sc}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}}\frac{d\varepsilon}{4\pi i}d^3\mathbf{r}. \tag{18}$$

Here Ω_+ and Ω_- are the thermodynamic potential densities far from the interface, on the positive and negative sides of it relative to the normal $\mathbf{n}$; S is the cross section of the containing vessel. Since $\hat{\mathcal{G}}^{\text{sc}}_\varepsilon$ is proportional to the velocity, Eq. (18) takes the form

$$\Omega_+-\Omega_-=\Gamma\mathbf{un} \tag{19}$$

and thus defines the frictional force $\mathbf{F}_{\text{fr}}=-\Gamma\mathbf{u}$ acting on the moving AB interface.

Using once again the relation $\mathbf{n}\nabla\hat{\mathcal{H}}=\mathbf{n}\nabla\hat{\mathcal{G}}_\varepsilon^{-1}$ and the equation of motion (13) for the Green's functions, one can show that

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr\,Sp}\int d^3\mathbf{r}[\mathbf{n}\nabla\hat{\mathcal{H}}(\mathbf{r})\hat{\mathcal{G}}_\varepsilon{}^{sc}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}}=\frac{i}{32m^2T}\mathrm{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)\\ &\times\mathrm{Sp\,Tr}\int\int dS_i'\,dS_k\{\nabla_k(\mathbf{n}\nabla)\\ &\times[\hat{\mathcal{G}}^R(\mathbf{r},\mathbf{r}')-\hat{\mathcal{G}}^A(\mathbf{r},\mathbf{r}')]\nabla_i'(\mathbf{u}\nabla')\hat{\mathcal{G}}^A(\mathbf{r}',\mathbf{r})\\ &-(\mathbf{n}\nabla)[\hat{\mathcal{G}}^R(\mathbf{r},\mathbf{r}')-\hat{\mathcal{G}}^A(\mathbf{r},\mathbf{r}')]\nabla_i'(\mathbf{u}\nabla')\nabla_k\hat{\mathcal{G}}^A(\mathbf{r}',\mathbf{r})\\ &-\nabla_k(\mathbf{n}\nabla)\nabla_i'[\hat{\mathcal{G}}^R(\mathbf{r},\mathbf{r}')-\hat{\mathcal{G}}^A(\mathbf{r},\mathbf{r}')]\\ &\times(\mathbf{u}\nabla')\hat{\mathcal{G}}^A(\mathbf{r}',\mathbf{r})+(\mathbf{n}\nabla)\nabla_i'[\hat{\mathcal{G}}^R(\mathbf{r},\mathbf{r}')\\ &-\hat{\mathcal{G}}^A(\mathbf{r},\mathbf{r}')](\mathbf{u}\nabla')\nabla_k\mathcal{G}^A(\mathbf{r}',\mathbf{r})+\nabla_k(\mathbf{n}\nabla)\mathcal{G}^R(\mathbf{r},\mathbf{r}')\nabla_i'(\mathbf{u}\nabla')\\ &\times[\hat{\mathcal{G}}^A(\mathbf{r}',\mathbf{r})-\hat{\mathcal{G}}^R(\mathbf{r}',\mathbf{r})]-(\mathbf{n}\nabla)\hat{\mathcal{G}}^R(\mathbf{r},\mathbf{r}')\nabla_i'(\mathbf{u}\nabla')\nabla_k[\hat{\mathcal{G}}^A(\mathbf{r}',\mathbf{r})\\ &-\hat{\mathcal{G}}^R(\mathbf{r}',\mathbf{r})]-\nabla_k(\mathbf{n}\nabla)\nabla_i'\hat{\mathcal{G}}^R(\mathbf{r},\mathbf{r}')(\mathbf{u}\nabla')[\hat{\mathcal{G}}^A(\mathbf{r}',\mathbf{r})-\hat{\mathcal{G}}^R(\mathbf{r}',\mathbf{r})]\\ &+(\mathbf{n}\nabla)\nabla_i'\mathcal{G}^R(\mathbf{r},\mathbf{r}')\nabla_k(\mathbf{u}\nabla')[\hat{\mathcal{G}}^A(\mathbf{r}',\mathbf{r})-\mathcal{G}^R(\mathbf{r}',\mathbf{r})]\}.\end{aligned} \tag{20}$$

Here the integrals over the spatial coordinates $\mathbf{r}$ and $\mathbf{r}'$ are over two infinitely remote surfaces such that the $\mathbf{r}$ coordinates lie inside the surface S'. It is evident from (20) that the frictional force acting on the moving AB interface is determined by the behavior of quasiparticles whose distance from the interface is large compared to its thickness, i.e., by the transmission and reflection (scattering) of the quasiparticles by the interface. This situation corresponds to the model considered in Ref. 3.

3. INTERACTION OF QUASIPARTICLES WITH THE INTERFACE; FRICTIONAL FORCE

To calculate the frictional force, we expand the Green's functions in terms of eigenfunctions of the Bogolyubov equations. We emphasize that since only the behavior of $\hat{\mathcal{G}}^{R(A)}_\varepsilon(\mathbf{r},\mathbf{r}')$ at large distances matters in (20), in the expansion it suffices to keep only the eigenfunctions belonging to the continuous spectrum:

$$\mathcal{G}^{R(A)}_{\alpha\beta}(\mathbf{r},\mathbf{r}')=\sum_s\int\frac{\check{\mathcal{U}}_\alpha{}^{(s)}(\mathbf{r})\check{\mathcal{U}}_\beta{}^{+(s)}(\mathbf{r}')}{E-\varepsilon\mp i\delta}\frac{dp_y\,dp_z\,dE}{(2\pi)^3}. \tag{21}$$

Here the sum on s is over all linearly independent functions with the same values of p_y, p_z, and E; the x axis is taken along the normal $\mathbf{n}$ to the interface. We have

$$\check{\mathcal{U}}_\alpha=\exp(ip_yy+ip_zz)\check{w}_\alpha(x),$$

where the spinor

$$\check{w}_\alpha=\begin{pmatrix}u_\alpha\\-v_\alpha\end{pmatrix},\quad \check{w}_\alpha{}^+=(u_\alpha{}^+,v_\alpha{}^+)$$

satisfies the Bogolyubov equation

$$\left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2}-\frac{p_x{}^2}{2m}\right)\check{w}_\alpha^{(s)}+\sum_\beta\mathcal{H}_{\alpha\beta}\check{w}_\beta^{(s)}=E\sigma_3\check{w}_\alpha^{(s)} \tag{22}$$

($p_x^2=p_F^2-p_y^2-p_z^2$) and the normalization conditions

$$\sum_s \int u_\alpha^{(s)}(x)\,u_\beta^{(s)}(x')\,\frac{dE}{2\pi}$$

$$=\sum_s \int v_\alpha^{(s)}(x)\,v_\beta^{(s)}(x')\,\frac{dE}{2\pi}=\delta_{\alpha\beta}\delta(x-x'),\tag{23}$$

$$\frac{1}{2}\sum_\alpha \int \check{w}_\alpha^{+(s)}(x)\,\sigma_3\check{w}_\alpha^{(s')}(x)\,dx=2\pi\delta(E-E')\,\delta_{ss'}.$$

In addition, Eq. (22) and its Hermitian conjugate imply the "flux density" conservation equation

$$\frac{\partial}{\partial x}\sum_\alpha\left[\check{w}_\alpha^{+(s)}(x)\,\frac{\partial \check{w}_\alpha^{(s')}(x)}{\partial x}-\frac{\partial \check{w}_\alpha^{+(s)}(x)}{\partial x}\,\check{w}_\alpha^{(s')}(x)\right]=0\tag{24}$$

for any pair of functions $\check{w}_\alpha^{(s)}$ and $\check{w}_\alpha^{(s')}$ belonging to the same quantum numbers p_y, p_z, E.

Since the states in the A and B phases far from the AB interface are unitary, i.e.,

$$\sum_\beta \Delta_{\alpha\beta}\Delta_{\beta\gamma}^{*}=|\Delta(\mathbf{p})|^2\delta_{\alpha\gamma},$$

the functions $\check{w}_\alpha$ at large distances must be expressible as

$$\check{w}_\alpha^{(s)}(x)=\sum_\beta\begin{pmatrix}\delta_{\alpha\beta}u_k^{(\rho)}(x)\\ -\dfrac{\Delta_{\alpha\beta}^{*}}{|\Delta(\mathbf{p})|}v_k^{(\rho)}(x)\end{pmatrix}e_\beta^{(\rho)}.\tag{25}$$

Here the pair of quantum numbers $s=(\rho,k)$, $\rho=1,2$, labels the linearly independent states in spin space, and $e_\alpha^{(\rho)}$ are two orthogonal spinors:

$$\sum_\alpha e_\alpha^{+(\rho)}e_\alpha^{(\rho')}=\delta_{\rho\rho'}$$

for which explicit expressions will not be required. Far from the interface the functions

$$\check{w}_k^{(\rho)}(x)=\begin{pmatrix}u_k^{(\rho)}(x)\\ -v_k^{(\rho)}(x)\end{pmatrix}$$

satisfy the equations

$$\begin{aligned}&(\partial^2/\partial x^2+p_x^2)u-2m|\Delta(\mathbf{p})|v=-2mEu,\\ &(\partial^2/\partial x^2+p_x^2)v+2m|\Delta(\mathbf{p})|u=2mEv,\end{aligned}\tag{26}$$

and can hence be expressed as a linear combination

$$\check{w}_i^{(\rho)}=\sum_{k=1,2}\left[T_{ik}^{(\rho)}\check{w}_k^{(0)}+S_{ik}^{(\rho)}\check{w}_k^{*(0)}\right]\tag{27}$$

of any four linearly independent solutions of Eqs. (26):

$$\check{w}_k^{(0)}=C_k\check{\beta}_k e^{ip_kx}\ \text{and}\ \check{w}_k^{*(0)}=C_k\check{\beta}_k e^{-ip_kx}\qquad(k=1,2),$$

where

$$p_{1,2}=[p_x^2\pm2m(E^2-|\Delta(\mathbf{p})|^2)^{1/2}]^{1/2},$$

$$C_{1,2}^2=|mE/p_{1,2}(E^2-|\Delta(\mathbf{p})|^2)^{1/2}|,$$

$$\check{\beta}_{1,2}=2^{-1/2}\begin{pmatrix}[1\pm(E^2-|\Delta(\mathbf{p})|^2)^{1/2}/E]^{1/2}\\ -[1\mp(E^2-|\Delta(\mathbf{p})|^2)^{1/2}/E]^{1/2}\end{pmatrix}.$$

In these formulas $|\Delta(\mathbf{p})|$ is the absolute value of the order parameter in phase A or B far from the interface. Together with their complex conjugates, the functions in (27) furnish a system of linearly independent solutions of the Bogolyubov equations (22). We note that the matrix elements $S_{ik}^{(\rho)}$ describe particle scattering with a momentum transfer of order $2p_x\approx2p_F$ by a potential $|\Delta(\mathbf{p})|\ll E_F$. Since the quasiparticles move semiclassically, the matrix elements $S_{ik}^{(\rho)}$, which correspond to "reflection over a barrier," are exponentially small of order $\exp(-p_F\xi)$ and may be set equal to zero. The matrix elements $T_{ik}^{(\rho)}$ describe Andreev reflection of quasiparticles by the AB interface.

Let us first consider states with energies satisfying the condition

$$E^2>\max\{|\Delta_A(\mathbf{p})|^2,\ |\Delta_B|^2\}$$

for a given $\mathbf{p}$. The system of linearly independent solutions for each $\rho=1,2$ then consists of four functions: the two functions $\check{w}_i^{(\rho)}$ in (27) and their conjugates $\check{w}_i^{(\rho)*}$ $(i=1,2)$. The normalization conditions (23) require that

$$\sum_m T_{im}^{*(\rho)}T_{km}^{(\rho)}=\delta_{ik}.\tag{28}$$

From the flux conservation conditions (24) we find that

$$T_{i1}^{*(\rho)}T_{k1}^{(\rho)}-T_{i2}^{*(\rho)}T_{k2}^{(\rho)}=\text{const.}\tag{29}$$

at the AB interface.

For states with energies

$$\min\{|\Delta_A(\mathbf{p})|^2,\ |\Delta_B|^2\}<E^2<\max\{|\Delta_A(\mathbf{p})|^2,\ |\Delta_B|^2\}$$

each $\rho=1,2$ corresponds to a pair of linearly independent functions:

$$\check{w}^{(\rho)}=\sum_{k=1,2}T_k^{(\rho)}\check{w}_k^{(0)}\ \text{and}\ \check{w}^{*(\rho)}.$$

The matrix elements T_k satisfy the normalization and flux conservation conditions:

$$T_1^*T_1=T_2^*T_2=1.\tag{30}$$

To calculate the frictional force, we substitute expansion (21) for the Green's functions into the right-hand side of Eq. (20). The integrals over the energies dE for each of the terms in the sum are calculated by moving the path of integration into the upper (lower) half plane, according to whether the function $\check{w}_k^{(0)}(x')$ or $\check{w}_k^{(0)*}(x')$ with $x'\to+\infty$ or $x;\to-\infty$ is analytic in the upper (lower) half plane of the complex variable E. In this process the contour of integration moves past the zeros of the denominators as $E=\varepsilon\pm i\delta$, respectively. As an illustration of one of the intermediate steps in the calculations, we obtain, e.g., for

$\varepsilon^2 > \max\{|\Delta_A(\mathbf{p})|^2, |\Delta_B|^2\}$:

$$\frac{1}{2}\operatorname{Tr}\operatorname{Sp}\int [\mathbf{n}\nabla\mathscr{H}\mathscr{G}_\varepsilon{}^{sc}(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}}\, d^3\mathbf{r} = \frac{i\mathbf{nu}S}{4m^2T}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)$$

$$\times\sum_{\rho iklm}\int\frac{dp_y\,dp_z}{(2\pi)^2}\left[\left(T_{ik}^{(\rho)}T_{lk}^{*(\rho)}\check{w}_k{}^{(0)+}\frac{\partial^2\check{w}_k{}^{(0)}}{\partial x^2}\right)_-^+\right]$$

$$\times\left[\left(T_{im}^{*(\rho)}T_{lm}^{(\rho)}\check{w}_m{}^{(0)+}\frac{\partial^2\check{w}_m{}^{(0)}}{\partial x^2}\right)_-^+\right], \tag{31}$$

where $(F)_-^+ = F(x = +\infty) - F(x = -\infty)$. We next calculate that

$$\sum_k T_{ik}^{(\rho)}T_{lk}^{*(\rho)}\check{w}_k{}^{(0)+}\frac{\partial^2\check{w}_k{}^{(0)}}{\partial x^2}$$

$$= -{}^1/{}_2 m(p_1-p_2)\,(T_{i1}^{(\rho)}\,T_{l1}^{*(\rho)}+T_{i2}^{(\rho)}\,T_{l2}^{*(\rho)})$$

$$-{}^1/{}_2 m(p_1+p_2)\,(T_{i1}^{(\rho)}T_{l1}^{*(\rho)}-T_{i2}^{(\rho)}T_{l2}^{*(\rho)}) \tag{32}$$

$$= -\frac{m^2[\varepsilon^2-|\Delta(\mathbf{p})|^2]^{1/2}}{p_x}\delta_{il}$$

$$-mp_x(T_{i1}^{(\rho)}T_{l1}^{*(\rho)}-T_{i2}^{(\rho)}T_{l2}^{*(\rho)}),$$

the last relation holding through terms of order Δ/E_F. Since by (29) we have

$$(T_{i1}^{(\rho)}T_{l1}^{*(\rho)}-T_{i2}^{(\rho)}T_{l2}^{*(\rho)})_-^+ = 0,$$

we find that the right-hand side of (31) is equal to

$$\frac{i\mathbf{un}S}{T}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)\int\frac{dp_y\,dp_z}{(2\pi)^2}\left[\left(\frac{p_1-p_2}{2}\right)_-^+\right]^2.$$

Recalling the identity

$$\left.\frac{dp_y\,dp_z}{(2\pi)^2p_x{}^2}\right|_{p_x>0} = \left.\frac{p_F}{2\pi}\frac{d\Omega_\mathbf{p}}{4\pi|p_x|}\right|_{p_x\gtrless 0},$$

one gets the final expression for the friction coefficient defined by (19):

$$\Gamma=\nu(0)\int\frac{d\Omega_\mathbf{p}}{4\pi}\int_{-\infty}^{+\infty}\frac{d\varepsilon}{4T}\operatorname{ch}^{-2}\left(\frac{\varepsilon}{2T}\right)\frac{m}{|p_x|}$$

$$\times\{(\varepsilon^2-|\Delta_{min}(\mathbf{p})|^2)\theta(\varepsilon^2-|\Delta_{min}(\mathbf{p})|^2)$$

$$+(\varepsilon^2-|\Delta_{min}(\mathbf{p})|^2)\theta(\varepsilon^2-|\Delta_{min}(\mathbf{p})|^2)\theta(|\Delta_{max}(\mathbf{p})|^2-\varepsilon^2)$$

$$+(\varepsilon^2-|\Delta_{max}(\mathbf{p})|^2)\theta(\varepsilon^2-|\Delta_{max}(\mathbf{p})|^2)-2[\varepsilon^2-|\Delta_{min}(\mathbf{p})|^2]^{1/2}$$

$$\times[\varepsilon^2-|\Delta_{max}(\mathbf{p})|^2]^{1/2}\theta(\varepsilon^2-|\Delta_{max}(\mathbf{p})|^2)\}. \tag{33}$$

Here $\nu(0) = mp_F/2\pi^2$ is the density of states, and the quantities

$$|\Delta_{min}(\mathbf{p})| = \min\{|\Delta_A(\mathbf{p})|, |\Delta_B|\},$$

$$|\Delta_{max}(\mathbf{p})| = \max\{|\Delta_A(\mathbf{p})|, |\Delta_B|\}$$

are chosen to be the maximum and minimum values of $|\Delta_A(\mathbf{p})|$ and $|\Delta_B|$ for each $\mathbf{p}$; they depend on the direction of $\mathbf{p}$.

4. DISCUSSION

In discussing Eq. (33) we first note that the viscosity coefficient Γ is readily seen to be essentially positive, so that the *AB* interface moves toward higher thermodynamic potentials. Moreover, Γ is very small because the leading terms in (32) cancel by virtue of the flux conservation condition (29). We have $\Gamma \sim \nu(0)\Delta^2/v_F$ in order of magnitude; this is smaller by a factor of $(\Delta/E_F)^2$ than the value Γ would have if the excitations were scattered in the ordinary way (i.e., with a change in sign of the momentum p_x). This result agrees with the estimates given in Ref. 3. Since the scattering matrix T_{ik} is unitary, the remaining terms in Γ are independent of the detailed structure of the *AB* interface and depend only on the properties of the superfluid phases far from the interface (namely, on the orientation of the anisotropy vector **l** in the *A* phase relative to the interface, and on the absolute values of the order parameters in the *A* and *B* phases).

Equation (33) for Γ is similar in form to the result obtained in Ref. 3. However, it does not involve the quasiparticle transmission coefficient across the *AB* interface, which remained undetermined in Ref. 3.

Because the viscosity coefficient is so small, the velocity u should increase rapidly as we move away from the phase equilibrium curve $T_{AB}(P)$, on which $\Omega_A = \Omega_B$. Indeed, in the experiment in Ref. 2, u began to exceed the critical velocity v_c quite quickly.

The phase equilibrium curve $T_{AB}(P)$ also depends on the magnetic field and may extend over a very wide range of temperatures. It will therefore be useful to study how Γ depends on temperature over the entire interval from 0 K to T_c.

At low temperatures, for which $T \ll |\Delta_A|, |\Delta_B|$, most of the contribution to the angular part of the $d\mathbf{p}$ integral comes from directions parallel to the anisotropy vector **l**. If we set $|\Delta_A(\mathbf{p})| = \Delta_A \sin\vartheta$ is the angle between **l** and **p**, we have

$$\Gamma=7\pi^4\nu(0)T^4/30v_F\Delta_A{}^2|\cos\chi|,$$

where χ is the angle between **l** and the normal **n** to the interface. This formula breaks down when **l** is parallel to the *AB* interface ($\chi = \pi/2$), in which case the logarithmic divergence in (33) for small p_x becomes important. In our model, the cutoff for this divergence should lie at momenta $p_x^2 \sim m(\varepsilon^2 - |\Delta(\mathbf{p})|^2)^{1/2}$. For $\chi = \pi/2$ we then get

$$\Gamma = \frac{6\zeta(3)\nu(0)T^3}{\pi v_F\Delta_A}\ln\left(\frac{E_F}{T}\right)$$

at low temperatures. We see that when **l** is parallel to the interface, the friction coefficient is proportional to T^3 rather than to T^4 and falls off more slowly with temperature.

It is difficult to calculate Γ in closed form for large T. For $T \to T_c$ an estimate gives

$$\Gamma \sim \frac{\nu(0)T_c{}^2}{v_F}\ln\left(\frac{E_F}{\Delta}\right)\left(1-\frac{T}{T_c}\right)^{3/2}.$$

Such orders of magnitude are in qualitative agreement with the experimental results[2] (see Ref. 3 for more on this point).

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