Bremsstrahlung accompanied by excitation and ionization of the scattering atoms

V. A. Astapenko, V. M. Buĭmistrov, and Yu. A. Krotov
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The cross section for bremsstrahlung by a charged relativistic particle in the field of an atom is computed with allowance for the polarization and excitation of the atom. The Lamb–Wheeler result [Phys. Rev. **55**, 858 (1939)] is obtained in the particular case when the polarization contribution is negligible. The approach developed makes possible the elimination of a certain difficulty inherent in the Bethe-Heitler theory: As the nuclear charge goes to zero, the cross section computed here goes over into the cross section for bremsstrahlung in collisions with free electrons. Because of the screening approximation, this passage to the limit is not possible in the Bethe-Heitler approximation. Experiments on the polarization effects in bremsstrahlung are discussed.

§ 1. INTRODUCTION

Bremsstrahlung is one of the fundamental physical effects, and has therefore been the subject of theoretical and experimental studies for a long time.[1] The important features of bremsstrahlung are correctly described by classical theory.[2,3] Sommerfeld,[1] Bethe and Heitler,[4] and Sauter[5] constructed the quantum theory of bremsstrahlung shortly after the development of quantum mechanics. Latter investigations by Ter-Mikaelyan,[6] Landau and Pomeranchuk,[7] and Migdal[8] established important limitations on the applicability of this theory in condensed media. As for the bremsstrahlung from scattering of a charged particle by an atom, here the theory did not undergo any significant changes.

The situation changed relatively recently when another limitation connected with the so-called screening approximation was elucidated.[9–12] The point is that Bethe and Heitler and Sauter considered in their investigations the scattering of an electron in a given external field.

In the consistent theory the "traditional" bremsstrahlung amplitude, i.e., the amplitude of bremsstrahlung in a screened static field, is supplemented by another term—the amplitude of the polarization bremsstrahlung. Underlying this new radiation mechanism is the dynamical polarization of the atomic electrons by the incoming particle. The polarization oscillations lead to the emission of photons, similarly to what happens in excitation of dynamical polarization by a free electromagnetic field.

Of greatest importance to us now is the fact that an atomic electron can change its state during collision with a charged particle. This change occurs in one quantum event, i.e., a photon is emitted at the expense of the energy of the incoming particle, and the atom is excited or ionized. This process was first considered long ago by Lamb and Wheeler for a relativistic incoming particle.[13] But Lamb and Wheeler's paper[13] is, if one may put it that way, of a middle-of-the-road nature, i.e., account is taken in it of the fact that the atomic electron can change its state, but the interaction of this electron with the free electromagnetic field is ignored. Thus, the polarization effect discussed above is not taken into consideration. The task of the present paper consists in the systematic analysis of "inelastic" bremsstrahlung, i.e., that radiation process in which the emission of a photon is accompanied by a change in the state of the atom. As to "elastic" bremsstrahlung, in which the state of the atom does not change, and "inelastic" bremsstrahlung emission by nonrelativistic particles,[1)] they are considered in Refs. 9–12 and 14–17.

There immediately arises the question of comparison of the theory with experiment, a question which has the same bearing on both "elastic" and "inelastic" bremsstrahlung. On the face of it, it seems somewhat strange that the significant change in the formulas for the bremsstrahlung cross section went unnoticed for a long time. This question is discussed in Refs. 9–12. Here we shall dwell on one of the strongest manifestations of polarization bremsstrahlung, namely, the bremsstrahlung from a proton in the low-frequency spectral region, where the photon energy is much lower than the initial energy of the incoming particle. It has long become almost platitudinous to state that a heavy particle radiates less than a lighter one in a ratio equal to the inverse square of their masses. This is quite understandable when the emission occurs as a result of the oscillation of the dipole moment of a heavy incoming particle, since the second derivative of the dipole moment is inversely proportional to the particle mass. But this is not at all the case when we are dealing with polarization bremsstrahlung, i.e., when the radiation is emitted by the dipole moment of the atomic electrons. Let us turn to the graphic picture of scattering. Let a proton with velocity **v** collide with a hydrogen atom in the laboratory system. Let us go over to the system of coordinates fixed to the proton. In this system the atom impinges with the same velocity **v** on the proton. In the dipole—with respect to the photon—approximation, bremsstrahlung is not produced by a collision between identical particles (protons) in both the classical and quantum theories. There remains the emission of the electron in the field of the proton. But this is now the radiation of a light particle (with the same velocity and, consequently, with a lower energy). As far as we know, Refs. 14 and 18 contain the first published consistent theory of the bremsstrahlung from a proton in a collision with an atom. Of great importance is the recently published work by Ishii and Morita.[19] It is essentially the application of the formulas obtained in Ref. 14 to a specific target atom, namely, aluminum. An anomalously intense low-frequency x-ray background has indeed been observed in the bombardment of different targets by protons, and has been described in a large number of papers (see, for example, Refs. 20 and 21). The contribution of Ishii and Morita[19] lies in the fact that they separated out this effect from the other types of background radiation. The results of their calculation, which was carried out in a definite kilovolt photon frequency interval for a pro-

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ton energy of several MeV, are in agreement with experiment. It is interesting that this is an interval in which other previously considered processes lead to a background radiation intensity that is too low. The mechanism of the continuous x-ray background has been studied in a large number of theoretical and experimental investigations (see, for example, Refs. 19–23). In the case of the bremsstrahlung from protons, it is important to estimate the role of the bremsstrahlung from the secondary electrons (the so-called δ electrons). The bremsstrahlung is produced in this case in two stages: first there occurs ionization of a atomic electron by the proton and then the δ electron, on being scattered by other atoms, emits a bremsstrahlung photon. Since the binding energy of the atomic electrons is much lower than the energy of the megavolt proton, it can be assumed that the proton energy is transferred to almost free electrons. The laws of conservation of energy and momentum limit the maximum energy transfer to a value equal to $2mv^2$. The polarization bremsstrahlung can predominate at higher frequencies, which are not emitted by the δ electrons.

§ 2. POLARIZATION BREMSSTRAHLUNG

Let us now proceed to the exposition of the investigation. Let a relativistic particle of energy ε_i collide with an atom in a state with energy E_i. As a result of this collision, a photon is emitted, the energy of the particle becomes equal to ε_f, and the atom can go over at the same time into another state, with energy E_f. The law of conservation of energy for this process has the following form:

$$\varepsilon_i+E_i=\varepsilon_f+E_f\pm\hbar\omega. \tag{1}$$

The plus sign in (1) pertains to emission of a photon, while the minus sign pertains to the absorption of a photon, i.e., to the inverse bremsstrahlung effect. It is interesting that a change in the limiting frequency of the bremsstrahlung photon is possible in such inelastic—with respect to the state of the atom—collisions. Indeed, when an electron (or another incoming particle) collides with an excited atom, the energy of the photon can increase by an amount equal to the energy $E_f - E_i$ of the excited atom (the atom then goes over from the upper energy level E_f to the lower level E_i). The limiting photon frequency $\omega_{\rm lim}$ is then equal to

$$\hbar\omega_{\rm lim}=\varepsilon_i+E_i-E_f \tag{2}$$

instead of the previous condition $\omega_{\rm lim} = \varepsilon_i/\hbar$.

The purpose of the present paper is to compute the cross section for bremsstrahlung by a relativistic particle in the field of an atom with consistent allowance for the polarization bremsstrahlung (PB) and the possible excitation, including ionization, of the atom in the course of the emission. In this way we shall fully take account of the dynamics of the atomic electrons during the emission of the bremsstrahlung. We assume that the atom is nonrelativistic ($Z\ll 137$, where Z is the nuclear charge), and that the incoming particle (charge e_1, mass m_1, and velocity v) is a Born particle.[2)] We also assume that the change in the momentum of the incoming particle in the process in question is small: $|\mathbf{p}_f - \mathbf{p}_i| = |\mathbf{q}| \ll |\mathbf{p}_{f,i}|$, where $\mathbf{p}_{i,f}$ are the initial and final momenta of the incoming particle. Therefore, in the zeroth approximation, we can consider the motion of the particle to be free, and set the change in the particle's energy

$$\varepsilon_f-\varepsilon_i\approx\mathbf{qv}\approx q^0.$$

In the case when the exchange, which naturally occurs when the incoming particle is an electron, can be ignored, it is convenient to represent the emission of PB as a process of scattering by the atomic electrons of a virtual photon having a four-momentum $q = (q^0, \mathbf{q})$ and constituting the field of the particle into a bremsstrahlung photon with $k = (\omega, \mathbf{k})$. Then we can, for the purpose of computing the PB cross section, replace the incoming particle by the virtual-photon field produced by it in the course of the bremsstrahlung. Let us note that the bremsstrahlung in a static field (SB) is computed with the aid of a similar method, namely, the method of equivalent quanta, in which the equivalent photons of the field of the atom are scattered into bremsstrahlung photons by the incoming particle. The electromagnetic potential of a virtual photon in the axial ($A_0 = 0$) gauge can, on the basis of the standard rules of quantum electrodynamics, be expressed in terms of the photon propagator and the incoming-particle current as follows:

$$\mathbf{A}^{\rm ip}(x) = \mathbf{A}^{\rm ip}(q)\,e^{iqx},\ x=\{t,\,\mathbf{x}\},\ qx=q^0t-\mathbf{qx},$$
$$A_n^{\rm ip}(q)=e_1D_{nm}(q)\,v_m,\qquad D_{nm}(q)=\frac{4\pi}{q^2}\left(\delta_{nm}-\frac{q_nq_m}{(q^0)^2}\right), \tag{3}$$

where the normalization volume has been set equal to unity. In (3) we have taken account of the fact that, for $|\mathbf{q}|\ll|\mathbf{p}_{f,i}|$, we can replace the transition current of the particle by the particle current. Let us note that in the general case we should use the exact transition current matrix element, the entire analysis remaining unchanged.

The replacement of the incoming particle by the field produced by it eliminates the sole relativistic degree of freedom (not considering the photons); therefore, the problem can be solved with the use of the nonrelativistic quantum-mechanical formalism, which greatly simplifies the analysis. The expounded method of computing the PB can be justified from first principles.[24] Further, we shall consider bremsstrahlung photon frequencies $\omega\ll m$ (where m is the electrom mass), a condition which is also necessary for the nonrelativistic description of the process.

Thus, the atomic electron–electromagnetic field interaction Hamiltonian has the form

$$\hat V=-\frac{e}{2m}\sum_{j=1}^{N}\{\hat{\mathbf{p}}_j\hat{\mathbf{A}}(x_j)+\hat{\mathbf{A}}(x_j)\hat{\mathbf{p}}_j-e\hat{\mathbf{A}}^2(x_j)\}, \tag{4}$$

where e is the electron charge, $\hat{\mathbf{p}}_j$ is the momentum operator of the atomic electrons, N is the number of atomic electrons, and $\hat{\mathbf{A}}$ is the vector potential operator of the total electromagnetic field: $\hat{\mathbf{A}} = \hat{\mathbf{A}}^{ph} + \hat{\mathbf{A}}^{ip}$. The field of the particle is given by the expression (3), $\hat{\mathbf{A}}^{ph}$ is the quantum mechanical vector potential of the photon field.

$$\hat{\mathbf{A}}^{\rm ph}(x)=\sum_{\mathbf{k},\sigma}\left(\frac{2\pi}{\omega}\right)^{1/2}\{\mathbf{e}_{\mathbf{k}\sigma}\hat a_{\mathbf{k}\sigma}e^{-ikx}+\mathbf{e}_{\mathbf{k}\sigma}^{*}\hat a_{\mathbf{k}\sigma}^{+}e^{ikx}\}, \tag{5}$$

$\omega = |\mathbf{k}|$, σ is the polarization index, $e_{\mathbf{k}\sigma}$ is the unit polarization vector, and $\hat a_{\mathbf{k}\sigma}^{+}$ and $\hat a_{\mathbf{k}\sigma}$ are the photon creation and annihilation operators.

Next, we shall compute the PB amplitude in second-order perturbation theory. Retaining only the terms containing $\hat{\mathbf{A}}^{ph}\hat{\mathbf{A}}^{ip}$, which describe the process of interest to us, we obtain the following expression for the PB amplitude:

$$T_{fi}^{\rm PB}=\left(\frac{2\pi}{\omega}\right)^{1/2} e^{*}_{\mathbf{k}\sigma,h} A^{\rm ip}_{m}(q)\ (q^0)^2\langle f|c_{hm}(k,q)|i\rangle. \qquad (6)$$

Here $\hat{c}_{hm}(k, q)$ is the operator describing the scattering of the electromagnetic field by the atomic electrons, which it is convenient to represent in the form

$$\hat{c}_{hm}(k,q) = \frac{e^2}{m(q^0)^2}\left\{-\hat{n}(\mathbf{q}_1)+im\int_{-\infty}^{+\infty} d\tau e^{i\omega\tau}T[\hat{j}_h(\mathbf{k},\tau)\hat{j}_m(\mathbf{q},0)]\right\},$$

$$\hat{n}(\mathbf{q}_1)=\sum_{j=1}^{N}\exp(-i\mathbf{q}_1\mathbf{x}_j),\quad \hat{\mathbf{j}}(\mathbf{q})=\frac{1}{2m}\sum_{j=1}^{N}\{\hat{\mathbf{p}}_j e^{-i\mathbf{q}\mathbf{x}_j}+e^{-i\mathbf{q}\mathbf{x}_j}\hat{\mathbf{p}}_j\}, \qquad (7)$$

$$q_1=q+k,\ \hat{o}(\mathbf{k},\tau)=\exp(i\hat{H}_a\tau)\hat{o}(\mathbf{k},0)\exp(-i\hat{H}_a\tau).$$

In Eq. (7) T is the symbol for chronological ordering, $\hat{\mathbf{j}}(\mathbf{q})$ and $\hat{n}(\mathbf{q})$ are the Fourier transforms of the current- and charge-density operators, respectively, of the atomic electrons, and $\hat{H}_a$ is the Hamiltonian of the unperturbed atom. Let us note that the matrix element c_{fi}^{hm} of the scattering operator can be rewritten as

$$c_{fi}^{hm}(k,q) = -\frac{e^2}{m(q^0)^2}\left\{n_{fi}(\mathbf{q}_1)\delta^{hm}+m \times\sum_n\left[\frac{j_{fn}{}^h(\mathbf{k})j_{ni}{}^m(\mathbf{q})}{\omega_{fn}+\omega+i0}+\frac{j_{fn}{}^m(\mathbf{q})j_{ni}{}^h(\mathbf{k})}{\omega_{in}-\omega+i0}\right]\right\}, \qquad (8)$$

where $\omega_{fn} = E_f - E_n$ and $\omega_{in} = E_i - E_n$ are the natural frequencies of the atom. Then, in the dipole—with respect to **k** and **q**—approximation, (8) coincides with the usual representation of the scattering tensor.[25]

The PB amplitude (6) describes not only the "elastic," but also the "inelastic" processes, in which the emission of bremsstrahlung is accompanied by a change in the state of the atom. The differential PB cross section, summed over all the final atomic states and photon polarizations, has the form

$$d\sigma_\Sigma^{\rm PB}(k,q)=\frac{2\pi}{v}\sum_{f,\sigma}\delta(\omega_{fi}+q^0)\,|T_{fi}^{\rm PB}|^2\frac{d\mathbf{k}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}. \qquad (9)$$

Using the expression (6) for $T_{fi}^{\rm PB}$, we can, after making a number of transformations, represent the PB cross section in the form

$$d\sigma_\Sigma^{\rm PB}(k,q)=\frac{(2\pi)^2}{v}\frac{(q^0)^4}{\omega}(\delta_{hl}-n_hn_l) \times A^{\rm ip}_n(q)A^{\rm ip}_m(q)\,L^{(i)}_{hmln}(q,k)\frac{d\mathbf{k}\,d\mathbf{q}}{(2\pi)^6}. \qquad (10)$$

Here $L^{(i)}_{hmln}$ is the tensor describing the interaction of the atom with the electromagnetic field:

$$L^{(i)}_{hmln}(k,q)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt e^{iq_1^0t}\langle i|\hat{c}_{hm}(k,q)\hat{c}_{ln}^{*}(k,q,t)|i\rangle. \qquad (11)$$

In deriving (10) from (9) and (6), we used the following rule for summation over the photon polarizations:

$$\sum_\sigma e^{*}_{\sigma\mathbf{k},h}e_{\sigma\mathbf{k},l}=\delta_{hl}-n_hn_l,\qquad \mathbf{n}=\mathbf{k}/\omega,$$

the integral representation of the δ function, and the completeness of the set of atomic wave functions.

The expression (10) is the most general. Let us consider the various limiting cases of it. Let initially $|\mathbf{q}| < 1/R$, where R is the atomic radius. Then we can simplify the expression for the PB cross section by using the dipole—with respect to **k** and **q**—approximation for the scattering operator $\hat{c}_{hm}(\omega)$. In this case, as can be seen from the energy conservation law, we are limiting ourselves to the consideration of the frequencies $\omega < v/R$. Then it is convenient to retain the explicit summation over f. Substituting (6) into (9), and integrating over $d\mathbf{q}$ right up to $|\mathbf{q}| = 1/R$ and over the solid angle $\mathbf{\Omega}_n$ of emission of the photon, we find for the spectral PB cross section the expression

$$d\sigma_\Sigma^{\rm PB}(\omega)=\frac{16}{9}\frac{e_1^2}{v^2}\frac{d\omega}{\omega} \times\sum_{f,\omega_{fi}<v/R-\omega}\omega^4\langle f|\hat{c}_{ml}(\omega)|i\rangle\langle f|c_{ml}(\omega)|i\rangle^{*}\ln\frac{\gamma v}{R(\omega+\omega_{fi})}, \qquad (12)$$

where $\gamma = \varepsilon/m_1$, $\omega < v/R$, and $|\mathbf{q}| < 1/R$.

In (12) the summation is over the discrete and continuum states, which are limited by the condition $E_f < E_i + v/R - \omega$. This condition essentially implies that the energy that is lost by the incoming particle corresponds to "distant collisions" (i.e., the change in the particle momentum should be smaller than the atomic momentum). Then the dipole approximation used here for the operator $\hat{c}_{hm}$ is valid. Let us note that other situations are possible when the states $|f\rangle$ ignored in (12) make the dominant contribution to the bremsstrahlung. For example, for low frequencies $\omega < \omega_1$ (ω_1 is the minimum excitation frequency for the atom), as noted in Ref. 26, the PB emission accompanied by ionization of the atom can, when $\omega_{fi} > v/R - \omega$, be the dominant process in the emission of PB. Thus, the expression (12) is mainly suitable for the study of PB emission accompanied by excitation of the atom in a state of the discrete spectrum, a situation which is of interest to us here when $\omega < v/R$. The formula (12) gives, in particular, the expression for the "elastic" PB cross section (the term with $f = i$) obtained in Refs. 15 and 16. Indeed, for the nondegenerate atomic state $|i\rangle$ proposed here we have the equality

$$\langle i|\hat{c}_{ml}(\omega)|i\rangle=\delta_{ml}\alpha_i(\omega), \qquad (13)$$

where $\alpha_i(\omega)$ is the dynamical polarizability of the atom in the state $|i\rangle$. From (12) and (13) we obtain the formula (8) of Ref. 16.

Let us consider the resonance case.[9,27] Let the frequency ω be such that the following inequality is fulfilled for the two energy levels $E_f < E_n$ of the discrete spectrum:

$$\omega\gg|\omega-\omega_{nf}|\gg\Gamma_{nf},$$

where Γ_{nf} is the width of the $n\to f$ transition line. Then in the expression for the matrix element we can single out one resonance term that makes the dominant contribution, and neglect the imaginary part of the scattering tensor in comparison with the real part. There also remains one resonance term in the sum over f in Eq. (12). (We assume that the oscillator strengths, f_{in} and f_{fn}, of the transitions in question are large enough for the indicated approximations to be valid.) After summing over the components of the angular momentum of the resonance states with allowance for the equality $J_i = 0$ (J is the total angular momentum quantum number), we obtain for the resonance bremsstrahlung (RB) cross section the expression

$$\frac{d\sigma^{\mathrm{RB}}}{d\omega}=\frac{4}{3}\frac{e_1^2}{v^2}r_e^2\left(\frac{\omega}{\Delta}\right)^2\frac{f_{in}}{\omega_{ni}}(2J_f+1)f_{fn}\ln\frac{\gamma v}{R(\omega+\omega_{fi})},$$
$$\Delta=\omega-\omega_{nf},\ r_e=e^2/m. \tag{14}$$

This formula is a generalization of the expression for RB emitted without excitation of the atom [the formula (10) in Ref. 16].

Let us turn our attention to the angular dependence of the bremsstrahlung cross section in the region of the resonance, where $|\Delta|\sim\Gamma_{nf}$. The formula for it can be obtained by substituting (6) into (9), and integrating the resulting expression over the momentum transfer:

$$d\sigma_i^{\mathrm{PB}}(\omega,\mathbf{n})=2\frac{e_1^2}{v^2}|\omega^2\alpha_i(\omega)|^2\frac{d\omega}{\omega}\ln\frac{\gamma v}{R\omega}(1+\cos^2\theta)\,d\Omega_{\mathbf{n}}; \tag{15}$$

here θ is the angle between the direction of emission of the photon and the velocity of the incoming particle. We assume, for simplicity, that the initial state of the atom coincides with the final state (i.e., that $f = i$). Thus, the radiation exhibits an anisotropy determined by the direction of motion of the incoming particle. This is the essential difference between bremsstrahlung and the radiation emitted by an atom as result of the population of the atomic levels. In the latter case, during the collisions, or, perhaps, for other reasons, the atom "forgets" the mode of excitation, and the radiation becomes isotropic. In this sense we can say that the emission of PB is a "direct' process, while the emission of radiation by a preexcited atom is a "cascade" one.

Let us now consider the PB in the frequency range $\omega > I$ (I is the ionization potential of the atom). Then, as in Ref. 16, we can simplify the expression for $\hat{c}_{hm}$ by expanding the terms in (9) in powers of ω_{fn}/ω and ω_{in}/ω. Retaining the zeroth-order term of the expansion, we obtain the following approximate expression for the scattering operator:

$$\hat{c}_{hm}(k,q)\approx-\frac{r_e}{(q^0)^2}\hat{n}(\mathbf{q}_1)\left\{\delta_{hm}+\frac{q_hq_m}{2m\omega}\right\}. \tag{16}$$

Substituting (16) into (10), (11), we find

$$d\sigma_\Sigma^{\mathrm{PB}}(k,q)\approx\frac{(2\pi)^2}{v\omega}r_e^2\left[\mathbf{n},\ \mathbf{A}^{\mathrm{ip}}(q)+\frac{\mathbf{q}(\mathbf{q}\,\mathbf{A}^{\mathrm{ip}}(q))}{2m\omega}\right]^2S_i(q_1)\frac{d\mathbf{k}\,d\mathbf{q}}{(2\pi)^6},$$
$$m>\omega>I. \tag{17}$$

Here we have introduced the quantity

$$S_i(q_1)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt\,e^{iq_1^0t}\langle i|\hat{n}(-\mathbf{q}_1)\hat{n}(\mathbf{q}_1,t)|i\rangle, \tag{18}$$

which we shall call the dynamical form factor of the atom (for the state $|i\rangle$),in accordance with the terminology adopted for the description of the effects in a medium. In the particular case when $f = i$ we have in place of (17) the expression

$$d\sigma_i^{\mathrm{PB}}(k,q)=\frac{(2\pi)^2}{\omega v}r_e^2\left[\mathbf{n},\ \mathbf{A}^{\mathrm{ip}}(q)+\frac{\mathbf{q}(\mathbf{q}\,\mathbf{A}^{\mathrm{ip}}(q))}{2m\omega}\right]^2\times\delta(q_1^0)n_{ii}(\mathbf{q}_1)\frac{d\mathbf{k}\,d\mathbf{q}}{(2\pi)^6},\quad m>\omega>I. \tag{19}$$

The expression (19) is a generalization of the formulas for the "elastic" PB considered in Ref. 16 to the case of the many-electron atom.

In order to compute the spectral cross sections for the total and "elastic" PB with the aid of the formulas (17) and (19), we must know the explicit forms of $S_i(q_1)$ and $n_{ii}(\mathbf{q}_1)$ as functions of $\mathbf{q}$. Since these functions are not known in the general case, we proceed from approximate expressions that allow us to follow the basic features. For the static form factor n_{ii} we use the approximation of exponential screening (when the electrostatic field of the screened nucleus falls off exponentially with distance):

$$n_{ii}(\mathbf{q}_1)=N/(1+R^2\mathbf{q}_1^2). \tag{20}$$

Then from (19), with the function $n_{ii}(\mathbf{q}_1)$ given by (20), we obtain the spectral "elastic" PB cross section for a relativistic particle in three frequency bands:

$$\frac{d\sigma_i^{\mathrm{PB}}}{d\omega}=\frac{16}{3}N^2r_e^2\frac{e_1^2}{\omega}\ln\frac{\gamma}{\omega R},\quad\frac{1}{R}>\omega>I, \tag{21a}$$
$$\frac{d\sigma_i^{\mathrm{PB}}}{d\omega}=2N^2r_e^2\frac{e_1^2}{\omega}(\omega R)^{-2}\ln\gamma,\quad\frac{\gamma^2}{R}>\omega>\frac{1}{R}, \tag{21b}$$
$$\frac{d\sigma_i^{\mathrm{PB}}}{d\omega}=4N^2r_e^2\frac{e_1^2}{\omega}(\omega R)^{-2}\left(\frac{\gamma^2}{\omega R}\right)^2,\quad m>\omega>\frac{\gamma^2}{R}. \tag{21c}$$

The stronger the inequalities characterizing the frequency bands are, the more accurately these formulas describe the process. The expressions (21a) and (21b) for the PB cross section are not very sensitive to the nature of the screening of the nucleus. Notice that the formula (21a) can be obtained from (9) for the term with $f = i$ if we take account of the fact that, for $1/R > \omega > I$, $c_{ii}^{ml}(\omega)\approx -Nr_e\delta^{ml}/\omega^2$, since the dipole approximation is valid in the frequency band specified in (21a). The frequency band specified in (21b) obtains for an ultrarelativistic incoming particle, i.e., for an incoming particle with $\gamma\gg 1$. In it, we must take account of the photon momentum, specifically, the possibility of compensation of $\mathbf{q}$ by $\mathbf{k}$, when $|\mathbf{q}+\mathbf{k}|<1/R$ and $n_{ii}\approx N$. This compensation occurs at small angles of particle scattering and photon emission, when $|\mathbf{q}|\approx|\mathbf{q}_\parallel|\approx|\mathbf{k}|$, where $\mathbf{q}_\parallel$ is the component of $\mathbf{q}$ parallel to the velocity of the incoming particle. Then $|\mathbf{q}_1|\approx 2\omega\sin\theta/2$, and the condition $|\mathbf{q}_1|<R^{-1}$, with allowance for the fact that $\omega > R^{-1}$, can be rewritten in the form: $\theta < 1/\omega R\sim\lambda/R$ (λ is the wavelength of the bremsstrahlung photon). Thus, the PB emitted by relativistic particles in the frequency range specified in (21b) is confined to a cone with aperture angle λ/R.

A distinctive feature of the "elastic" PB cross section (21) is its quadratic dependence on the number of atomic electrons.[18] When the state of the atomic electrons does not change, their total charge Ne, remaining localized in the atom, manifests itself as the charge of a single particle of radius R. In the frequency range given in (21a) coherence obtains for all the angles of emission and the entire volume of localization of the electronic charge: the atomic electrons emit a bremsstrahlung photon as a point particle. In the range given in (21b) the radiation emitted by the electron cloud remains coherent in the angle region $\theta < \lambda/R$, which explains the appearance of the factor $(\omega R)^{-2}$ in the expression for the cross section. For frequencies $\omega > \gamma^2/R$, we always have $|\mathbf{q}_1| > 1/R$, which follows from the energy conservation law, and the "elastic" PB cross section, (21c), is strongly suppressed, as can be seen from the formula, (20), for the static form factor. Notice that the formula (21a) for $N = 1$ essentially describes the Thomson scattering of the virtual photons of the field of the incoming particle by an atomic electron. The factor $N^2 r_e^2$ in the formulas (21) is the cross section for Thomson scattering of a photon by an electron, and the factors containing R take account of the fact that, strictly speaking, the virtual photons are scattered not by a point charge, but by an electron cloud of radius R. As has already been indicated, in a many-electron atom a single charge Ne does the scattering. As the wavelength decreases, and the dipole approximation breaks down, the effective charge of the electron cloud, as it were, decreases, since the cloud no longer oscillates as a unit.

For the derivation of the expression for the total spectral PB cross section, let us use for S_i the following expression, which is the simplest approximation to this quantity:

$$S_i(q_1)=N^2\theta(1-R|\mathbf{q}|)\delta(q_1^0)+N\theta(R|\mathbf{q}|-1)\delta(q_1^0+\mathbf{q}^2/2m). \tag{22}$$

The formula (22) can be obtained from the definition (18) by replacing q_1 by q, using the explicit forms of the operators $\hat{n}(-\mathbf{q})$ and $\hat{n}(\mathbf{q}, t)$,and taking account of the fact that the most probable values of the coordinates of the atomic electrons are those satisfying the condition $|\mathbf{x}| < R$. Then, for $|\mathbf{q}| < 1/R$, the exponential functions in the definitions of $\hat{n}(\mathbf{q})$ and $\hat{n}(\mathbf{q}, t)$ can each be approximately replaced by unity, after which the integration over t in the expression (18) yields the first term in the formula (22). To obtain the second term, we use the equality

$$\exp\{i\hat{H}_a t\}\sum_j \exp\{-i\mathbf{q}\mathbf{x}_j\}=\sum_j \exp\{-i\mathbf{q}\mathbf{x}_j\}$$
$$\times\exp\left\{it\left[\sum_{j'}\frac{(\mathbf{p}_{j'}-\delta_{jj'}\mathbf{q})^2}{2m}+v(\mathbf{x}_j)\right]\right\}. \tag{23}$$

Here $v(\mathbf{x}_j)$ is the potential energy for the interaction of the atomic electrons with the nucleus and with each other. The formula (23) expresses the fact that e^{-iqx} is a displacement operator in momentum space. Since the second term in (22) takes account of momentum transfers such that $|\mathbf{q}| > 1/R$, we can neglect $\hat{\mathbf{p}}_j$ in comparison with $\mathbf{q}$ in the first part of the equality (23). As a result we obtain the approximate expression

$$\langle i|\hat{n}(-\mathbf{q}_1)\hat{n}(\mathbf{q}_1, t)|i\rangle$$
$$\approx\left\langle i\left|\sum_{jj'}\exp(i\mathbf{q}(\mathbf{x}_j-\mathbf{x}_{j'}))\right|i\right\rangle\exp(i\mathbf{q}^2t/2m).$$

Since $|\mathbf{q}| > 1/R$, the terms with $j \neq j'$ will, after being averaged over the $|i\rangle$, make a small contribution to the sum because of the rapid oscillations of the corresponding exponential functions. After integrating with respect to t we obtain the second term in the formula (22). The δ function in (22) describes the energy conservation law for the "elastic" PB ($q_1^0 = 0$) and the process accompanied by ionization of an atomic electron.

Using the approximation (22) for the total spectral PB cross section, we find

$$\frac{d\sigma_\Sigma^{\rm PB}}{d\omega}\simeq\frac{16}{3}\frac{e_1^2}{\omega v^2}r_e^2\left\{\theta\left(\frac{v}{R\omega}-1\right)\left[N^2\ln\frac{\gamma v}{\omega R}+N\ln(mvR)\right]\right.$$
$$\left.+\theta\left(1-\frac{v}{\omega R}\right)\theta\left(\frac{mv^2}{2\omega}-1\right)N\ln\frac{\gamma mv^2}{\omega}\right\},$$
$$m>\omega>I,\quad \varepsilon_i\gg\omega,\quad v>\frac{1}{mR}. \tag{24}$$

The expression (24) is valid for nonrelativistic but Born velocities $\mathbf{v}$ as well.

The first, square-bracketed term in (24) describes the frequency band $v/R > \omega > I$, where the PB emission process either can be accompanied by ionization of the atom (if $|\mathbf{q}| > 1/R$), when the contribution of the atomic electrons is incoherent and the cross section for the process is proportional to N, or can occur without the atom's changing its state (if $|\mathbf{q}| < 1/R$), when the entire electronic charge radiates coherently and the PB cross section is proportional to N^2. In the region $mv^2/2 > \omega > v/R$ the energy conservation law [the δ function in the second term of the formula (22)] allows only those $\mathbf{q}$ for which $|\mathbf{q}| > 1/R$; therefore, the process accompanied by ionization of the atom, whose cross section is proportional to N, predominates.

The situation becomes more complicated in the case of bremsstrahlung of a multishell atoms: at a fixed frequency ω the PB emission process occurring on each of the various shells either can be accompanied by ionization of the atom, or can occur without the atom's changing its state; therefore, the description of the process with the aid of the average values of I and R turns out to be too crude. Hence the formula (24) should be used separately for each subshell with its own I and R. Let us discuss here the question of the polarization bremsstrahlung from a heavy incoming particle, e.g., a proton ($m_1 \gg m$). From the formula (24) it follows that the upper PB frequency limit for it is determined by the energy of an electron having the same velocity as the incoming particle, i.e., this cutoff frequency ω^h is much lower than the initial energy of the particle:

$$\omega^h=mv^2/2\ll\varepsilon_i.$$

The point is that, when $|\mathbf{q}| > 1/R$, the momentum of the incoming particle is transferred mainly to the knocked-on electron, and when $\omega > \omega^h$ the simultaneous fulfillment of the energy and momentum conservation laws for this process is impossible. The latter fact is noted in Refs. 14 and 19 for the case of a nonrelativistic heavy incoming particle. It is also pointed out in Ref. 19 that the radiation emitted by the secondary electrons, the upper cutoff frequency of which is determined by the maximum energy ($2mv^2$) transferred to the ionized electron, predominates in the frequency range $2mv^2 > \omega > mv^2/2$. A numerical computation carried out for an aluminum target in that paper, in which exact allowance is made for the effect of the momentum transfer $\mathbf{q}$ from the incoming particle on the bremsstrahlung cross section, shows that, for frequencies $\omega > 2mv^2$, the dominant contribution to the radiation emitted by a heavy particle is made by the "elastic" PB (in the terminology of the authors, the "atomic" bremsstrahlung). This does not accord with the formula (24). The reason is that, in the parameter-value range considered in Ref. 19, the dipole—with respect to **q**—approximation turns out to be too crude for the computation of the "elastic" PB in the formula (24). And it is precisely in (24) that the "elastic" PB's nondipole—with respect to **q**—part, which is described in the relativistic case by the formula (21c), has been neglected. At the boundary of the frequency band $\omega \gtrsim v/R$ this correction can exceed the contribution of any of the other emission mechanisms (e.g., the bremsstrahlung from the secondary electrons) because of the coherence of the photon emission in the "elastic" PB. Thus, the above analytic formulas, as well as numerical computations, indicate that the polarization effects make a substantial contribution to the x-ray background that arises in the bombardment of a target by protons.

It is important to note that we have obtained for the total PB cross section an expression, (24), that allows passage to the zero nuclear charge ($Z=0$) limit, where $R\to\infty$. Then the square-bracketed terms in the formula (24) vanish because of the presence of the θ function. The remaining expression coincides with the cross section for bremsstrahlung by N slow free electrons in a collision with a relativistic particle [the formula (97.5) in Ref. 25]. Physically it is clear why the "elastic" PB cross section does not have the correct $Z=0$ limit. Indeed, in this case the atomic electrons acquire the excess momentum, and their state should (when they are not bound to a nucleus) change: the process becomes basically "inelastic."

It follows from the expression (24) that the atomic electrons for $N>1$ can, in the description of the PB, be considered to be free in the region $\omega > v/R$, the condition $\omega > I$ being inadequate for this purpose. Indeed, only when $N=1$ will the sum of the square-bracketed terms in the formula (24) be equal to the last term in the formula. And this means that the PB is equivalent to radiation emitted by the free electron that scatters the incoming particle.

§ 3. BREMSSTRAHLUNG EMITTED IN A STATIC FIELD

Let us now consider the static term in the expression for the bremsstrahlung amplitude, a term which takes account of the excitation of the atomic electrons. Let us proceed from the static bremsstrahlung (SB) amplitude obtained in Ref. 16—formula (2)—for a hydrogen-like ion. This expression is easily generalized to the case of the many-electron atom, for which purpose it is sufficient to take as the transition current the atomic-electron transition current:

$$\mathbf{j}_{fi}(\mathbf{q}_1)=\left\langle f\,\Big|\sum_{j=1}^{N}(\exp(-i\mathbf{q}_1\mathbf{x}_j)\hat{\mathbf{p}}_j/2m+(\hat{\mathbf{p}}_j/2m)\exp(-i\mathbf{q}_1\mathbf{x}_j))\,\Big|\,i\right\rangle,$$

$$j_{fi}^{0}(\mathbf{q}_1)\equiv n_{fi}(\mathbf{q}_1)=\left\langle f\,\Big|\sum_{j=1}^{N}\exp(-i\mathbf{q}_1\mathbf{x}_j)\,\Big|\,i\right\rangle.$$

Let us use here the Coulomb gauge ($\operatorname{div}\mathbf{A}=0$) for the electromagnetic potential, in which we can neglect the space components of the 4-potential in the case of nonrelativistic atomic electrons and a stationary nucleus. Then for the SB amplitude we obtain

$$T_{fi}^{\mathrm{SB}}=4\pi e\frac{e_1^2}{q_1^2}[Z\delta_{fi}-n_{fi}(\mathbf{q}_1)]\Gamma^{\nu}(p_{f,i},k)e^{*}_{\mathbf{k}\sigma,\nu}, \tag{25}$$

$$\Gamma(p_{f,i};k)=\frac{\bar{u}_f}{(2\varepsilon_f)^{1/2}}\left\{\gamma\frac{\gamma(p_f+k)+m_1}{(p_f+k)^2-m_1^2}\gamma^0+\gamma^0\frac{\gamma(p_i-k)+m_1}{(p_i-k)^2-m_1^2}\gamma\right\}\frac{u_i}{(2\varepsilon_i)^{1/2}}. \tag{26}$$

Here γ and γ^0 are the Dirac matrices, and u_i and $\bar{u}_f$ are bispinors. The function introduced above does not depend on the state of the atom. It describes the scattering of the time component of the electromagnetic potential of the atom into a bremsstrahlung photon by the incoming particle.

From (25) we obtain for the spectral SB cross section, summed over all the final atomic states and the bremsstrahlung-photon polarizations, the expression

$$\frac{d\sigma_{\Sigma}^{\mathrm{SB}}}{d\omega}=\frac{1}{v}\int\frac{\omega^2\,d\Omega_{\mathbf{n}}\,d\mathbf{q}}{(2\pi)^6}\frac{2\pi}{\omega}\int dt\,e^{iq_1^0t}[\mathbf{n}\Gamma]^2\left(\frac{4\pi e_1^2e}{q_1^2}\right)^2\times\langle i|\,(Z-\hat{n}(-\mathbf{q}_1))\,(Z-\hat{n}(\mathbf{q}_1,t))\,|i\rangle. \tag{27}$$

It is assumed that $|i\rangle$ is a nondegenerate state, and that the population of the other states can be ignored. If the excitation energy of the atomic electrons is negligible in comparison with the photon frequency ω, then we can set $\hat{n}(\mathbf{q}_1,t)=\hat{n}(\mathbf{q}_1)$ in (27), and obtain as a result

$$\frac{d\sigma_{\Sigma}^{\mathrm{SB}}}{d\omega}=\frac{1}{v}\int\frac{\omega^2\,d\Omega_{\mathbf{n}}\,d\mathbf{q}}{(2\pi)^6}\frac{2\pi}{\omega}\delta(q_1^0)\,[\mathbf{n}\Gamma]^2\times\left(\frac{4\pi e_1^2e}{q_1^2}\right)^2\langle i|\,|Z-\hat{n}(\mathbf{q}_1)\,|^2|i\rangle. \tag{28}$$

The expression (28) coincides with the result obtained by Lamb and Wheeler,[13] who were the first to consistently take account of the contribution of the excitation of the atomic electrons to the static bremsstrahlung. Notice that, in the case of large $|\mathbf{q}_1|$, specifically, for $|\mathbf{q}_1|\gg R^{-1}$, the $|i\rangle$-averaged quantity in (28) can be rewritten in the form

$$\langle i|\,|Z-\hat{n}(\mathbf{q}_1)\,|^2|i\rangle=Z^2+\langle i|\,|\hat{n}(\mathbf{q}_1)\,|^2|i\rangle-Z\langle i|2\operatorname{Re}\hat{n}(\mathbf{q}_1)\,|i\rangle\approx Z^2+N. \tag{29}$$

In deriving the last equality in (29) we took into account the fact that

$$\langle i|\,|\hat{n}(\mathbf{q}_1)\,|^2|i\rangle\approx N,\quad \langle i|\operatorname{Re}\hat{n}(\mathbf{q}_1)\,|i\rangle\approx 0$$

for $|\mathbf{q}_1|>1/R$. Then for $Z=N$ (neutral atom), the substitution of (29) into (28) yields the Landau-Rumer result.[28]

Let us discuss the physical picture of the SB with allowance for the excitation of the atomic electrons. For $|\mathbf{q}_1|<1/R$, the momentum transfers are acquired by the atom as a whole. This leads to the well-known picture of a nucleus screened by the atomic electrons. But if $|\mathbf{q}_1|>1/R$, then the scattering by the nucleus and the one by the atomic electrons are two different processes that almost do not interfere with each other: The momentum is transferred either to the nucleus or to an atomic electron; in the latter case the atom is ionized. As a result, the total SB cross section is equal to the sum of the cross sections for emission of bremsstrahlung in collisions with the screened nucleus and the atomic electrons: the Landau–Rumer result.

Let us compare the spectral PB and SB cross sections.

The expressions for the corresponding cross sections have the simplest form in the semiclassical ($\varepsilon_i \gg \omega$) and ultrarelativistic ($\gamma \gg 1$) limits in the frequency region $m > \omega > I$. When the incoming particle is an electron, the dominant contributions to the PB and SB are made in the frequency region $v/R > \omega > I$ in the case when $N, Z > 1$ by the "elastic" cross section terms[18]:

$$\frac{d\sigma_i^{\text{PB}}}{d\omega} = \frac{16}{3} N^2 r_e^2 \frac{e^2}{\omega} \ln \frac{\gamma}{R\omega}, \tag{30a}$$

$$\frac{d\sigma_i^{\text{SB}}}{d\omega} = \frac{16}{3} Z^2 r_e^2 \frac{e^2}{\omega} \ln (mR), \tag{30b}$$

which (for $N = Z$) differ only in the expressions under the logarithm sign although their radiation directivity patterns differ essentially from each other. Let us write out the expressions for the "inelastic" PB and SB cross sections in the frequency region $m > \omega > v/R$, where the dominant contribution to the PB is made by the processes accompanied by ionization of the atomic electrons[18]:

$$\frac{d\sigma_{\text{H}}^{\text{PB}}}{d\omega} = \frac{16}{3} N r_e^2 \frac{e^2}{\omega} \ln \frac{\varepsilon}{\omega}, \tag{31a}$$

$$\frac{d\sigma_{\text{H}}^{\text{SB}}}{d\omega} = \frac{16}{3} N r_e^2 \frac{e^2}{\omega} \ln (mR) \qquad (\omega < \gamma^2/R). \tag{31b}$$

Thus, the "elastic" PB and SB cross sections are comparable in magnitude in the region $v/R > \omega > I$, where (for $\gamma \gg 1$) the former increases logarithmically with increasing incoming-particle energy, whereas the latter does not depend on the energy of the incoming particle. The "inelastic" PB and SB cross sections are comparable right up to frequencies $\omega \sim m$ in the case of a relativistic incoming particle. For $\omega > m$ it follows from the energy conservation law that $|\mathbf{q}| > 1/R$, and the atomic electrons can, with a high degree of accuracy, be free in the PB process. The PB is then emitted by N slow free electrons during the collision with the relativistic incoming particle. The cross section for such radiation in the case when $\omega > m$ is, as is well known from quantum electrodynamics, ω/m times smaller than the cross section for emission by a fast electron (if the incoming particle is an electron).

All the foregoing is valid also for the bremsstrahlung from an ultrarelativistic positron in a collision with an atom when the sign of the polarization term in the expression for the amplitude is changed. But, as in the case of the electron, because of the different angular dependences of the radiation, the interference between the PB and the SB can be neglected.

If the incoming particle is a heavy one, e.g., a proton, then the SB cross section ["elastic" and "inelastic:" formulas (30b) and (31b)] is inversely proportional to the square of the proton mass [as follows from (25)–(27)], and is therefore negligibly small. The polarization term in the expression for the total bremsstrahlung cross section leads, as in the nonrelativistic problem,[14,23,29] to a situation in which a proton emits anomalously intense bremsstrahlung in the region $\omega < m$ in a collision with an atom.

§ 4. CONCLUSION

Thus, we have derived here the most general expression for the cross section for bremsstrahlung by a relativistic charged particle in a collision with an atom, i.e., an expression which takes account of both the polarization mechanism and the scattering of the incoming particle in the static field of the nucleus and the atomic electron cloud, as well as the possibility of excitation of the atom in a single quantum-mechanical event involving the emission of a bremsstrahlung photon. In the case of a nonrelativistic particle the possibility of excitation and ionization of the atom during bremsstrahlung has been considered in a number of papers,[9,14,23,30] and it has been shown that this process makes a substantial contribution to the bremsstrahlung. As to the case of a relativistic incoming particle, bremsstrahlung with simultaneous excitation of the atom has been considered only in the static-field approximation by Lamb and Wheeler.[13] This result is obtained in the present article as part of the total bremsstrahlung amplitude.

Thus, allowance for the polarization and excitation of the atom in the course of bremsstrahlung leads to important results for a relativistic incoming particle.

Allowance for the polarization contribution in the directivity pattern for the bremsstrahlung by a relativistic particle gives rise to dipole radiation that, in the low-frequency region ($\omega < m$), is integrally comparable to the narrow-beam SB by a fast electron in a collision with an atom in both the elastic and inelastic cases. And what is more, in the lower part of this region (namely, for $\omega < v/R$) the dominant contribution to the PB is made by the "elastic" channel, whereas for $\omega > v/R$ the dominant contribution to the PB is made by the processes accompanied by excitation and ionization of the atom.

The "elastic" SB and PB are respectively proportional to the squares of the nuclear charge and the number of atomic electrons, whereas the "inelastic" terms of the total bremsstrahlung cross section—for the SB, as well as the PB—are proportional to the first power of the number of atomic electrons, and are consequently important in bremsstrahlung processes that occur on light atoms. Furthermore, the expressions under the logarithm sign in the integrated cross sections for PB and SB (both "elastic" and "inelastic") depend differently on the energy of the incoming particle.

The significant difference in the directivity patterns of the SB and PB in the ultrarelativistic limit allows us to ignore the contribution of their interference to the total cross section for bremsstrahlung in the field of an atom. Narrow-beam SB in the field of an atom possesses a large coherence length in the ultrarelativistic limit, and can, under certain conditions, be strongly suppressed in a medium.[6–8] The practically isotropic PB is less susceptible to the density effects.

For a relativistic incoming particle there also arises an interesting possibility that we do not have in the nonrelativistic case. Specifically, above we showed that, in the ultrarelativistic ($\gamma \gg 1$) limit, the photon momentum, which for small emission angles (i.e., for $\theta < \lambda/R$) can compensate the momentum transfer, must be taken into account in the frequency range $\gamma^2/R > \omega > 1/R$. As a result, the PB also acquires directivity in the indicated region of variation of the parameters.

For a heavy incoming particle, the PB, both in the case when it does not depend on the particle mass and in the case when the particle is nonrelativistic, is much greater than the contribution of the SB in the low-frequency region.

The foregoing analysis is valid for $Z|ee_1|/v < 1$, when the interaction between the incoming particle and the atom can be considered to be weak (the Born approximation). Let us note that the opposite case, i.e., the case of large $|e_1|$, is considered for a nonrelativistic incoming particle in Ref. 23, where only the interaction of the charged particles with the photon is assumed to be weak.

Finally, let us make one important comment. Polarization bremsstrahlung in the field of an atom has its "counterpart" in a plasma, where the role of the atom is played by an ion surrounded by a Debye "jacket." This is the so-called transition bremsstrahlung, proposed and investigated in Refs. 31 and 32 by Tsytovich and Akopyan. The question of the relation between polarization bremsstrahlung in the field of an atom and transition bremsstrahlung in a plasma is investigated in detail in Ref. 24. Let us note that both the transition bremsstrahlung and the polarization bremsstrahlung fit into the general scheme of transition scattering.[33]

[1)] We use quotation marks for the terms "elastic" and "inelastic" bremsstrahlung since bremsstrahlung is basically an inelastic process because of the transformation of the energy of the incoming particle into bremsstrahlung-photon energy.

[2)] We use the system of units in which $\hbar = c = 1$ is used, and summation over repeated indices is implied.

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