Theory of an amplification effect in the case of total internal reflection of wave beams from inverted media

B. E. Nemtsov and V. Ya. Éĭdman

Scientific-Research Radiophysics Institute
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A theoretical analysis is made of the reflection of spherical waves and wave beams from a homogeneous inverted medium. The integration contours in integrals describing the reflected field are determined by the causality condition. An asymptotic analysis of the fields of reflected waves shows that amplification due to reflection involves excitation in the incident signal of unstable lateral waves that grow exponentially in space. It is shown that in the limit of a plane wave the field described by the Fresnel reflection coefficient is supplemented by the field of an unstable lateral wave. The amplitude and the range of existence of this wave largely depend on the spectral profile of the beam from which the transition is made to the plane wave limit (configurational instability). This means that the solution of the problem of reflection of plane waves from an inverted medium, obtained using the Fresnel formulas, does not correspond to any realistic formulation of the problem.

Some experiments on the reflection of laser radiation from inverted media have revealed that the reflected signal exceeds the incident signal considerably (by several orders of magnitude).[1–3] Several authors have attempted to explain this observation using the formulas for the Fresnel reflection coefficients (see Ref. 4 and the literature cited there). However, as shown in Ref. 2, the Fresnel reflection coefficients of an active medium cannot be much greater than unity so that the experimental results[1–3] cannot be explained by an analysis of reflection of plane waves. Moreover, this effect cannot be accounted for by allowance for the finite width of a beam[4] when a standard analysis of this problem is made[5] in which the integrals with respect to the wave number, which describe the reflected field, are taken along the real axis and further calculations are based on the expansion of the functions in the integrand as a series near the maximum of the spatial spectrum of the beam. Therefore, in spite of the fact that amplification as a result of reflection (called superluminescence in Ref. 1) had been discovered some time ago, a satisfactory explanation is still lacking.

We shall develop a theory of the reflection of monochromatic signals of various profiles by an amplifying medium. The justification for considering the reflection of monochromatic waves is that the convective instability occurs in the "usual" inverted media employed in quantum electronics.[6] This means that on application of a signal the waves in the medium itself decay with time at any fixed point in space and after a certain time interval only the monochromatic fields oscillating at the frequency of the source remain. We shall first consider the reflection of a spherical wave excited by a vertical dipole. Using the causal formulation of this problem,[1] we shall establish the rules for bypassing the singularities of the integrand in the plane of a complex variable $\varkappa$ ($\varkappa$ is the projection of the wave vector on a plane parallel to the boundary of the medium). It is then found that the integration contour $L_\varkappa$ is shifted away from the real axis and bypasses in a suitable manner the branching points of the integrand function.

An asymptotic analysis of the integrals governing the reflected field shows that, in addition to a specularly reflected signal whose behavior is described by the Fresnel reflection coefficient, an additional field corresponding to a lateral wave is formed. The field of a lateral wave excited at the interface with the inverted medium grows exponentially in space because the beams corresponding to lateral waves propagate partly through the active medium.[5] We shall assume, as usual, that a lateral wave is excited at observation angles exceeding the critical value, so that the amplification effect can only occur in the part of the space defined by this angle.

In the second part of the present paper we shall analyze the reflection of a beam from a boundary with an amplifying medium. We shall show that the reflected field includes not only the specularly reflected beam, but also a perturbation which is not described by the Fresnel formulas. This perturbation is due to the excitation of fundamentally unstable modes of the inverted medium. These unstable modes are excited even in the limit of an infinitely wide beam, and their amplitude and profile are extremely sensitive to the spectral profile of the incident wave (configurational instability[9]). This important circumstance means that the solution of the problem of the reflection of a plane wave by an interface with an inverted medium given by the Fresnel formulas does not correspond to any realistic formulation of the problem.

An analysis of the profile of the reflected beam will be used to show that the exponential growth of the field in space is due to the excitation of a lateral wave. In the case of beams with a Gaussian profile the amplitude of the lateral waves increases strongly on approach of the angle of incidence to the critical value. In this case the rise of the field on increase in the beam width occurs also in the range of angles of observation less than the critical value.

It therefore follows that the experimentally observed amplification of waves on reflection can be explained fully

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without invoking additional complicating factors such as inhomogeneities of the population inversion, nonlinearities of the medium, finite dimensions of the system, etc.

REFLECTION OF SPHERICAL WAVES FROM AN INVERTED MEDIUM

We shall assume that the source of spherical waves is a dipole oriented at right-angles to the interface between media (Fig. 1). We shall consider the specific case when a medium containing the source is not inverted and has a permittivity ε_1, whereas the second medium has amplifying properties and has a complex permittivity $\hat{\varepsilon}_2$. We shall assume that both media are nonmagnetic so that $\mu = 1$.

The equations for the only nonzero z component of the vector potential **A** in the first and second media are

$$\Delta A_1 - \frac{\hat{\varepsilon}_1}{c^2}\frac{\partial^2}{\partial t^2}A_1 = -\frac{4\pi}{c}j, \tag{1}$$

$$\Delta A_2 - \frac{\hat{\varepsilon}_2}{c^2}\frac{\partial^2}{\partial t^2}A_2 = 0, \tag{2}$$

where A_1 and A_2 are related in the usual way to the intensities of the electric and magnetic field **E** and **H**:

$$\mathbf{H}_{1,2} = \operatorname{rot}\mathbf{A}_{1,2}, \quad \mathbf{E}_{1,2} = -\nabla\varphi_{1,2} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_{1,2},$$
$$\operatorname{div}\mathbf{A}_{1,2} + \frac{1}{c}\hat{\varepsilon}_{1,2}\frac{\partial}{\partial t}\varphi_{1,2} = 0. \tag{3}$$

The relationships in the system (3) can be used to obtain the boundary conditions for the time-dependent Fourier components $A_{1,2}$:

$$A_1 = A_2|_{z=0}, \quad A_{1z}'/\varepsilon_1 = A_{2z}'/\varepsilon_2. \tag{4}$$

Here, $\varepsilon_{1,2}$ are the frequency-dependent permittivities $\hat{\varepsilon}_{1,2}$ In accordance with the formulation of the problem, we shall select j in Eq. (1) to be

$$j = I_0 f(t)\, d\delta(x)\delta(y)\delta(z-l). \tag{5}$$

Here, $f(t)$ is a function representing the time dependence of the current in the dipole, I_0 is the characteristic amplitude of the current, and d is the dipole length.

Solving the system (1)–(5) by the Fourier transformation with respect to time and the coordinates x and y, we find that

$$A_i = \frac{iI_0 d}{4\pi c}\int_{L_\omega} f(\omega)\exp(-i\omega t)\,d\omega \times \int_{-\infty}^{\infty} \varkappa H_0^{(1)}(\varkappa\rho)p_1^{-1}\exp(ip_1|l-z|)\,dz, \tag{6}$$

$$A_r = \frac{iI_0 d}{4\pi c}\int_{L_\omega} f(\omega)\exp(-i\omega t)\,d\omega \times \int_{-\infty}^{\infty} \varkappa H_0^{(1)}(\varkappa\rho)p_1^{-1}\exp(ip_1(l+z))V(\varkappa,\omega)\,d\varkappa, \tag{7}$$

$$p_{1,2} = (k_1^2 - \varkappa^2)^{1/2}, \quad \rho = (x^2+y^2)^{1/2},$$

where A_i is the field of the incident wave; A_r is the field of the reflected wave; $V(\varkappa,\omega) = (\nu p_1 - p_2)/(\nu p_1 + p_2)$ ($\nu = \varepsilon_2/\varepsilon_1$) is the reflection coefficient of the active medium; $f(\omega)$ is the Fourier transform of $f(t)$; $k_{1,2} = \omega\varepsilon_{1,2}^{1/2}/c$; $p_{1,2} = i\varkappa$ in the limit $\mathrm{Re}\varkappa \to \infty$; $H_0^{(1)}(x)$ is a Hankel function. In the causal formulation of the problem we have $f(t) \equiv 0$ up to a certain moment t_0. Without limiting the generality of our treatment, we can assume that $t_0 = 0$. Up to that moment there are no fields A_i and A_r. This condition is satisfied if the integration contour in the integrals of Eqs. (6) and (7) is selected so that it passes above all the singularities in the integrands of Eqs. (6) and (7). This condition is obeyed automatically if L_ω is lifted sufficiently far into the upper half-plane of ω. The branching points $\varkappa_{1,2} = \omega\varepsilon_{1,2}{}^{1/2}/c$ in the complex plane of α are then bypassed from below, whereas the branching points $\varkappa_{3,4} = -\omega\varepsilon_{1,2}{}^{1/2}/c$ are bypassed from above (Fig. 2).

These rules solve completely the problem of selection of the integration contours in the $(\omega, \varkappa)$ space. However, in practice, particularly when solving the problems of reflection of quasimonochromatic signals, it is more convenient to represent the integrals (6) and (7) in such a form that the integration contour in the complex plane of ω passes along the real axis. This can be done if in the process of deformation of the initial contour L_ω toward the real axis we ensure that the singularities in the complex plane of $\varkappa$ do not intersect the contour. When deforming the contour L_ω toward the real axis, we can quite readily encounter a situation that L_ω comes in contact with the singularities of the functions in the integrand in the upper half-plane of the complex variable ω.

If we consider the integral with respect to $\varkappa$ in Eqs. (6) and (7) as a function $G(\omega)$, we can distinguish two cases: $G(\omega)$ has singularities in the upper half-plane of the variable ω and $G(\omega)$ has no such singularities in that half-plane. The former case corresponds to amplifying media in which an absolute instability occurs. In investigations of the reflection by such media we have to solve completely the initial problem allowing for the effects of the switching-on signal, because in the final analysis the exponential growth of the

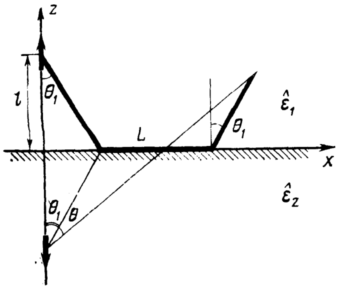

FIG. 1.

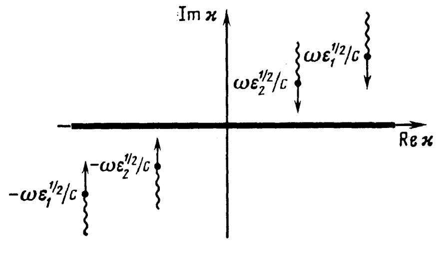

FIG. 2

fields in time is related to these effects. The second case corresponds to active media with a convective instability. In media of this kind the unstable perturbations drift away rapidly and the intrinsic fields excited by the switching on of the source decay at each fixed point in space. This means that in the case of systems with the convective instability the perturbations oscillating at the frequency of the source finally predominate. Since a convective regime is the one mainly realized in quantum electronics when active media are used,[6] we shall consider the problem of reflection in this specific case.

Bearing in mind this discussion, we shall drop the contour L_ω on the real axis so that the integration contour $G_\varkappa$ in the $\varkappa$ plane is of the form shown in Fig. 3. In plotting the results shown in Fig. 3 an allowance is made for the fact that an inverted medium is characterized by $\mathrm{Im}\varepsilon_2 < 0$ (Ref. 4), so that the branching point $\omega\varepsilon_2^{1/2}/c$ is located below the real axis and the branching point $-\omega\varepsilon_2/c$ is above the real axis.

Using this representation of the integrals we shall now solve the problem of reflection of monochromatic signals from an inverted medium. Then, we have to substitute $f(\omega) = 2\pi\delta(\omega - \Omega)$ in Eqs. (6) and (7) and to replace integration along L_ω with integration along the real axis. Consequently, the reflected field is described by [the factor $\exp(-i\Omega t)$ is omitted]

$$A_r = i(I_0 d/2c) \int_{G_\varkappa} \varkappa V H_0^{(1)}(\varkappa\rho) p_1^{-1} \exp[ip_1(z+l)]\, d\varkappa. \tag{8}$$

In an asymptotic analysis of the reflected field it is convenient to adopt a new variable $\eta = \varkappa/k_1$ and to use the Hankel-function representation for large values of the argument. After transformations, Eq. (8) becomes

$$A_r = (k_1/2\pi\rho)^{1/2} (I_0 d e^{i\pi/4}/c) \int_{G_\eta} \eta^{1/2} V(\eta) \xi^{-1}(\eta) \times \exp\{ik_1 R(\eta \sin\theta + \xi(\eta)\cos\theta)\}\, d\eta, \tag{9}$$

$$V(\eta) = \frac{n^2\xi(\eta) - \zeta(\eta)}{n^2\xi(\eta) + \zeta(\eta)}, \quad \xi^2 + \eta^2 = 1, \quad \zeta^2 + \eta^2 = n^2. \tag{10}$$

Here, $R = [(z+l)^2 + \rho^2]^{1/2}$; $\sin\theta = \rho/R$; $\cos\theta = (z+l)/R$; $n = \nu^{1/2}$ is the relative value of the refractive index of the media in contact. The integration contour G_η and the positions of the branching points of the integrands are shown in Fig. 4 for the most interesting case when $\mathrm{Re}\,n < 1$, which corresponds to the possibility of total reflection. We shall confine our attention to the case when $\mathrm{Re}\,n < 1$.

In addition to the branching points of the integrands $\eta = \pm n, \pm 1$, we can expect the existence of poles of the reflection coefficient V. However, under the conditions corresponding to experiments on active media, when

$$|\mathrm{Im}\, n| \ll \mathrm{Re}\, n, \tag{11}$$

and for the method of taking cuts shown in Fig. 4 there are no poles. We shall now verify this.

Poles of V should satisfy the equation

$$n^2\xi + \zeta = 0. \tag{12}$$

If we square Eq. (12), we find that in the case of possible poles

$$\eta_{1,2} = \pm n(1+n^2)^{-1/2}. \tag{13}$$

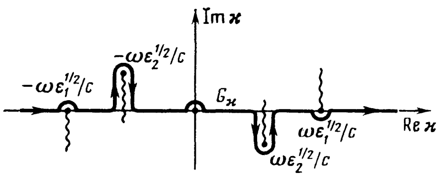

FIG. 3

It is clear from Eq. (13) that when the condition (11) is obeyed then $\mathrm{Re}\eta_{1,2} < \mathrm{Re}\,n$, and $|\mathrm{Im}\eta_{1,2}| \ll |\mathrm{Re}\eta_{1,2}|$, i.e., $\eta_{1,2}$ lies between the cuts passing through the points $\pm n$. It follows from the definition of the roots, given after Eq. (7), that $\mathrm{Re}\xi_{1,2} > 0$, $\mathrm{Re}\zeta_{1,2} > 0$ and therefore $\eta_{1,2}$ do not satisfy Eq. (12).

We shall investigate Eq. (9) by the steepest-descent method. The saddle point is

$$\eta_0 = \sin\theta, \tag{14}$$

and the global steepest descent contour Γ_η is found from the conditions

$$\mathrm{Im}\, i(\eta\sin\theta + \xi\cos\theta) = 1, \tag{15}$$

$$\mathrm{Re}\, i(\eta\sin\theta + \xi\cos\theta) \leqslant 0. \tag{16}$$

The condition (16) determines the selection of the steepest-descent path. It follows from Eqs. (15) and (16) that Γ_η intersects the real axis at the saddle point at an angle $-\pi/4$ (Fig. 4). Using this circumstance and the inequality of Eq. (11), we can readily show that the global path does not intersect the branching point n for angles of observation satisfying the condition

$$\sin\theta < \mathrm{Re}\, n - |\mathrm{Im}\, n| \equiv \sin\theta_0. \tag{17}$$

In this part of space there is only a specularly reflected spherical wave

$$A_r(\theta < \theta_0) = I_0 d c^{-1} V(\eta = \sin\theta) R^{-1} \exp(ik_1 R). \tag{18}$$

If the inequality $\theta > \theta_0$ is obeyed, then in the process of deformation of the initial contour G_η we find that Γ_η acquires a loop surrounding the branching point n. Therefore, in addition to the contribution made to the reflected field by the saddle point, we have to include also the contribution of the integral along the edges of the cut. A simple analysis shows that for angles in the range $\theta_0 < \theta <$ Ren the integral along the cut edges does not give rise to exponentially growing waves and the integral along the loop describes the part of the field which grows in space only when

$$\sin\theta > \mathrm{Re}\, n \equiv \sin\theta_1. \tag{19}$$

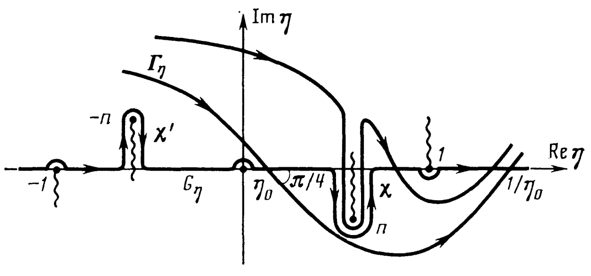

FIG. 4

The formula for A_r then becomes

$$A_r=\frac{I_0 d}{cR}V(\eta=\sin\theta)\exp(ik_1R)$$
$$+\frac{2iI_0d\exp\{ik_1R\cos(\theta-\theta_1)+|\mathrm{Im}\,n|k_1R\sin(\theta-\theta_1)/\cos\theta_1\}}{ck_2R^2(1-n^2)q^{3/2}\sin^{1/2}\theta},$$
$$q=\frac{\sin(\theta-\theta_1)}{\cos\theta_1}+i\frac{|\mathrm{Im}\,n|\cos\theta}{[1-(\mathrm{Re}\,n)^2]^{1/2}}. \tag{20}$$

The second term in Eq. (20) describes the field of a lateral wave growing exponentially in space. The exponential factor in Eq. (20) can be rewritten in the form $\exp\{|\mathrm{Im}k_2|L\}$, where L is the length of the optical path of the lateral wave inside the amplifying medium. In fact, the beam corresponding to a lateral wave travels in the first medium at the total reflection angle [when the condition (11) is obeyed, the total reflection angle is θ_1], and then it passes inside the amplifying medium along the interface and emerges at the point of observation at the angle θ_1 (Fig. 1). It follows from Fig. 1 that $L=R\sin(\theta-\theta_1)/\cos\theta_1$.

This investigation of the field of the reflected spherical wave shows that, in addition to the field of a specularly reflected wave, there is a field of a lateral wave growing in space and it does not generally disappear when the source is moved to infinity. This fact alone means that the solution of the problem of the reflection of a plane wave by an interface with an inverted medium described by the Fresnel formulas is incorrect In fact, as we move the source to infinity, the front of a spherical wave becomes plane, but the solution nevertheless is given by Eq. (20) containing not only the Fresnel term, but also an exponentially growing factor.

REFLECTION OF A WAVE BEAM FROM AN INVERTED MEDIUM

In the experiments described in Refs. 1–3 a beam and not a spherical wave was incident on the interface with an inverted medium; it would therefore be of interest to determine the characteristics of reflection of sufficiently wide beams from an inverted medium.

Let us assume that a beam is incident from the uninverted medium on the interface. The electric field $\mathbf{E}$ of the beam has one nonzero component parallel to the interface and it is formed on an aperture separated by a distance l from the interface. In the plane $z=l$ the field E is described by the expression

$$E_i(z=l,\,x)=Q(x)\exp(ik_1x\sin\varphi-i\Omega t),\qquad k_1=\Omega\varepsilon_1^{1/2}/c. \tag{21}$$

Here, φ is the angle of incidence on the interface and Q is a function describing the beam profile (the case $Q=1$ corresponds to the incidence of a plane wave on the interface). We can find the reflected wave by the usual procedure[5] involving expansion of the incident field along plane waves:

$$E_i(z,x)=(2\pi)^{-1}\int_{G_\varkappa}d\varkappa E(\varkappa)\exp(i\varkappa x+ip_1|l-z|), \tag{22}$$

where $E(\varkappa)$ is the Fourier transform of $E(l,x)$ [the factor $\exp(-i\Omega t)$ is omitted].

We shall now seek the fields of the reflected and transmitted waves in the form

$$E_r(z>0)=(2\pi)^{-1}\int_{G_\varkappa}d\varkappa V_1(\varkappa)E(\varkappa)\exp\{i\varkappa x+ip_1(l+z)\}, \tag{23}$$

$$E_{tr}(z<0)=(2\pi)^{-1}\int_{G_\varkappa}d\varkappa W_1(\varkappa)E(\varkappa)\exp(i\varkappa x+ip_1l-ip_2z). \tag{24}$$

The relationships (22)–(24) satisfy the field equations and the functions V_1 and W_1 readily satisfy the boundary conditions. The boundary conditions of continuity of the tangential components of the electric and magnetic fields are satisfied if

$$V_1=(p_1-p_2)/(p_1+p_2), \tag{25}$$

$$W_1=1+V_1. \tag{26}$$

However, we can still select arbitrarily the integration contour $G_\varkappa$. It has to be selected in such a way that the result obtained by solving the initial problem is not in conflict with the principle of causality. The procedure for finding $G_\varkappa$ based on this principle is described in the first part of the present paper. However, in practice it is frequently more convenient to adopt a simpler method (well known from plasma physics[11]) of determination of the integration contour $G_\varkappa$. When this method is applied to the reflection problems, the wave fields are selected so that they vanish in the limit $t\to-\infty$. When the field is described by a formula of the (21) type, this is achieved by selecting the positive imaginary part $\mathrm{Im}\Omega$. If we are dealing with propagation of waves in stable systems, the correct result will also be obtained for $\mathrm{Im}\Omega\to 0$. In the case of inverted media, $\mathrm{Im}\,\Omega$ cannot be selected to be infinitesimally small: conversely, $\mathrm{Im}\Omega$ should exceed all the increments in the system. This rule will be satisfied automatically if we assume initially that $\mathrm{Im}\Omega\to\infty$. In this case the integration contour in Eqs. (22)–(24) should be selected along the real axis, when the branching points $\Omega\varepsilon_{1,2}^{1/2}/c$ lie in the upper half-plane and are bypassed from below, whereas the points $-\Omega\varepsilon_{1,2}^{1/2}/c$ lie in the lower half-plane and are bypassed from above. Naturally, the values of the roots $p_{1,2}$ considered in the limit $\varkappa\to\infty$ are as usual equal to $i|\varkappa|$.

After establishing these bypassing rules, we can take $\mathrm{Im}\Omega$ to zero retaining the relative positions of the singularities and the contour. This has the effect that the contour is shifted away from the real axis and is of the form shown in Fig. 3. The solution of the field equations obtained in this way is in agreement with the causal formulation of the problem.

We shall now investigate the field of the reflected wave. We shall do this by going over, as before, to a new variable $\eta=\varkappa/k_1$ which represents the sine of the angle of incidence of a plane wave. We then have the following expression for E_r:

$$E_r=(2\pi)^{-1}k_1\int_{G_\eta}d\eta V_1(\eta)E(k_1\eta)\exp\{ik_1R(\eta\sin\theta+\xi\cos\theta)\}. \tag{27}$$

Since the problem of the reflection coefficient of an inverted medium has been discussed frequently in the literature (see Refs. 2, 4, and 10, and the citations given there), we shall consider briefly this topic. Firstly, we must stress that $V_1(\eta)$ is an analytic function of the complex variable η, so that the value of V_1 on the real axis of η, representing the reflection of plane waves, should be found from analytic properties of the reflection coefficient. We shall write down V_1 in the form

$$V_1(\eta)=\frac{\xi-i\zeta'}{\xi+i\zeta'},\quad \zeta'=(\eta^2-n^2)^{1/2}, \tag{28}$$

where $\zeta'=\eta$ for $\eta\to\infty$ and $\xi>0$ for $|\eta|<1$. The root ζ' can be represented in the form $\zeta'=(\rho_1\rho_2)^{1/2}\exp\{i(\varphi_1-\varphi_2)/2\}$, where $\rho_{1,2}$ are the lengths of the radius vectors drawn from the branching point $\pm n$ to the point η, and $\varphi_{1,2}$ are the angles between the vectors and the real axis (Fig. 5). Using this representation of the root, we shall write V_1 in the form

$$|V_1|^2=\frac{a^2+2ab\sin(\varphi_1-\varphi_2)/2+b^2}{a^2-2ab\sin(\varphi_1-\varphi_2)/2+b^2}, \tag{29}$$

$$a=\xi,\quad b=(\rho_1\rho_2)^{1/2},\quad |\eta|<1.$$

It is clear from Eq. (29) that if $|V_1|^2>1$, then

$$\sin(\varphi_1-\varphi_2)/2>0. \tag{30}$$

If the cuts joining the branching points $\pm n$ to infinity are made vertically, the condition (30) is satisfied for $\eta>\mathrm{Re}\,n$ or for plane waves incident at an angle φ which obeys the condition

$$\sin\varphi>\mathrm{Re}\,n. \tag{31}$$

If the opposite inequality is obeyed, then $|V_1|^2<1$ since φ_1 now varies between the limits $-\pi$ and $-3\pi/2$ and we have $\varphi_2<|\pi+\varphi_1|$. Therefore, on opposite sides of the cut the reflection coefficient is different and in crossing the cut along the loop χ the value of $|V_1|^2$ changes abruptly.

However, it should be stressed that the conclusions, which can be found in one form or another in Refs. 2, 4, and 10, have no meaning because the value of the reflection coefficient depends on the method used to make the cuts. In fact, if a cut is not made vertically but is shifted for example to the left (Fig. 5, cut C_2), then $|V_1|^2>1$ also for the angles satisfying the condition $\sin\varphi<\mathrm{Re}\,n$. If we shift the cut far to the right, we can achieve a situation when $|V_1|^2$ is less than unity for any angle of incidence. This means that in investigations of the reflection of waves from an inverted medium we cannot restrict our analysis simply to homogeneous plane waves, but we must include also inhomogeneous plane waves determined by the integral along a loop which covers the branching points χ.

We shall now consider the field of the reflected wave given by Eq. (27). The integral along the contour G_η can be represented by a sum of an integral along the real axis and integrals along the loops χ and χ_1 (Fig. 4) surrounding the branching points $\pm n$, i.e.,

$$E_r\sim\int_{G_\eta}=\int_{-\infty}^{\infty}+\int_{\chi}+\int_{\chi_1}\equiv E_1+E_2+E_3. \tag{32}$$

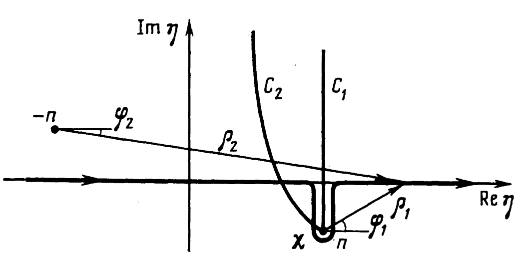

FIG. 5

In the subsequent analysis we shall assume, for the sake of simplicity, that the point of observation is in the projector zone, i.e., we shall assume that the following inequality is obeyed:

$$k_1L^2\gg R, \tag{33}$$

where L is the characteristic width of the beam and R is the distance from the center of the image of the beam to the point of observation. When the condition (33) is obeyed, then $E(k_1\eta)$ is the sharpest function in the integral of Eq. (27) with its maximum at $\eta=\sin\varphi$ so that in the calculation of E_1 it is sufficient to expand the remaining functions in the integrand as a series in the vicinity of $\eta=\sin\varphi$. This gives

$$E_1=V_1(\eta=\sin\varphi)Q[x-(z+l)\,\mathrm{tg}\,\varphi-\Delta]\times\exp\{ik_1[x\sin\varphi+(z+l)\cos\varphi]\}. \tag{34}$$

Here, $\Delta=k_1^{-1}\partial\psi/\partial\eta|_{\eta=\sin\varphi}$, and ψ is the phase of the reflection coefficient $[V_1=\exp(-i\psi)]$.

We can show that the term E_1 does not describe the amplification of the reflected signal observed in the experiments. This does not yet mean that there is no amplification effect in the adopted formulation of the problem (compare with Refs. 2, 4, and 10). Amplification occurs because of excitation of exponentially growing lateral waves (terms E_2 and E_3) by the incident wave. In considering this problem it is sufficient to deal with the integral E_2, because in the case of sufficiently rapidly decreasing spectra the value of E_3 is fairly small.

We shall represent the integral E_2 in the form

$$E_2=\alpha\int_n^{n_1}\lambda E(k_1\eta)\exp\{ik_1R(\eta\sin\theta+\xi\cos\theta)\}d\eta,$$

$$\alpha=-2ik_1(2n)^{1/2}/\pi(1-n^2)^{1/2},\quad \lambda=(\eta-n)^{1/2}. \tag{35}$$

Here, $n_1=\mathrm{Re}\,n$ and $\lambda>0$ in the limit $\eta\to\infty$. The inequality (11) was used to derive Eq. (35).

In the subsequent analysis it is necessary to select a specific distribution of the intensity in a beam. Let us assume that, for example, the beam envelope is a Gaussian curve, i.e.,

$$Q(x)=\exp(-x^2/L^2). \tag{36}$$

Then, if the inequality

$$k_1R|n_2|^2/2\cos^3\theta_1\ll 1 \tag{37}$$

(where $\sin\theta_1=\mathrm{Re}\,n$) is obeyed, the argument of the exponential function in the integrand of Eq. (35) can be expanded as a series near the point $\eta=n_1$, retaining on the first two terms, so that E_2 becomes

$$E_2=\pi^{1/2}\alpha L|n_2|^{1/2}\exp(3\pi i/4)\exp\{ik_1R\cos(\theta-\theta_1)\}\times\int_0^1 z^{1/2}\exp\left\{\gamma^2\left(z-1+i\frac{\sin\varphi-n_1}{|n_2|}\right)^2-\beta(z-1)\right\}dz, \tag{38}$$

where $\gamma=k_1L\,|n_2|/2$, and $\beta=k_1R\,|n_2|\sin(\theta-\theta_1)/\cos\theta_1$.

It is clear from Eq. (38) that the field of the reflected wave rises exponentially on increase in the beam width L if the angle of incidence φ is close to the total reflection angle $\theta_1=\sin^{-1}n_1$. More exactly, exponential growth occurs if

$$|\sin\varphi - n_1| < |n_2|, \qquad (39)$$

so that the field reaches its maximum value when the angle of incidence is $\varphi = \theta_1$ and the other conditions are kept fixed (fixed point of observation, beam parameters, and gain).

An asymptotic calculation of the integral of Eq. (38) will be made for the most interesting case when $|\beta| \gg 1$. Cosequently, the field intensity governed by the integral along the edges of the cut χ is

$$E_2 = \pi^{1/2}\alpha L |n_2|^{1/2} \exp(3\pi i/4) \exp\{ik_1 R\cos(\theta-\theta_1)\}$$
$$\times \begin{cases} -\dfrac{\exp\{\gamma^2(\bar{a}-1)^2\}}{\beta+2\gamma^2\bar{a}} & \text{for} \quad \theta<\theta_1, \quad |\beta|\gg\gamma^2, \\ \dfrac{\pi^{1/2}\exp(\beta+\gamma^2\bar{a}^2)}{2(2\gamma^2\bar{a}+\beta)^{1/2}} & \text{for} \quad \theta<\theta_1, \quad |\beta|\ll\gamma^2, \\ & \text{and also for} \quad \theta>\theta_1. \end{cases} \qquad (40)$$

Here $\bar{a} = 1 + i(n_1 - \sin\varphi)/|n_2|$. We can see from Eq. (40) that for angles of observation θ smaller than the total reflection angle θ_1 the field falls exponentially on increase in the beam width and the rate of fall decreases in the limit $\varphi \to \theta_1$. As the angle of incidence approaches the critical value ($\theta \lesssim \theta_1$), the field rises and the rise is particularly strong for beams incident at angles close to θ_1. In the case of angles of observation greater than the total reflection angle the field rises exponentially in space (factor $\exp\beta$), as determined by the excitation of a lateral wave. It should be stressed that the amplitude of a lateral wave grows exponentially on increase in L if the inequality of Eq. (39) is obeyed and falls when the opposite inequality is satisfied. This is due to the fact that in the limit $\varphi \to \theta_1$ the beam excites most strongly a lateral wave since the maximum of the angular spectrum of the beam is located at angles close to the angle of total reflection.

It follows from this analysis that, as in the case of reflection of spherical waves, the amplification is due to excitation of lateral waves. It should be mentioned in this connection that the field described by Eq. (40) does not tend to zero at all when the beam width tends to infinity. On the contrary, E_2 rises exponentially when Eq. (39) is obeyed. This circumstance means that in solving the problem of reflection of waves from an inverted medium we cannot limit ourselves to the stimulated solutions determined by the Fresnel formulas, but must consider right from the beginning a beam of finite width. It is then found that on going to the plane wave limit ($L \to \infty$) the field of the reflected wave depends strongly on the angular spectrum of the beam. For example, if we select a beam with a rectangular profile, i.e., if

$$Q(x) = \begin{cases} 1, & |x|<L \\ 0, & |x|>L \end{cases},$$

then the exponential growth of the field on increase in L is observed for any angles of incidence φ, in contrast to a Gaussian beam for which the growth occurs only in the range of angles φ satisfying the inequality (39). This is a manifestation of a configurational instability which appears on reflection of waves from unstable media.[9] The essence of this effect is that on reflection of signals from an unstable medium there are considerable differences in the reflected field even if the form of the signals is very similar and the differences appear in the amplitudes of unstable waves and also in the form of the part of space where unstable waves exist.[9]

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[1)]The need for the causal formulation of the problems of reflection from unstable media was pointed out in papers on the theory of stability of a tangential discontinuity[7–9] and of electrodynamics.[4–10]

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