Averaged description of waves in the Korteweg–de Vries–Burgers equation

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We generalize Whitham's averaged equations for the amplitude, velocity, and period of waves described by the Korteweg–de Vries equation to the case where there is a small dissipation present. We find the stationary solution of the equations obtained. We study the conditions at the boundaries of the region occupied by oscillations.

I. INTRODUCTION

The aim of the present paper consists in taking into account the effect of a small dissipation on the dynamics of nearly periodic waves described by the Korteweg–de Vries (KdV) equation.

The high effectiveness of the method proposed by Whitham for describing nearly periodic solutions of partial differential equations was made clear in the past few years. The idea of this method consists in that one constructs to begin with a strictly periodic solution of the equation. Afterwards one assumes that the arbitrary constants occurring in that solution are slowly varying functions of the coordinates and time, for which one derives equations.[1,2] This program has been studied most completely so far for waves described by the KdV equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0. \qquad (1)$$

One can reduce Whitham's equations for this case to a symmetric "Riemannian" form. They have been used to solve the "collisionless shock wave" problems which are of considerable physical interest.[3] These equations possess important mathematical properties, as they are in a well defined sense completely integrable".[4,5] However, Eq. (1) describes undamped waves: there is no dissipation in it. Usually, when the dispersion law $\omega(k)$ for small k can be expanded in powers of k, the damping of the wave is given by $\mathrm{Im}\,\omega \propto k^2$ In that case the waves are described by the equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=\nu\frac{\partial^2 u}{\partial x^2}, \qquad (2)$$

which in the literature is sometimes called the Korteweg–de Vries–Burgers equation.[6] (It changes into the Burgers equation if we neglect the term with the third derivative.) The presence of dissipation in itself destroys the strict periodicity of the waves. For the waves to remain nearly periodic it is necessary that the dissipation be small, i.e., that the parameter ν satisfy the condition

$$\nu\lambda\ll 1 \qquad (3)$$

(λ is the wavelength), and we shall assume in what follows that it is satisfied.

We recall some properties of the periodic solutions of Eq. (1), which are necessary for what follows. If the function u depends on x and t in the combination $x - Ut$ we get for it an ordinary differential equation which has a first integral:

$$u_{xx}=6B+Uu-u^2/2, \qquad (4)$$

and furthermore

$$\begin{aligned}{}^1/_2u_x^2&=-36A+6Bu+{}^1/_2Uu^2-{}^1/_6u^3\\ &\equiv{}^1/_6(6\alpha-u)(u-6\beta)(u-6\gamma),\quad \alpha\geqslant\beta\geqslant\gamma.\end{aligned} \qquad (5)$$

Here A, B, U, α, β, γ are constants and

$$A=-\alpha\beta\gamma,\quad B=-(\alpha\beta+\alpha\gamma+\beta\gamma),\quad U=2(\alpha+\beta+\gamma).$$

Instead of α, β, γ we shall in what follows normally use their combinations r_α :

$$r_1=3(\beta+\gamma),\quad r_2=3(\alpha+\gamma),\quad r_3=3(\alpha+\beta),\quad r_3\geqslant r_2\geqslant r_1. \qquad (6)$$

The periodic solution of Eq. (5) has the form

$$u(x,t)=\frac{2a}{s^2}dn^2\left[\left(\frac{a}{6s^2}\right)^{1/2}(x-Ut),s\right]+U-\frac{2a}{3s^2}(2-s^2), \qquad (7)$$

where $dn(y,s)$ is a Jacobi function of modulus s, a determines the amplitude of the oscillations

$$2a=u_{max}-u_{min},$$

and a, s, and U can be expressed in terms of the r_α according to

$$a=r_2-r_1,\quad s^2=(r_2-r_1)/(r_3-r_1),\quad U={}^1/_3(r_1+r_2+r_3).$$

The function W, which was introduced by Whitham and which plays the role of an adiabatic invariant for Eq. (4),[1)] will be important in what follows:

$$\begin{aligned}W&=-\frac{1}{36}\oint u_x\,du=-\frac{1}{2\cdot 3^{5/2}}\int_{6\beta}^{6\alpha}[(6\alpha-u)(u-6\beta)(u-6\gamma)]^{1/2}du\\ &=-2^{3/2}\int_\beta^\alpha[(\alpha-\eta)(\eta-\beta)(\eta-\gamma)]^{1/2}d\eta.\end{aligned} \qquad (8)$$

One can express the wavelength λ in terms of W as follows:

$$\lambda=(\partial W/\partial A)_{B,U}\equiv W_A. \qquad (9)$$

The function W and its derivatives can be expressed in terms

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of complete elliptical integrals. These expressions are given in the Appendix.

2. DERIVATION OF THE EQUATIONS

We now assume that the quantities a, s, and U (or, what comes to the same, the r_α) are slowly varying functions of x and t and we obtain equations for these functions. The method developed in Ref.1 for that purpose is based on the fact that Eq. (1) has three conservation laws of the form[2)]:

$$\partial P_\alpha/\partial t+\partial Q_\alpha/\partial x=0, \tag{10}$$

where

$$P_1=u, \quad P_2=u^2/2, \quad P_3=u^3/6-u_x^2/2. \tag{11}$$

If we now average (10) over a fixed range of wavelengths λ in the vicinity of the point x then, in view of the commutability of the differentiation and averaging operations, Eq. (10) reduces to

$$\partial \bar{P}_\alpha/\partial t+\partial \bar{Q}_\alpha/\partial x=0. \tag{12}$$

In first approximation we can assume in the averaging that the parameters of the solution (7) are constants so that $\bar{P}$ and $\bar{Q}$ turn out to be simply given functions of the r_α, and (12) are the required Whitham equations for the latter. These equations turn out to be first order homogeneous in the derivatives of the r_α.

We now take damping into account, i.e., we change from Eq. (1) to Eq. (2). The term with ν leads, firstly, to a change in the expressions for P_α and Q_α. This change is, however, unimportant since the extra terms, containing ν are small by virtue of Eq. (3). Moreover, Eq. (2) has only one conservation law of the form Eq. (10) (to such a form one can reduce the equation itself). This means that there are on the right-hand side of the equations for P_1 and P_2 terms proportional to ν, and after averaging they take the form

$$\partial \bar{P}_\alpha/\partial t+\partial \bar{Q}_\alpha/\partial x=-\nu\bar{R}_\alpha, \quad \alpha=2, 3, \tag{13}$$

where the expressions for $\bar{P}_\alpha$ and $\bar{Q}_\alpha$ are unchanged. The quantities $\nu\bar{R}_\alpha$ do not contain derivatives of r_α but on the other hand they are proportional to ν. All terms in Eqs. (13) can thus, generally speaking, be of the same order.

We now actually carry out the projected calculations. Differentiating Eq. (11) with respect to t, taking the expression for u_t from Eq. (2), and averaging we get, dropping small terms in the fluxes and expressions which vanish on averaging:

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial}{\partial x}\overline{(u^2/2)}=0,$$

$$\frac{\partial}{\partial t}\overline{(u^2/2)}+\frac{\partial}{\partial_x}(\overline{u^2/3}-\overline{uu_{xx}}-\overline{u_x^2/2})=\nu(\overline{uu_{xx}}),$$

$$\frac{\partial}{\partial t}(\overline{u^3/6}-\overline{u_x^2/2})+\frac{\partial}{\partial x}(\overline{u^4/8}+\overline{u^2u_{xx}/2}+\overline{u_{xx}^2/2}+\overline{u_xu_t})$$

$$=-\nu(\overline{u^2u_{xx}/2}+\overline{u_{xx}^2}), \tag{14}$$

where u_t on the left-hand side of the last equation is assumed to be the expression from Eq. (1).

The averaging in Eq. (14) is carried out according to

$$\overline{F(u)}=-\frac{1}{\lambda}\int_0^\lambda F\,dx=\frac{1}{\lambda}\oint\frac{F(u)}{u_x}du$$

using Eqs. (4) and (5). Introducing the wave number

$$\varkappa=\lambda^{-1}=W_A^{-1},$$

we write Eqs. (14) in the form ($W_A=\partial W/\partial A$, $W_B=\partial W/\partial B$, $W_U=\partial W/\partial U$)

$$\frac{\partial}{\partial t}(\varkappa W_B)+\frac{\partial}{\partial x}(\varkappa UW_A-B)=0,$$

$$\frac{\partial}{\partial t}(\varkappa W_U)+\frac{\partial}{\partial x}(\varkappa UW_U-A)=\nu\varkappa W,$$

$$\frac{\partial}{\partial t}[\varkappa(AW_A+BW_B+UW_U-W)]$$

$$+\frac{\partial}{\partial x}[\varkappa U(AW_A+BW_B+UW_U-W)$$

$$-B^2/2-AU]=\nu\varkappa UW. \tag{15}$$

When there is no dissipation, i.e., when $\nu=0$, Eqs. (15), as should be the case, change to Eqs. (45) of Ref. 1. At first sight, only the first of Eqs. (15) has the form of a conservation law. One checks, however, easily that from Eq. (15) there still follows a conservation law for the "number of waves" in the form[3]

$$\frac{\partial\varkappa}{\partial t}+\frac{\partial}{\partial x}(U\varkappa)=0. \tag{16}$$

Using this relation we can rewrite Eq. (15) in the more compact form corresponding to Eqs. (47) of Ref. 1:

$$\frac{DW_A}{Dt}-W_A\frac{\partial U}{\partial x}=0, \quad \frac{DW_B}{Dt}-W_A\frac{\partial B}{\partial x}=0,$$

$$\frac{\partial W_U}{Dt}-W_A\frac{\partial A}{\partial x}=-\nu W, \tag{17}$$

where $D/Dt=\partial/\partial t+U(\partial/\partial x)$. Equations (17) are convenient for solving several simple problems. We shall, in particular, use them in the next section to find a stationary solution. To study more general cases and for a numerical integration it is necessary to have equations for the "Riemannian" variables r_α. The change to these variables requires very cumbersome calculations. It is advisable first to obtain equations for α, β, and γ. It is then useful to take into account that each of these quantities satisfies the equation

$$\alpha^3-{}^1/_2U\alpha^2-B\alpha+A=0$$

and so on. Finally, the equations have the form

$$\frac{\partial r_\alpha}{\partial t}+v_\alpha\frac{\partial r_\alpha}{\partial x}=\nu\rho_\alpha \tag{18}$$

(no summation over α is implied!). The dissipative terms ρ_α are equal to

$$\rho_1=-\frac{3W}{2(K-E)}\left(\frac{6s^2}{a}\right)^{1/2}, \quad \rho_2=\frac{3W}{2(E-s'^2K)}\left(\frac{6s^2}{a}\right)^{1/2},$$

$$\rho_3=\frac{3W}{2E}\left(\frac{6s^2}{a}\right)^{1/2}, \quad s'^2=1-s^2. \tag{19}$$

We note that $\rho_1\geqslant\rho_3\geqslant\rho$. The Riemannian velocities, on the other hand, have the same form as when there is no dissipation:

$$v_1=U-\frac{2}{3}(r_2-r_1)\frac{K}{K-E},\qquad v_2=U-\frac{2}{3}(r_2-r_1)\frac{s'^2K}{E-s'^2K},$$
$$v_3=U+\frac{2}{3}(r_3-r_2)\frac{K}{E}. \tag{20}$$

(We corrected a misprint which had slipped in in Ref.3 and somewhat transformed the expression for v_3.) Here $K=K(s)$ and $E=E(s)$ are complete elliptical integrals of the first and second kind. We note that Eqs. (18), in contrast to the usual Whitham equations do not have solutions with constant r_α, meaning that there are no strictly periodic solutions of Eq. (2).

Equations (18) are invariant under the transformation (C is an arbitrary constant)

$$r_\alpha\to r_\alpha+C,\quad x\to x+Ct, \tag{21}$$

in view of the invariance of the original equation under the transformation

$$u\to u+C,\quad x\to x+Ct. \tag{22}$$

Moreover, Eqs. (18) are invariant under the additional transformation

$$r_\alpha\to Cr_\alpha,\quad t\to t/C. \tag{23}$$

3. STATIONARY SOLUTIONS

Sagdeev considering a set of magnetohydrodynamic equations for a plasma taking friction into account found in 1961 (in some reference frame) a stationary solution corresponding to a shock wave.[7] For small dissipation the structure of the wave had an oscillating, almost periodic nature. One can therefore use the equations obtained in the preceding section to study it and this enables us to find the complete solution of the problem. This gives a simple, though instructive, example of their application.

We shall seek for the set (17) a solution which is stationary in some reference frame, i.e., which moves with a constant velocity. This means that we must put in (17) $U=\text{const}$ and look for a solution that depends on $x-Ut=X$. Then $D/Dt=0$, and the first of Eqs. (17) is satisfied identically, and it follows from the second one that $\partial B/\partial x=0$, $B=\text{const}$. The third equation then gives $W_A\,\partial A/\partial x=-W$ or, using the fact that U and B are constant

$$W(X)=W(X_0)\exp[\nu(X-X_0)]. \tag{24}$$

We show that the solution obtained determines completely the structure of the shock wave front. Indeed, without loss of generality we can take the magnitude of the jump at the shock wave front to be unity, i.e., put $u=u^+=0$, as $X\to+\infty$ and $u=u^-=1$, as $X\to-\infty$. After that using (4) we can express B in terms of the averaged values $\bar{u}$ and $\overline{u^2}$:

$$6B=\bar{u}(U-\bar{u}^2/2)-S=\text{const}. \tag{25}$$

Here $S=(\overline{u^2}-\bar{u}^2)/2$ is a quantity proportional to the energy of the oscillations. Using the fact that there are no oscillations as $X\to\pm\infty$, $S=0$ and $u|_{X\to\infty}=u^+=0$, $\bar{u}|_{X\to-\infty}=u^-=1$, we find from Eq. (25) that

$$B=0,\quad U=1/2. \tag{26}$$

Using Eqs. (25) and (26) and the formulae for $\bar{u}$ and S given in the Appendix, we can easily express all quantities of interest to us—the average value $\bar{u}$, the amplitude a of the oscillations, and the wave number $\varkappa$—in terms of the single parameter s^2:

$$\bar{u}=(1-f_1f_2^{-1/2})/2,\qquad a=f_2^{-1/2}/2,$$
$$\varkappa=f_2^{-1/4}/4\sqrt{3}\,sK,\qquad S=(1-f_1^2f_2^{-1})/8,$$
$$f_1(s)=\frac{2}{3s^2}(2-s^2-3E/K),\qquad f_2(s)=\frac{4}{9s^4}(1-s^2+s^4). \tag{27}$$

The functions (27) are shown in Fig. 1(a). It is interesting that the energy S of the oscillations remains always a small quantity—its maximum value $S_m=0.125$ is reached for $s^2\approx0.96$.

The way the parameter s^2 depends on the coordinate X is given by Eq. (24):

$$\nu(X-X_0)=\ln[(1-s^2+s^4)E-(1-s^2)(1-s^2/2)K]-{}^5/_4\ln(1-s^2+s^4). \tag{28}$$

It is shown in Fig. 1(b) (X_0 is an arbitrary point fixing the beginning of the wave). We have thereby found the behavior of all average quantities. Substituting Eqs. (26)–(28) into Eq. (7) we can determine also the oscillating structure of the wave front which is shown in Fig. 2 for $\nu=10^{-2}$

The structure of the stationary shock front in the Korteweg–de Vries–Burgers equation thus has a universal character and is completely described by Eqs. (27), (28), and (7). The average value $\bar{u}$ increases very steeply near the leading front of the wave $X\approx X_0$:

$$\bar{u}\approx-6/\ln[\nu(X-X_0)].$$

The oscillations have a maximum amplitude $a=3/4$. Their structure has a soliton character but a distinct splitting off of a chain of solitons occurs only for rather small values of $\nu\lesssim10^{-4}$. The distance between the first and the second solitons is $\lambda_s\approx\sqrt{2}\,(3.75+\ln\nu^{-1})$. Away from the leading

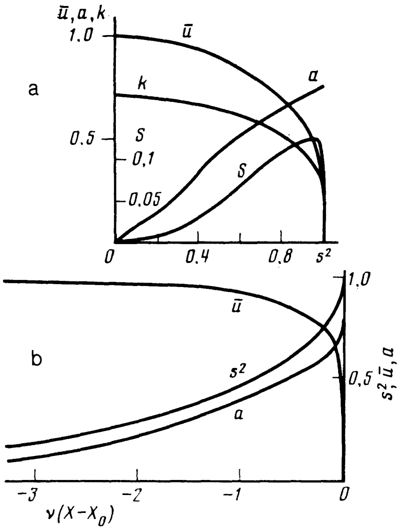

FIG. 1. Distribution of average values in a stationary shock front: a) $\bar{u}$, $k=2\pi\varkappa$, a, and S as functions of the parameter s^2; b) $\bar{u}$, a, and s^2 as functions of the distance $X-X_0$.

front the oscillations take on rapidly a sinusoidal character. Their amplitude then decreases exponentially, and similarly $\bar{u}$ approaches its asymptotic value $u = 1$ exponentially:

$$a = {}^3/_4 s^2 \approx (6/5\pi)^{1/2} \exp[-\nu(X_0 - X)/2],$$
$$\bar{u} - 1 \approx -17/15\pi \exp[-\nu(X_0 - X)].$$

The wavelength of the oscillations tends then to[5] $\lambda_0 = 2\pi\sqrt{2}$.

It is important to emphasize that the width of the shock wave front is $\Delta X \sim \nu^{-1}$. It is, in fact, independent of the magnitude of the jump $\bar{u}$. This significantly distinguishes hydrodynamics with dispersion and a small dissipation from the usual hydrodynamics, where the width of the shock front increases with increasing viscosity ν and decreasing discontinuity: $\Delta X \sim \nu/\bar{u}$. The reason for this difference is that the dissipative process in the case considered by us is connected with oscillations and not with a change in the average value $\bar{u}$, as is directly clear from Eqs. (17).

The formation of a shock wave can be considered as the result of the decay of an initial discontinuity of the quantity u, assuming, for instance, that at $t = 0$ we have $u = u^-$ for $x < 0$ and $u = u^+ = 0$ for $x > 0$. The present authors have solved the problem of the decay of such a discontinuity.[3] During the decay a region is formed which spreads, is bounded on both sides, and is filled with oscillations. The leading front, on which the condition $2a = 3U$ is always satisfied for solitons, moves with a velocity $U = 2u^-/3$. If one takes dissipation into account, the solution of Ref. 3 must be regarded as describing the initial stage of a process which subsequently reaches asymptotically the solution (27), (28). The velocity of the leading front then decreases from $2u^-/3$ to $u^-/2$, with a corresponding reduction in the soliton amplitude. On the other hand, the trailing edge simply moves asymptotically to $X \to -\infty$.

In concluding this section we show that the simple form of the solution (24) is connected with the existence of a general mechanics theorem about the change of an adiabatic invariant under the action of linear friction.

We consider a particle of mass m performing a finite motion in a field and we evaluate the change in energy due to a small friction force $f = -\gamma\dot{x}$. The average change in energy over a period is equal to the average of the work done by the friction force (T is the period of the oscillations):

$$\frac{d\bar{E}}{dt} = \overline{\dot{x}f} = -\frac{\gamma}{T}\int_0^T \dot{x}^2 dt = -\frac{\gamma}{mT}\oint p\,dx. \qquad (29)$$

The integral in this formula is equal to the adiabatic invariant I, multiplied by 2π. Using the fact that $T = 2\pi(dI/dE)$ we can write Eq. (29) in the form

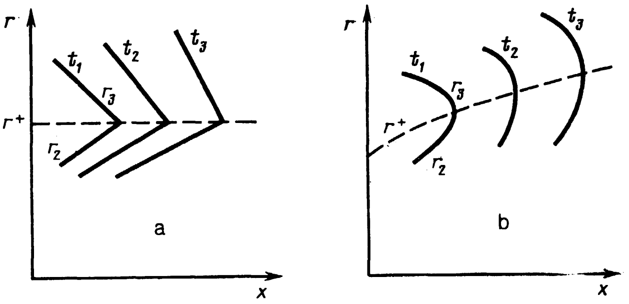

FIG. 3. Intersection of the Riemann variables $r_2(x)$ and $r_3(x)$ at various times t.

$$dI/dt = -\gamma I/m \quad \text{or} \quad I = I_0 \exp(-\gamma t/m). \qquad (30)$$

We note now that Eq. (4) has the form of the equation of motion of a particle [if we consider $(-x)$ as the time], and if we change to Eq. (2), there is added on the right-hand side of Eq. (4) a "linear friction" of the form νu_x. On the other hand, the integral W is proportional to the adiabatic invariant of the system described by Eq. (4), so that Eq. (24) is completely analogous to Eq. (30).

4. EFFECT OF DISSIPATION ON THE STRUCTURE OF THE LEADING FRONT

A characteristic property of the Whitham equations is the possible existence of unique features in their solutions. We have in mind here the already mentioned leading front on which the wave number vanishes, and the trailing edge on which the amplitude of the oscillations vanishes. These features, which were first discovered in Ref. 3 in an analysis of some self-similar solutions of the equations, were studied in their general form in Ref. 8 (see also Ref. 9, Ch. IV, Sec. 4). Avilov and Novikov[10] have investigated numerically the formation of the initial problem for solutions with such singularities and established the admissible classes of initial conditions.

We consider now the effect of dissipation on the structure of the leading front. (We shall show that there is no effect of the dissipation on the trailing edge.) First, though, we recall the properties of that front when there is no dissipation. On the leading front $s^2 = 1$, i.e., $r_3^+ = r_2^+ = r^+$ (we shall label all quantities on the leading front with a superscript $+$). Then

$$u_{min}^+ = \bar{u}^+ = r_1^+,$$
$$v_2^+ = v_3^+ = U^+ = (2u_{max} + u_{min})/3 = (r_1 + 2r_3)/3.$$

Of most importance is the problem of how $1 - s^2$ or $r_3 - r_2$

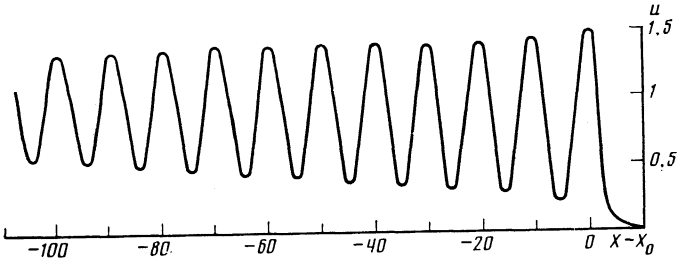

FIG. 2. Oscillatory structure of a stationary shockfront (for $\nu = 10^{-2}$).

tends to zero as $x \to x^+$. As r_2 and r_3 are different functions satisfying different equations it may appear that the most natural way for the curves $r_2(x)$ and $r_3(x)$ to intersect is "at an angle," as shown in Fig. 3(a). In fact, this case is singular. Indeed, from the equations written in the form

$$(\partial x/\partial t)_{r_2}=v_2, \quad (\partial x/\partial t)_{r_3}=v_3, \tag{31}$$

it is clear that as time goes on the values of r_2 and r_3 are transferred with velocities v_2 and v_3. Since $v_2^+ = v_3^+$, it is clear that under such a deformation of the curve of Fig. 3(a) the value of r^+ at which the intersection occurs will remain constant, which is impossible in the general case. In order that the value of r^+ change with time it is necessary that the curves $r_2(x)$ and $r_3(x)$ merge "smoothly" without a kink and that is possible only if at the point where they join they have a vertical tangent [see Fig. 3(b)]:

$$(\partial x/\partial r_2)_t=(\partial x/\partial r_3)_t=0 \text{ at } r_2 = r_3 = r^+. \tag{32}$$

We note that in both cases considered the front moves with a velocity U^+:

$$dx^+/dt=U^+. \tag{33}$$

In the first case this follows directly from Eq. (31) by virtue of the fact that r^+ is constant on the front. In the second case we have

$$\frac{dx^+}{dt}=\left(\frac{\partial x}{\partial t}\right)_{r_2}+\left(\frac{\partial x}{\partial r_2}\right)_t\frac{dr^+}{dt}=U^+$$

by virtue of Eqs. (31) and (32).

We now elucidate the way $1-s^2$ tends to zero on the basis of Eqs. (18), i.e., when dissipation is taken into account. We assume then, as will be confirmed by calculations, that condition (32) remains valid also when there is dissipation present, so that the curves have the shape of Fig. 3(b). We note first of all that it follows from (19) that as $r_2 \to r_3$

$$\rho_2 \approx \rho_3 \approx -8a^2/45. \tag{34}$$

Since ρ_2 and ρ_3 are finite, it follows at once that the front velocity is given as before by Eq. (33). Indeed, near the front $r_2 = f_2[x-x^+(t),t]$. The term containing dx^+/dt in the equation has thus the form

$$\frac{\partial r_2}{\partial x}\frac{dx^+}{dt}.$$

But by virtue of Eq. (32) this term tends to infinity as $x \to x^+$, so that the finite dissipative term ρ_2 is unimportant for determining dx^+/dt.

The following expressions hold for the velocities v_2 and v_3 near the front:

$$v_{2,3} \approx U^+ \mp \frac{2}{3}a\frac{s'^2L/2}{1\mp s'^2L/4}=U^++\delta v_{2,3},$$

$$s'^2=1-s^2, \quad L=\ln(16/s'^2).$$

One checks easily that $|r_2-r_3| \gg |r_2+r_3-2r^+|$. Substituting

$$\left(\frac{\partial r_2}{\partial t}\right)_x \approx \left(\frac{\partial r_2}{\partial x}\right)_t\frac{dx^+}{dt}+\frac{dr^+}{dt}$$

into the equation for r_2 we get after dividing by δv_2

$$\left(\frac{\partial r_2}{\partial x}\right)_t=\left[-\frac{3}{as'^2L}+\frac{3}{4a}\right]\left(\frac{dr^+}{dt}+\frac{8}{45}\nu a^2\right).$$

Subtracting a similar equation for r_3 we get an equation for s'^2:

$$\frac{\partial s'^2}{\partial x}=-\frac{6}{a^2s'^2L}\left(\frac{dr^+}{dt}+\frac{8}{45}\nu a^2\right),$$

whence we finally have

$$s'^4\left(\ln\frac{16}{s'^2}+\frac{1}{2}\right)=-\frac{12}{a^2}\left[\frac{dr^+}{dt}+\frac{8}{45}\nu a^2\right](x-x^+). \tag{35}$$

For $\nu=0$ this formula goes over into Eq. (9) of Ref. 9 (Ch. IV, Sec. 4).[6] We note that when there is no dissipation the case $dr^+/dt=0$ is singular, as the right-hand side of Eq. (35) vanishes and one must take into terms of higher order in $(x-x^+)$. Just such a situation occurred in the problem of the decay of an initial discontinuity in Ref. 3. However, for finite ν this case is not at all special. Indeed, one can easily verify that the behavior of s'^2 on the leading front of the solution (27), (28) completely agrees with Eq. (35).

As to the trailing edge, the presence of dissipation does in no way affect in this case the way the amplitude of the oscillations tends to zero. Indeed, the functions ρ_1 and ρ_2 are for small a proportional to a and such terms are unimportant as compared to the finite quantity dr^-/dt, so that Eq. (37) from Ref. 9 (Ch. IV, Sec. 4) remains valid.

It is interesting that the presence of the trailing front violates the "number of waves" conservation expressed by Eq. (16). Indeed, the velocity v^- of the trailing edge is less than the velocity U^- of the wave on the front. Therefore in a unit time the length of the oscillation region increases by $(U^- - v^-)$, and $\varkappa^-(v^- - U^-) = [\frac{2}{3}(r_3^- - r_1^-)]^{3/2}\pi$ wave periods are generated.

APPENDIX

We give the formulae which express the function W and its derivatives with respect to A, B, and U in terms of the complete elliptical integrals $K(s)$ and $E(s)$:

$$W=-\frac{64}{15}\left(\frac{a}{6s^2}\right)^{5/2}((1-s^2+s^4)E-(1-s^2)(1-s^2/2)K],$$

$$W_A=\lambda=2\left(\frac{6s^2}{a}\right)^{1/2}K,$$

$$W_B=-\frac{\lambda}{6}\bar{u}=-\frac{\lambda}{6}\left[U-\frac{2a^2}{3s^2}\left(2-s^2-3\frac{E}{K}\right)\right],$$

$$W_U=-\frac{\lambda}{72}\overline{u^2}=-\frac{\lambda}{72}(\bar{u}^2+2S),$$

$$S=\frac{\overline{u^2}-\bar{u}^2}{2}=\frac{2a^2}{3s^4}\left[s^2-1+2(2-s^2)\frac{E}{K}-3\frac{E^2}{K^2}\right]. \tag{36}$$

Between these quantities there exist simple relations:

$$^5/_2W=3AW_A+2BW_B+UW_U, \quad UW_B=BW_A+6W_U.$$

[1)]The KdV equation is written in Ref. 1 in the form $\eta_t+6\eta\eta_x+\eta_{xxx}=0$ so that our $u=6\eta$. The notation in the present paper is, however, chosen such that the function $W(A,B,U)$ has the same form as in Ref. 1.

[2)]In actual fact there follow from Eq. (1) an infinite number of relations such as Eq. (10). This is, however, unimportant for what follows.

[3)]The fact that Eq. (16) must remain valid also when a small dissipation is taken into account was noted in Ref. 2 (§ 14.10).

[4)]One makes the transition to an arbitrary jump in u^- simply by multiplying by u^- the average velocity $\bar{u}$, the amplitude of the oscillations a, the Riemann variables r_α, and so on. We note that if $u^+ \neq 0$ then $U=(u^+$

$+u^-)/2$, $B=-u^-u^+/12$, and the magnitude of the jump in $u^- - u^+$.

[5)] S. P. Novikov has informed us that with V. V. Avilov and I. M. Krichever he has also obtained the averaged Eqs. (18) and found their stationary solution.

[6)] There is a wrong coefficient 6 instead of 12 in Eq. (35) of Ref. 9.

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