Stochastic acceleration of relativistic particles in a magnetic field

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In this paper we study the dynamics of relativistic charged particles in the field of a fast electromagnetic wave $\omega \gg kc$ and a transverse magnetic field. The particle equations of motion are reduced to a Poincaré mapping. We find the solution of the problem of the evolution of the particle distribution function and show that when $\omega = kc$ there occurs an unbounded stochastic particle acceleration and the energy grows with time according to the power law $E \propto t^{3/7}$. When $\omega > kc$ there exists a limitation on the magnitude of the stochastic acceleration. We discuss the characteristic features of the regular and the stochastic particle dynamics, the structural properties of the phase space, and typical bifurcations of phase trajectories.

1. INTRODUCTION

Charged resonance particles are accelerated when they move in the field of a plane wave and in a transverse magnetic field. Particles moving with a velocity close to the phase velocity of the wave in a magnetic field can be reflected many times from the wave front and their energy can increase.[1] In the nonrelativistic case the same mechanism leads to particles leaving the potential well of the wave and to a weakening of the interaction.[2] However, the dynamics of detrapped particles may turn out to be stochastic, and a different kind of particle acceleration then arises. The stochastic mechanism for the acceleration of charged particles in a magnetic field and in the field of a plane wave was studied in Ref. 3 in connection with the problem of ion heating by lower-hybrid oscillations. In that paper the case of strong magnetic fields was considered, when the particle passed twice in each cyclotron period through a resonance region and interacted with the wave. Successive changes in velocity in the resonance regions turn out to be uncorrelated with one another and there occurs diffusion in velocity and a stochastic particle heating. The stochastic heating of detrapped particles in vanishingly small magnetic fields was studied in Refs. 4 and 5. The slow damping of plasma waves due to the stochastic heating of detrapped particles was studied in the same papers. The stochastic acceleration process in the case when the wave frequency is close to the frequency of cyclotron harmonics was considered in Ref. 6. An interesting feature of the nonlinear particle dynamics in that case is the formation of a stochastic web in the phase plane.[7] A small group of particles is then stochastisized; they diffuse along the channels of the stochastic web and their energy increases on the average without limit.

In the relativistic case, trapping in a potential well is possible for a group of particles with an initial velocity close to the phase velocity of the wave; this is accompanied by unbounded regular particle acceleration along the wave front.[8-10] Such a regular acceleration method for charged particles trapped by a plasma wave, based on using a transverse magnetic field, is called surfatron.[8] The radiation which arises for such an electron motion was studied in Refs. 10 and 11. The radiation is caused by the oscillations of the trapped particles in directions at right angles to the wave front during their relativistic motion along the wave front. Thus, various physical effects which arise when charged particles move in a plasma wave and in a transverse magnetic field are rather complicated and characterized by an interlacing of regular and stochastic mechanisms for transferring energy between the particles and the waves. A review of many results in this field is contained in Ref. 12.

In the present paper we study the dynamics of charged particles in the field of a fast electromagnetic wave with a phase velocity $\omega/k \gg c$ (c is the velocity of light *in vacuo*) in a transverse magnetic field. The trapping effect and regular particle acceleration do not occur in this situation. The stochastic detrapped-particle acceleration mechanism considered in Refs. 4 and 5 becomes here more efficient[13,14] due to the nonlinearity caused by the energy dependence of the cyclotron frequency of the motion of relativistic particles in a magnetic field. It is important to emphasize that in the case where the phase velocity of the wave equals the velocity of light, $\omega = kc$, there exists for the stochastic acceleration of relativistic-particles a universal mechanism which is not connected with the magnitude of the amplitude of the field of the electromagnetic wave. We obtain a solution of the problem of the evolution of the distribution function of the particles subject to stochastic acceleration. For large times the distribution function ceases to depend on the initial conditions and becomes self-similar, and the average particle energy E increases with time according to a power law $E \propto t^{3/7}$. The dynamic equations of motion of the ultrarelativistic particles then reduce to a simple Poincaré mapping.

We show that because of the Kolmogorov-Arnold-Moser theorem about the conservation of invariant tori in the phase plane, there exist in the region of small energy values relatively large islands of stability in the vicinity of lower-order cyclotron resonances. We study the effect of dynamic correlations connected with the existence of such islands on the diffusion of charged particles.

2. MAPPING FOR RELATIVISTIC PARTICLES

The Hamiltonian H describing the dynamics of relativistic charged particles in an electromagnetic field with vector potential $\mathbf{A}$ has the following form:

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$$H=[m^2c^4+(c\mathbf{P}-e\mathbf{A})^2]^{1/2}, \tag{2.1}$$

where $\mathbf{P}$ is the generalized momentum. The Hamiltonian (2.1) is a function of time:

$$\dot{H}=-\frac{ec}{H}\frac{\partial}{\partial t}\mathbf{PA}+\frac{e^2}{2H}\frac{\partial}{\partial t}\mathbf{A}^2. \tag{2.2}$$

In correspondence with (2.1), the canonical equations of motion have the form

$$\dot{\mathbf{r}}=(c/H)(c\mathbf{P}-e\mathbf{A}), \quad \dot{\mathbf{P}}=(ec/H)\nabla\mathbf{PA}-(e^2/2H)\nabla\mathbf{A}^2. \tag{2.3}$$

We consider the motion of a relativistic charged particle in a constant magnetic field $B_0\|z$, at right angles to which, along the x axis, there propagates a linearly polarized electromagnetic plane wave of frequency ω and wave number k. We choose the vector potential $\mathbf{A}$ of the electromagnetic field as follows:

$$\mathbf{A}=\mathbf{e}_y\{B_0x+A_\perp\sin(kx-\omega t)\}, \tag{2.4}$$

where $A_\perp$ is the amplitude of the vector potential of the electromagnetic wave. Since the vector potential (2.4) and the Hamiltonian (2.1) are independent of the coordinates y and z, the generalized-momentum components P_y and P_z are integrals of the motion. For the sake of simplicity we shall assume in what follows that these integrals are equal to zero. The Hamiltonian (2.1) then takes the following form:

$$H=\{m^2c^4+c^2p^2+e^2[B_0x+A_\perp\sin(kx-\omega t)]^2\}^{1/2}. \tag{2.5}$$

The particle equation of motion (2.3) is also considerably simplified:

$$\dot{x}=c^2p/H,$$

$$\dot{p}=-e^2[B_0+kA_\perp\cos(kx-\omega t)][B_0x+A_\perp\sin(kx-\omega t)]/H, \tag{2.6}$$

where $p\equiv p_x$. Accordingly, we have for the change in the total energy

$$\dot{H}=-(e^2\omega A_\perp/H)\cos(kx-\omega t)[B_0x+A_\perp\sin(kx-\omega t)]. \tag{2.7}$$

We write the solution of the unperturbed problem ($A_\perp=0$, $\dot{H}=0$) in the form

$$x=\rho\sin\theta, \quad \theta=\Omega(t-t_0), \tag{2.8}$$

where $\rho=v_\perp/\Omega$ is the Larmor radius, $v_\perp$ the rotational velocity, and t_0 the initial time.

If $A_\perp\neq 0$ a particle moving along a Larmor circle starts to interact with the wave and for ultrarelativistic particles with energies $E\gg(mc^2, eA_\perp)$ the interaction turns out to be most effective only in the vicinity of a single point on the Larmor circle, namely where the particle moves in the direction of the wave propagation. Due to the relativistic Doppler effect the particle reaches the uniform-field region and effectively gathers energy from or gives off energy to the wave. On the remaining part of the trajectory the motion in the rapidly oscillating field is adiabatic. Neglecting on the right-hand side of (2.7) the second term in the square brackets and integrating over the cyclotron period we find the change in energy of a relativistic particle over that time interval:

$$\Delta E=-e^2B_0A_\perp\omega\int dt(x/E)\cos(kx-\omega t). \tag{2.9}$$

Assuming the phase velocity of the wave to be equal to the velocity of light ($\omega=kc$), we evaluate the integral on the right-hand side of (2.9) by the stationary-phase method. In the vicinity of the time $t=t_n$ corresponding to the nth passage of the particle through the nonadiabatic region we expand the trajectory in a series:

$$x(t)\approx x(t_n)+\dot{x}(t-t_n)+\ddot{x}(t-t_n)^2/2+\dddot{x}(t-t_n)^3/6+\dots. \tag{2.10}$$

Assuming the change in the particle energy during a single collision time to be small, we differentiate (2.8) with respect to the time and, using (2.10), we write down the phase of the integrand on the right-hand side of (2.9):

$$kx(t)-\omega t\approx k\rho\sin\theta-k\rho\Omega t_n\cos\theta+(k\rho\Omega\cos\theta-\omega)t$$
$$-(k\rho\Omega^2/2)(t-t_n)^2\sin\theta-(k\rho\Omega^3/6)(t-t_n)^3\cos\theta. \tag{2.11}$$

For ultrarelativistic particles $\rho\Omega\approx c$ and the stationary phase points are given by the equation $\cos\theta\approx 1$. We then get for the phase (2.11) the following approximate expression:

$$kx(t)-\omega t\approx-\omega t_n-(\omega\Omega^2/6)(t-t_n)^3. \tag{2.12}$$

In the vicinity of a stationary phase point $x\approx c(t-t_n)$, and we therefore get for the change in energy (2.9) the estimate

$$\Delta E=-(e^2\omega cB_0A_\perp/E)\int dt(t-t_n)\cos[\omega t_n+(\omega\Omega^2/6)(t-t_n)^3]. \tag{2.13}$$

Since, owing to the fast oscillations of the integrand in (2.13), only the region $\omega\Omega^2(t-t_n)^3/6<1$ contributes to the integral, we can extend the integration limits to infinity. As a result we obtain

$$\Delta E=3^{1/6}2^{2/3}\Gamma(2/3)e^2B_0A_\perp\omega c(\omega\omega_H^2)^{-2/3}(mc^2)^{-4/3}E^{1/3}\sin\omega t_n, \tag{2.14}$$

where $\Gamma(z)$ is the gamma function. The time interval $\Delta t=t_{n+1}-t_n$ between consecutive collisions between the particle and the wave is equal to the period of the cyclotron rotation and is determined by the particle energy:

$$\Delta t=2\pi/\Omega=2\pi E/\omega_H mc^2. \tag{2.15}$$

Equations (2.14) and (2.15) lead to the following mapping:

$$u_{n+1}=u_n+Q\sin\psi_n, \quad \psi_{n+1}=\psi_n+u_{n+1}^{3/2}, \tag{2.16}$$

where $u_n=(2\pi\omega E_n/\omega_H mc^2)$ and the phase $\psi_n=\omega t_n$. The mapping (2.16) conserves the phase volume and depends only on a single parameter

$$Q=\frac{2\cdot 3^{1/6}}{9}(12\pi)^{2/3}\Gamma\left(\frac{2}{3}\right)\frac{\omega}{\omega_H}\frac{eA_\perp}{mc^2}\approx 5{,}85\frac{\omega}{\omega_H}\frac{eA_\perp}{mc^2}. \tag{2.17}$$

3. STOCHASTICITY REGION

We study the boundary of the stochasticity appearance in (2.16) as follows.[15] To do this we must define the quantity

$$K=|d\psi_{n+1}/d\psi_n-1|={}^3/_2Qu_n^{1/2}|\cos\psi_n|.$$

When $K\gtrsim 1$ there arises a local instability leading to mixing in phase space. If we disregard the region of phases ψ in which chaos appears, the stochasticity condition determines the lower bound for the particle energy, starting from which the motion becomes chaotic and stochastic acceleration begins:

$$K_0=16\frac{eA_\perp}{mc^2}\left(\frac{\omega}{\omega_H}\right)^{4/3}\left(\frac{E}{mc^2}\right)^{1/3}>1. \tag{3.1}$$

The criterion (3.1) is, apart from a numerical coefficient of order unity on the left-hand side, the same as the criterion obtained in Ref.13 for the overlapping of the nonlinear resonances.

Inequality (3.1) determines the threshold value $E_{\rm thr}$ of the particle energy starting from which there appears stochastic acceleration:

$$E_{\rm thr}=\frac{mc^2}{16^3}\left(\frac{\omega_H}{\omega}\right)^4\left(\frac{mc^2}{eA_\perp}\right)^3. \tag{3.2}$$

It follows from this expression that the threshold energy increases with increasing external magnetic field strength like $E_{\rm thr}\propto B_0^4$. Most important, however, is the fact that for any, however small, value of the field strength of the electromagnetic wave there exists a threshold value for the particle energy above which the dynamics is chaotic and the particles have a chance of increasing their energy without limit.

We consider some properties of the mapping (2.16). We give in Fig. 1 the numerical results of the mapping (2.16) for $Q=0.5$ after 10^5 iterations for one trajectory. The initial condition was chosen in the chaos region. The stability islands not occupied by trajectories are stationary elliptical points. The coordinates of the stationary points of a single period $(u^{(0)},\psi^{(0)})$ are determined by the solutions of the following equations:

$$Q\sin\psi^{(0)}=0,\quad |u^{(0)}|^{3/2}=2\pi N,\quad N-\text{integer}. \tag{3.3}$$

Hence it follows that the phase $\psi^{(0)}=0$ or π, and $u^{(0)}=(2\pi N)^{2/3}$. The last equation is equivalent to the condition for cyclotron resonance

$$\omega=N\omega_H mc^2/E. \tag{3.4}$$

We study the stability of the stationary points as follows. We construct in the neighborhood of $\psi^{(0)}$ and $u^{(0)}$ the tangent mapping

$$\begin{bmatrix}\Delta u_{n+1}\\ \Delta\psi_{n+1}\end{bmatrix}=\hat{A}\begin{bmatrix}\Delta u_n\\ \Delta\psi_n\end{bmatrix}, \tag{3.5}$$

where the matrix $\hat{A}$ has the form

$$\hat{A}=\begin{bmatrix}1 & Q\cos\psi^{(0)}\\ (3/2)\,|u^{(0)}|^{1/2} & 1+(3/2)\,|u^{(0)}|^{1/2}Q\cos\psi^{(0)}\end{bmatrix}. \tag{3.6}$$

Solving the equation for the eigenvalues λ of the matrix $\hat{A}$:

$$\lambda^2-\lambda\,\mathrm{Sp}\,\hat{A}+1=0, \tag{3.7}$$

we find the stability condition for the stationary points of a single period

$$|2+{}^3/_2|u^{(0)}|^{1/2}Q\cos\psi^{(0)}|<2. \tag{3.8}$$

Hence it follows that if $Q>0$ the motion in the vicinity of the cyclotron resonances is at $\psi^{(0)}=0$ always unstable, and when $\psi^{(0)}=\pi$ stability occurs in the region

$$|u^{(0)}|<64Q^{-2}/9. \tag{3.9}$$

If inequality (3.9) is violated the motion in the vicinity of the cyclotron resonance becomes unstable. The instability consists in that the elliptical point becomes hyperbolic. Two new elliptical points of the doubled period are then generated. This is the ordinary island-doubling bifurcation. When the particle energy grows further there occur successive island-doubling bifurcations which are typical of Hamiltonian sys-

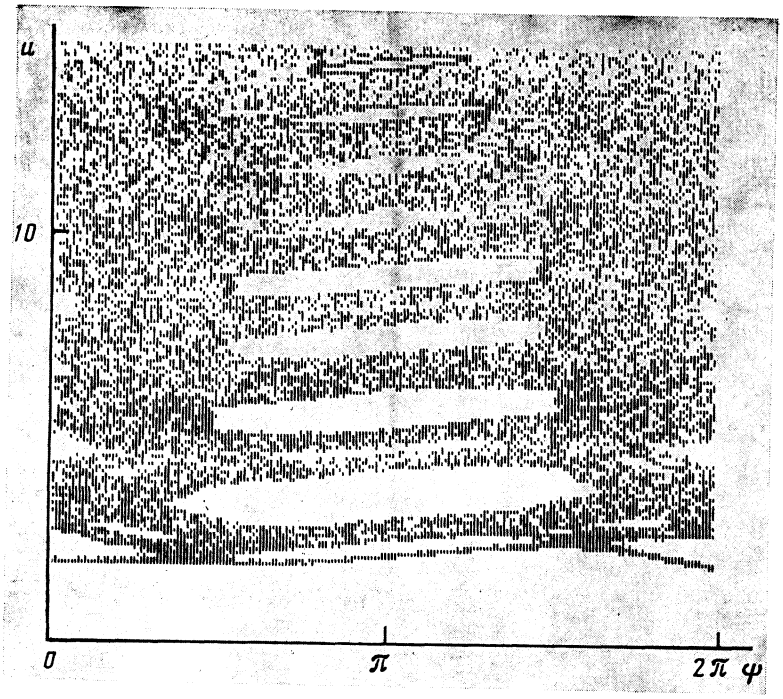

FIG. 1. The (u,ψ) phase plane of the mapping (2.16) for $Q=0.5$.

tems. When $K_0 = K_0^{(n)}$ the elliptical points with periods $\sim 2^n$ lose their stability and there appears a cycle with period $\sim 2^{n+1}$. The sequence of bifurcation values $K_0^{(n)}$ converges rapidly for sufficiently large n to a limit $K_0^{(\infty)}$, according to a geometric progression law, with exponent $\delta \approx 8.72$.[16,17] In the range of parameter values K_0 between two values corresponding to the sequence of doubling bifurcations in the vicinity of the elliptical points there appear and are split off necklaces of islands corresponding to higher-order resonances.

The structure of the phase space becomes thus particularly complicated in the region $1 < K_0 < K_0^{(\infty)}$ where the fraction of stable components of the motion plays an important role. Stochastic trajectories form in this region an extraordinarily complicated structured set, the so-called fat fractals.[18] The particle trajectories can stick for a long time in the region of relatively low energies and this is in turn strongly reflected in the particle diffusion which is considerably diminished.

When K_0 increases, i.e., in the region of large particle energy values, the measure of the regular component tends to zero and the motion becomes completely chaotic. For the sake of simplicity we further consider the case

$$K_0 = {}^3/_2 Q u^{1/2} \gg 1. \tag{3.10}$$

For the phase correlations we can give the following estimate[15]:

$$R(t) = \frac{1}{2\pi} \int_0^{2\pi} d\psi(0) \exp\{i[\psi(t) - \psi(0)]\} \sim \exp\left(-\frac{t}{\tau_c}\right), \tag{3.11}$$

where the time for the decoupling of correlations is

$$\tau_c = 2\Delta t / \ln K_0. \tag{3.12}$$

Here Δt is the time between two consecutive steps of the mapping, and is equal to the cyclotron period. Substituting Eq. (2.15) for Δt into (3.12) we get

$$\tau_c = 4\pi E / \omega_H mc^2 \ln K_0. \tag{3.13}$$

The time τ_c for the decoupling of correlations or the time for the loss of memory of the original conditions satisfies, according to (3.10), the inequality

$$\Omega \tau_c \ll 1. \tag{3.14}$$

4. RESULTS OF NUMERICAL CALCULATIONS

The mapping (2.16) which we obtained and studied in preceding sections of the present paper describes the dynamics of ultrarelativistic charged particles with energies $E \gg mc^2$ in the field of an electromagnetic wave propagating at right angles to an external magnetic field with a phase velocity equal to the velocity of light. In the more general formulation it is necessary to turn to a numerical analysis of the problem. We studied the set of Eqs. (2.6) with the Hamiltonian (2.5) numerically for various values of the parameters $\varepsilon = eA_\perp / mc^2$, $\nu = \omega/\omega_H$, and $\delta = 1 - kc/\omega$. Figures 2, a–c illustrate the results of a numerical analysis of the set of Eqs. (2.6). The points of the trajectory of the particle motion in the phase plane are taken at times $t_p = 2\pi p/\omega$ ($p = 1, 2, \ldots$) and the Poincaré mapping is thus constructed. The results of the numerical calculations confirm the conclusion [obtained from an analysis of the mapping (2.16)] that the dynamics of ultrarelativistic particles is chaotic and at $\omega = kc$ there occurs an unlimited stochastic acceleration of relativistic particles. At the same time, regions of regular motion exist for particles of sufficiently low energies $E \approx mc^2$ moving in the field of an electromagnetic wave with a relatively small field strength amplitude $\varepsilon \ll 1$.

The dynamics of low-energy particles becomes particularly interesting under conditions where there is resonance between the frequency of the electromagnetic wave and the nonrelativistic cyclotron frequency $\omega = n\omega_H$. In the phase portrait (Figs. 2, a–c) there appears then in the low-energy region a system of relatively large islands of stability, which possesses an nth order orientational symmetry. In Fig. 2a which corresponds to the case $\nu = 2$ one can see also stability islands corresponding to third- and fourth-order cyclotron resonances; we discussed in the preceding section in detail the stability of the motion in their neighborhood. In the same figure one can discern the smaller stability islands corresponding to second-order resonances. One observes a similar picture also for $\nu = 3$ (Fig. 2b). Here the system of stability islands in the low-energy region corresponds to third-order cyclotron resonance. In the region of larger energies there is also a system of rather large-scale islands corresponding to fourth-order resonances. The finer-scaled islands correspond to second-order resonances. Stochastic layers are formed in the intervals between the stability islands. For small values of the parameter ε the stochastic layers surrounding the cyclotron resonances are separated from one another by invariant curves. When the parameter K increases the invariant curves which exist between the layers vanish and something like a stochastic web appears (Fig. 2c). This web, however, disintegrates rapidly because of strong nonlinearity of the motion of relativistic particles in a magnetic field.

Such a picture of the particle motion, the existence of stability islands and of invariant curves in the region of low energy values, leads for $\varepsilon \ll 1$ to the fact that for particles which initially had rather low energies the adiabatic invariance is always conserved and the channel for unlimited stochastic acceleration remains closed. Only those particles can be accelerated which initially have a rather high energy. The situation is, however, changed when the amplitude of the field strength of the wave ε increases. We show in Fig. 2d the phase portrait of the set of Eqs. (2.6) for $\nu = 3$ and $\varepsilon = 0.3$. For such a value of the parameter ε the regular component of the motion practically vanishes and, independently of the initial conditions, almost all particles are in a regime of unlimited stochastic acceleration.

5. STOCHASTIC ACCELERATION

The diffusion of particles in the phase plane when the stochasticity condition (3.10) is satisfied can as usually be described by a Fokker-Planck-Kolmogorov equation. To derive the diffusion equation we follow a general prescription described in Ref. 15 and change in the Hamiltonian (2.5) to the action-angle variables (J, θ) of the unperturbed problem corresponding to the free rotation of a relativistic particle in a magnetic field:

$$x = \rho \sin\theta, \quad p = \rho m \omega_H \cos\theta, \tag{5.1}$$

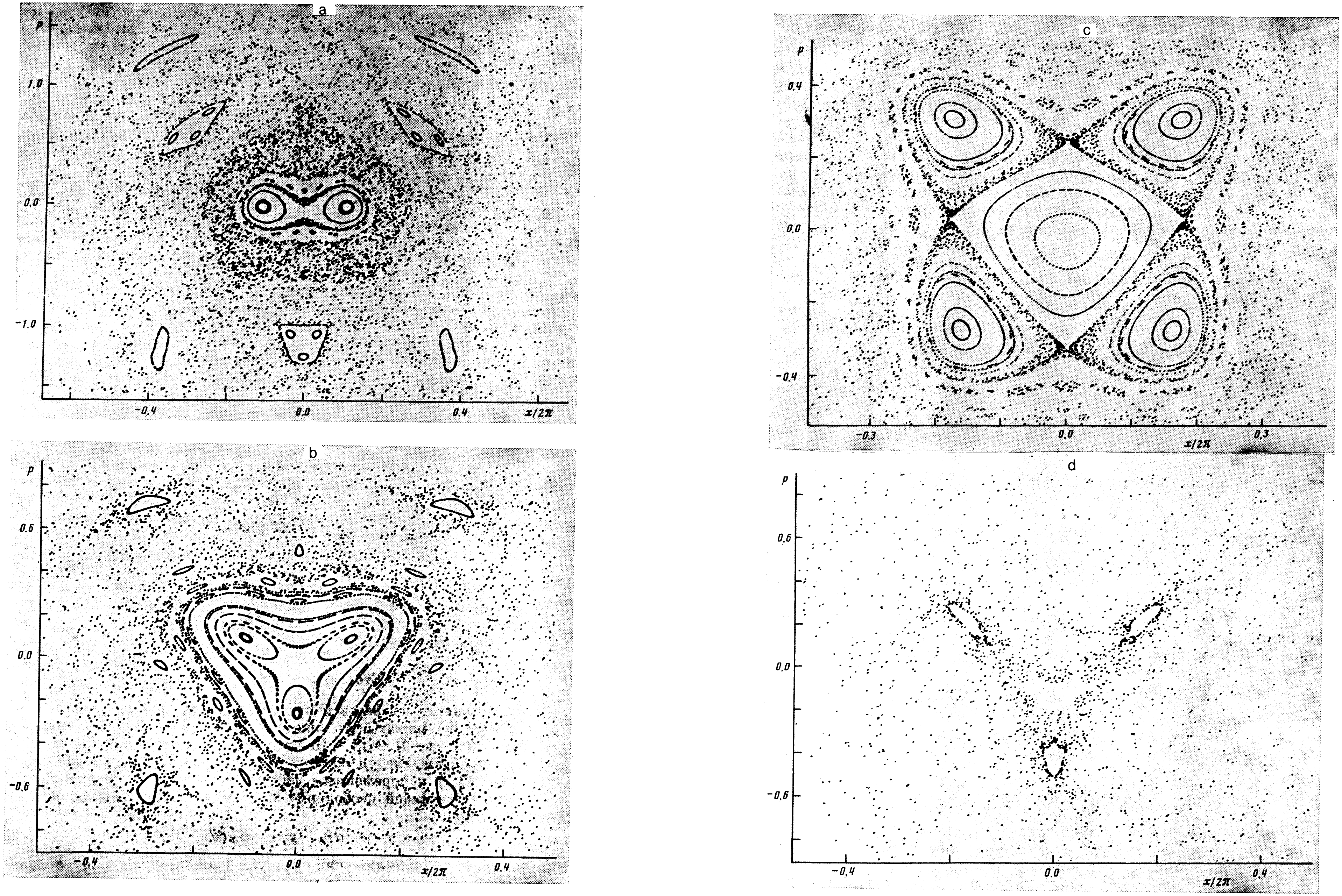

FIG. 2. The (p,x) Poincaré section for the set of Eqs. (2.6) for $\delta = 0$. a: $\varepsilon = 0.1$, $\nu = 2$; b: $\varepsilon = 0.1$, $\nu = 3$; c: $\varepsilon = 0.1$, $\nu = 3.77$; d: $\varepsilon = 0.3$, $\nu = 3$.

where $\rho = (2cJ/eB_0)^{1/2}$ is the Larmor radius. The Hamiltonian (2.5) has in the new variables the following form:

$$H(J,\theta,t)=[H_0^2+2e^2B_0A_\perp\rho\sin\theta\sin(k\rho\sin\theta-\omega t)-(e^2A_\perp^2/2)\cos(2k\rho\sin\theta-2\omega t)]^{1/2}, \tag{5.2}$$

where

$$H_0(J)=[m^2c^4+e^2A_\perp^2/2+2ecB_0J]^{1/2} \tag{5.3}$$

is the Hamiltonian of the unperturbed problem. For particles having a sufficiently large energy $E \gg eA_\perp$ Eq. (5.2) can be somewhat simplified and can be written as follows:

$$H(J,\theta,t)=H_0-e^2B_0A_\perp\rho H_0^{-1}\sum_{n=-\infty}^{+\infty}J_n'(k\rho)\cos(n\theta-\omega t), \tag{5.4}$$

where $J_n'(k\rho)$ is the derivative of the Bessel function with respect to its argument. Under condition (3.10) the particle motion is chaotic and characterized by fast phase mixing over a time τ_c and a slower diffusion with respect to the action. If we take the finite correlation decoupling time into account in the spirit of Ref. 15, the diffusion equation has the form

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial J}\left[D(J)\frac{\partial f}{\partial J}\right], \tag{5.5}$$

where $D(J)$ is the diffusion coefficient:

$$D(J)=(e^2B_0A_\perp\rho/2H_0)^2\sum_{m=-\infty}^{+\infty}m^2J_m'^2(k\rho)\int_0^\infty d\tau \times\exp\{i(m\Omega-\omega)\tau-\tau/\tau_c\}+\text{c.c.}, \tag{5.6}$$

and $\Omega=\partial H_0/\partial J$ the nonlinear frequency of the unperturbed motion. Performing the integration over time on the right-hand side of (5.6) we get

$$D(J)=(e^2B_0A_\perp\rho/2H_0)^2(i/\Omega)\times\sum_{m=-\infty}^{+\infty}\left\{\left(m-\frac{\omega}{\Omega}+\frac{i}{\Omega\tau_c}\right)^{-1}-\left(m-\frac{\omega}{\Omega}-\frac{i}{\Omega\tau_c}\right)^{-1}\right\}m^2J_m'^2(k\rho). \tag{5.7}$$

We can sum the series on the right-hand side of (5.7) using the identity

$$\sum_{m=-\infty}^{+\infty}\frac{J_{m+p}(a)J_m(a)}{m-q}=-\frac{\pi}{\sin\pi q}J_{p+q}(a)J_{-q}(a). \tag{5.8}$$

As a result we obtain the following expression for the diffusion coefficient

$$D(J)=i(e^2B_0A_\perp\rho/2H_0)^2\Omega^{-1}[d(a)-d(a^*)], \tag{5.9}$$

where

$$a=\frac{\omega}{\Omega}-\frac{i}{\Omega\tau_c},$$

$$d(a)=\frac{a}{2}-\frac{a^3}{(k\rho)^2}-\frac{\pi a^2}{\sin\pi a}J_a'(k\rho)J_{-a}'(k\rho). \tag{5.10}$$

In the limit $E \gg mc^2$ Eqs. (5.9) and (5.10) lead to the following expression for the diffusion coefficient:

$$D(J)=\frac{\pi}{2}\left(\frac{e^2B_0A_\perp\rho}{H_0}\right)^2\frac{\omega^2}{\Omega^3}J_{\omega/\Omega}'^2(k\rho)\times\frac{\text{sh}(2\pi/\Omega\tau_c)}{\text{ch}(2\pi/\Omega\tau_c)-\cos(2\pi\omega/\Omega)}. \tag{5.11}$$

The last factor on the right-hand side of (5.11) takes into account the slowing down of the diffusion due to the finite correlation decoupling time when one approaches the boundary of the stochasticity region, where $\Omega\tau_c\to\infty$. According to Eq. (3.13)

$$\Omega\tau_c=2/\ln K_0 \tag{5.12}$$

and in the region of global chaos, where $K_0 \gg 1$ and there are no stability islands, $\Omega\tau_c\to 0$. The factor which takes the finite correlation decoupling into account is in this region equal to unity so that we finally have for the diffusion coefficient the usual quasilinear expression

$$D(J)=\frac{\pi}{2}\left(\frac{e^2B_0A_\perp\rho}{H_0}\right)^2\frac{\omega^2}{\Omega^3}J_{\omega/\Omega}'^2(k\rho). \tag{5.13}$$

In the case where the phase velocity of the electromagnetic wave equals the velocity of light $\omega = kc$ Eq. (5.13) simplifies in the high-energy region $E \gg mc^2$ and the kinetic Eq. (5.5) takes the form

$$\frac{\partial f}{\partial t}=\nu_0\frac{\partial}{\partial J}\left[J^{5/6}\frac{\partial f}{\partial J}\right], \tag{5.14}$$

where $\nu_0=\zeta A_\perp^2\omega^{2/3}e^{7/6}c^{-5/6}B_0^{-5/6}$ and the numerical coefficient is

$$\zeta=\pi 2^{7/6}3^{-2/3}/\Gamma^2(1/3)\approx 0.472.$$

Hence it follows that for large t

$$\langle J\rangle\propto t^{6/7}, \tag{5.15}$$

and by virtue of (5.3) the average energy $\langle E\rangle$ grows with time as

$$\langle E\rangle\propto t^{3/7}. \tag{5.16}$$

In comparison with the known regular acceleration mechanisms, such as the surfatron,[8,9] when the particle energy grows linearly with time, $E\propto t$, the above considered stochastic mechanism leads to a slower energy growth for the accelerated particles. The stochastic acceleration mechanism also loses out against the self-resonance acceleration method when a circularly polarized wave propagating along the magnetic field with frequency $\omega = kc = \omega_H$ accelerates particles according to the law $E\propto t^{2/3}$. However, the conditions for the realization of the stochastic acceleration method are considerably less rigid. Unlimited stochastic acceleration takes place not only of resonance particles, but also of any particle with energy exceeding the threshold (3.2).

The solution of Eq. (5.14) with the initial condition $f(t=0)=f_0(J)$ has the form

$$f(J,t)=\frac{6J^{1/12}}{7\nu_0 t}\exp\left\{-\frac{36}{49}\frac{J^{7/6}}{\nu_0 t}\right\}\int_0^\infty dJ'\,J'^{1/12}\exp\left\{-\frac{36}{49}\frac{J'^{7/6}}{\nu_0 t}\right\}\times I_{-1/7}\left[\frac{72}{49}\frac{(JJ')^{7/12}}{\nu_0 t}\right]f_0(J'). \tag{5.17}$$

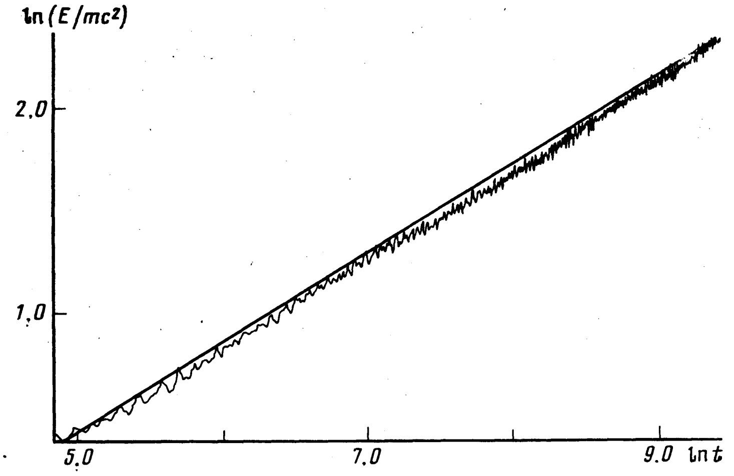

FIG. 3. Curve of the time-dependence of the average particle energy in a doubly logarithmic plot for $\varepsilon = 0.3$, $\nu = 2$, $\delta = 0$.

For large values of t the solution (5.17) reaches a self-similar form, which has forgotten the initial conditions:

$$f(J,t) = \frac{(^6/_7)^{5/7}}{\Gamma(^6/_7)} \frac{N}{(\nu_0 t)^{6/7}} \exp\left\{-\frac{36}{49} \frac{J^{7/6}}{\nu_0 t}\right\}, \qquad (5.18)$$

where $N = \int f\, dJ$ is the particle density. The particle energy distribution function has also a self-similar form:

$$f(E, t) \sim Et^{-6/7} \exp\{-\text{const}\, E^{7/3}/t\}. \qquad (5.19)$$

We carried out a numerical analysis of the set of Eqs. (2.6) to check the stochastic acceleration law (5.16). The calculation was performed for 800 trajectories with different initial conditions in the chaos region for $\omega = \omega_H = \nu = 2$ and different values of the parameters $\varepsilon = eA_\perp/mc^2$ and $\delta = 1 - kc/\omega$. The calculation time corresponded to 2×10^3 periods of the electromagnetic wave. We show in Fig. 3 the time-dependence of the average particle energy on a doubly logarithmic plot for $\varepsilon = 0.3$ and $\delta = 0$. The straight line corresponds to the stochastic acceleration law (5.16). An analysis of the numerical results showed that the deviations of the stochastic acceleration law from the self-similar law (5.16) for $\delta = 0$ and large times $\omega t/2\pi > 10^3$ are always small.

In the case when the phase velocity of the electromagnetic wave exceeds the velocity of light *in vacuo*, $\omega > kc$, the situation becomes different. We show in Fig. 4 the curves of the time-dependence of the average particle energy for $\nu = 2$ and $\varepsilon = 4$. Curve 1 corresponds to the case $\delta = 0$, and curve 2 to the case $\delta = 0.045$. The smooth curve corresponds to the stochastic heating curve (5.16). It is clear from the figure that starting from some time the stochastic acceleration ceases in the case when the phase velocity of the wave exceeds the velocity of light. The stochastic acceleration is limited by the exponential cutoff of the diffusion coefficient at high energies. Using the asymptotic form of the derivative of the Bessel function in (5.13) for large values of the argument and the index, we find the maximum value $E_{\max}$ of the energy up to which particles are accelerated:

$$E_{max} = mc^2 (\omega_H/\omega)\, \delta^{-3/2}, \qquad (5.20)$$

where $\delta = 1 - kc/\omega \ll 1$. The restriction on the magnitude of the stochastic acceleration is connected with the fact that in the case $\delta \neq 0$ for particles with energies exceeding the maximum (5.20) the regions of steep changes in the adiabatic invariant disappear and the particle moves in a fast oscillating field during the whole of the cyclotron period.

Yet another restriction on the magnitude of the maximum energy which particles can acquire in the stochastic acceleration process is connected with energy losses through synchrotron radiation. The energy ΔE_c emitted during a cyclotron period of a relativistic particle rotating in a magnetic field is given by the following expression:

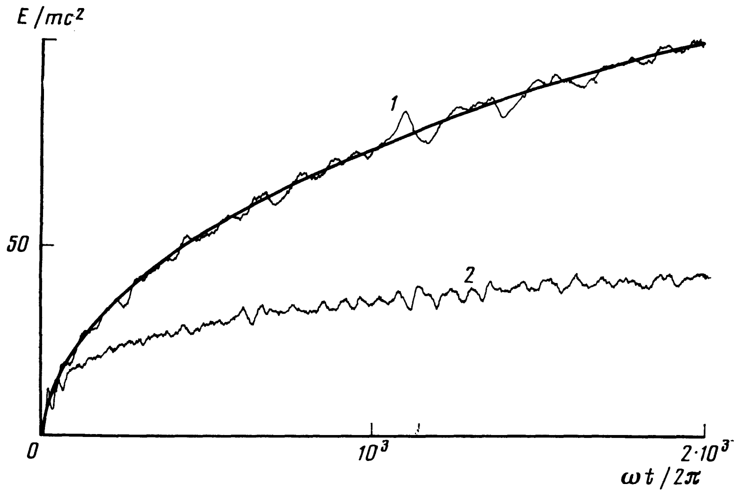

FIG. 4. Time-dependence of the average particle energy for $\varepsilon = 4.0$ and $\nu = 2$. The curves 1 and 2 correspond to $\delta = 0$ and 0.045.

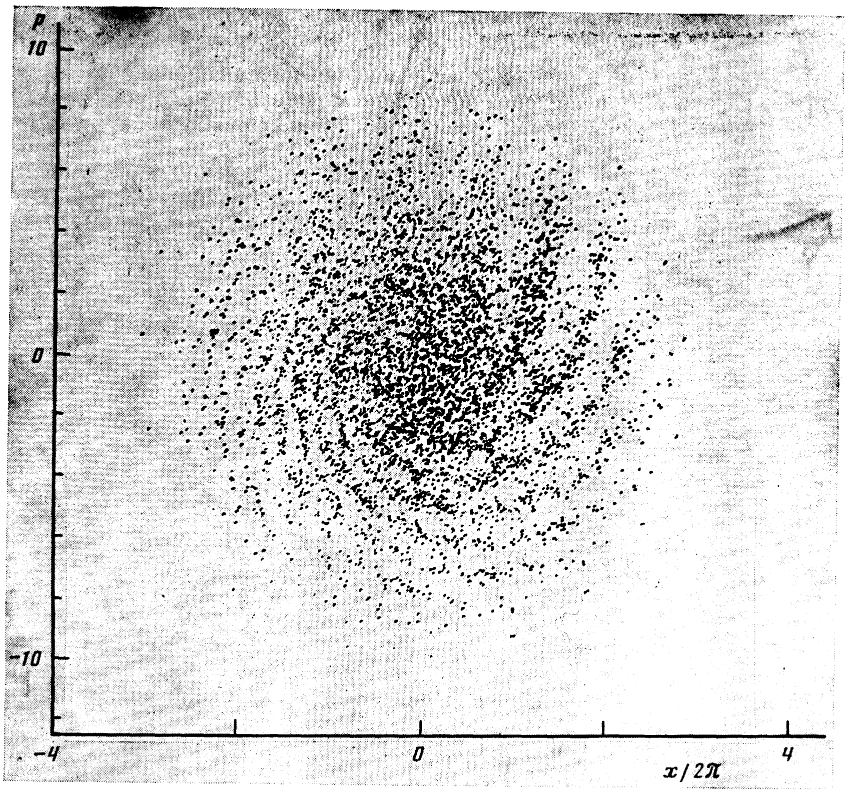

FIG. 5. Formation of a spiral structure in the (p,x) phase plane for $\varepsilon = 0.3$, $\nu = 2$, and $\delta = 0$ at the time $\omega t/2\pi = 100$.

$$\Delta E_c = (4\pi/3)\,(e^2\omega_H/c)\,(E/mc^2)^3. \tag{5.21}$$

The losses through synchrotron radiation start to play a significant role if the change of energy (5.21) becomes comparable with the magnitude of the energy change (2.14) due to the interaction of the particle with the wave. Equating these expressions we find the maximum value of the particle energy up to which stochastic acceleration is possible:

$$E_{max} = mc^2 (eA_\perp/mc^2)^{3/8} (\omega/\omega_H)^{1/8} (mc^3/e^2\omega_H)^{3/8}. \tag{5.22}$$

We note yet another interesting feature of the stochastic acceleration of relativistic particles, which manifests itself in the formation of spiral structures in the phase plane (see Fig. 5). Such spiral structures were observed in a numerical experiment within relatively short calculation times $\omega t/2\pi < 10^3$. For longer times the structures are, in general, washed out and the distribution function is smoothed out. The formation of a spiral structure is connected with the choice of the initial particle distribution function in phase. As the initial condition in the numerical calculations we chose a function which was evenly distributed along a wavelength. Therefore, for part of the particles the phase relations were most favorable for acceleration. As time goes on there is a phase mixing and the nonuniformity of the acceleration disappears.

In conclusion we note that one of the most important physical consequences of the acceleration consists in the following. In the self-consistent problem the increase in particle energy occurs at the expense of the waves losing that energy. We are thus led to the existence of a universal mechanism for the damping of an electromagnetic wave in a magnetic field.

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