The superhydrodynamics of ^{3}He-*A*

A. F. Andreev and M. Yu. Kagan

Institute for Physical Problems of the USSR Academy of Sciences

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Superfluid ^{3}He-*A* at zero temperature is described by superhydrodynamics, which, just as ordinary hydrodynamics, is uniquely derived from considerations of symmetry, but which in addition to ordinary variables contains anticommuting variables. The range of applicability of superhydrodynamics is determined and it is shown that the elimination of the anticommuting variables leads to the appearance of nonlocal (logarithmic) terms.

The dynamical (orbital) properties of ^{3}He-*A* at a temperature tending to zero are not described by hydrodynamics. This is due to the specific symmetry properties of the *A*-phase, one of the principal manifestations of which is the presence of zeroes of the energy gap at two diametrically opposed poles of the Fermi sphere. In fact, for this reason, the orbital dynamics of ^{3}He-*A* at $T = 0$ has been studied on the basis of microscopic models (Refs. 1–3).

It shown in the present paper that the dynamics of the ground state of ^{3}He-*A* can be described by superhydrodynamics, which, just like ordinary hydrodynamics, is derived by means of an expansion in terms of gradients solely on the basis of symmetry considerations, but which contains in addition to ordinary (bosonic) variables also anticommuting Grassmann (fermionic) variables. It is the presence of "fermionic goldstones" that is most characteristic for a system with the symmetry properties of ^{3}He-*A*. A self-consistent hydrodynamic description must of necessity contain all Goldstone degrees of freedom (including fermionic ones), since the elimination of part of these variables from the system of equations leads, as a rule, to nonhydrodynamic expressions, which cannot be expanded in terms of gradients. We note that the possibility that fermionic goldstones exist in principle had been previously discussed[4] in the framework of quantum field theory.

^{3}He-*A* has another specific property, viz., the presence of an anomalous term in the expression of the mass flow.[5–8] As was made clear in the papers of Volovik and co-workers (Refs. 1, 2), the main contribution to the anomalous current comes from portions of the momentum space which are remote from the zeroes of the energy gap. Therefore the anomalous current cannot even be encompassed by superhydrodynamics. It is, however, important to stress the fact that the existence of an anomalous current for arbitrarily large wavelengths of the motion is a feature of the simplified model used in the calculations of Refs. 1–3. In reality there exists a region of sufficiently large wavelengths where the anomalous current is small and superhydrodynamics is applicable. In this connection one cannot fail to note that the corresponding inequality to which the wavelength are subject is quite stringent.

In calculations of the anomalous current use has been made of a model which does not take account of the damping of Fermi quasiparticles which is specific for ^{3}He-*A*. The reason for this is that owing to the anisotropy of the *A*-phase a finite damping of all quasiparticles, except those situated near the zeroes of the gap, occurs even for zero temperature. There are several damping mechanisms—the decay of one Fermi excitation into three and the emission of orbital wave quanta by a Fermi excitation. In order to estimate the lifetime τ of quasiparticles with an energy close to the maximal gap, on account of decay into three excitations, it is sufficient to substitute T_c for the energy counted from the Fermi level in the known formula for quasiparticle damping. We obtain $\hbar/\tau \sim \varepsilon_F (T_c/\varepsilon_F)^2$. The corresponding mean free path is of the order of $l \sim a(\varepsilon_F/T_c)^2$. We note that this process becomes ineffective if the quasiparticle energy is close to the minimal value for a prescribed $P_\perp$ ($P_\perp$ is the projection of the quasiparticle momentum on a plane perpendicular to the anisotropy axis). The suppression is due to the reduction of the statistical weight of the final states which are compatible with the energy and momentum conservation laws. It is quite important that even in this case the mean free path l of the excitations is finite, on account of emission processes of orbital wave quanta. Indeed, quasiparticles of even minimal energy will have a finite speed on account of the anisotropy of the gap, and therefore can emit orbital quanta of sufficiently low energy which have a quadratic dispersion law. An arbitrarily small damping is exhibited only by quasiparticles situated sufficiently close to the zeroes of the energy gap.

The anomalous current arises (see Refs. 1–3) as a result of a peculiar quantization of the quasiparticle levels under the influence of their interaction with the gradients of the order parameter. A quantization really exists if the appropriate gradient interaction energy (which has the order of magnitude $T_c(a/\lambda)^{1/2}$, where a is the interatomic distance, λ is the wavelength, see Refs. 1–3) substantially exceeds the energy indeterminancy $\hbar/\tau$. This yields $\lambda \ll 1$. The results of the quoted papers (Refs. 1–3) thus refer in fact to the intermediate asymptotic region $\xi \ll \lambda \ll l$, where $\xi \sim \hbar v_F/T_c$ is the coherence length, since $l = \infty$ in the model used. If the opposite inequality $\lambda \gg l$ holds, the quasiparticle damping plays a dominant role and the anomalous term in the current is small. This is just the region where superhydrodynamics is applicable. As already noted the characteristic length parameter, $l \sim \xi(\varepsilon_F/T_c) \sim 1$ mm, is quite large.

1. THE BOSE PART OF THE ACTION

Since the known Lagrangian formulations for the Bose part of the problem (Refs. 8, 9) contain redundant parameters and supplementary conditions, we expose a method free of this drawback.

The variable determining the local-equilibrium state of ^{3}He-*A* at $T = 0$ are: the density $\rho(\mathbf{r},t)$ and the three angles

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$\varphi^a(\mathbf{r},t)$ defining the orientation of the trio of mutually orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{l} = \mathbf{e}_1 \times \mathbf{e}_2$. As order parameter we can choose then the complex vector $\boldsymbol{\Delta} = \mathbf{e}_1 + i\mathbf{e}_2$.

Let

$$\delta\theta = \lambda_a(\varphi)\, d\varphi^a$$

be the Cartan form defining the infinitesimal rotation $\delta\theta$ corresponding to two neighboring units φ^a and $\varphi^a + d\varphi^a$ in the group space $\{\varphi^a\}$ of the three-dimensional rotation group. Setting $\{\varphi^a\} = \boldsymbol{\varphi}$ where the vector $\boldsymbol{\varphi}$ is directed along the rotation axis and its magnitude is $\tan(\theta/2)$, where θ is the rotation angle, then we have (Ref. 10)

$$\lambda_{ai} = \frac{2}{1+\varphi^2}(\delta_{ai} + \varepsilon_{iba}\varphi^b). \tag{1}$$

Here ε_{iba} is the Levi-Civita tensor.

As is easily verified, the quantities (1) satisfy the "flatness" conditions

$$\frac{\partial\lambda_a}{\partial\varphi^b} - \frac{\partial\lambda_b}{\partial\varphi^a} + [\lambda_a\lambda_b] = 0. \tag{2}$$

Since the conditions (2) are covariant with respect to a change of the coordinates φ^a, they retain their form also for any other parametrization of the rotations.

The superfluid velocity $\mathbf{v}_s$ has the components

$$v_{si} = -\frac{\hbar}{2m}\mathbf{l}\frac{\delta\theta}{\delta x_i} = -\frac{\hbar}{2m}\mathbf{l}\lambda_a\frac{\partial\varphi^a}{\partial x_i}, \tag{3}$$

where m is the mass of the ^{3}He atom. The Mermin-Ho identity[11] for $\mathbf{v}_s$ is automatically fulfilled on account of the conditions (2).

Since the generator of the gauge transformation $\boldsymbol{\Delta} \to \boldsymbol{\Delta}e^{i\chi}$ is the quantity $-\hbar\rho/2m$, the derivative of the action density with respect to χ equals $-\hbar\rho/2m$. In order to meet this condition it is necessary that the density of the Lagrange function should have the following form:

$$L_B = \frac{\hbar\rho}{2m}\mathbf{l}\frac{\delta\theta}{dt} - E = \frac{\hbar\rho}{2m}\mathbf{l}\lambda_a\dot{\varphi}^a - E, \tag{4}$$

where E is to be interpreted as the energy density. The canonical momentum $\mathbf{j}$ of the unit volume equals

$$\mathbf{j} = -\nabla\varphi^a\frac{\partial L}{\partial\dot{\varphi}^a} = \rho\mathbf{v}_s. \tag{5}$$

On account of Galilean invariance we have

$$E = \frac{\rho v_s^2}{2} + E_0(\mathbf{l}, \partial_i\mathbf{l}, \rho). \tag{6}$$

Variation of the Lagrange function with respect to $\delta\rho$ and $\delta\varphi^a$, making use of the conditions (2), yields the hydrodynamic equations for the purely bosonic subsystem:

$$\dot{\rho} + \operatorname{div}\mathbf{j} = 0, \quad \dot{\mathbf{l}} + (\mathbf{v}_s\nabla)\mathbf{l} + \frac{2m}{\hbar\rho}\left[\mathbf{l}, \frac{\partial E_0}{\partial\mathbf{l}} - \partial_i\frac{\partial E_0}{\partial\partial_i\mathbf{l}}\right] = 0,$$

$$-\frac{\hbar}{2m}\mathbf{l}\frac{\delta\theta}{dt} + \frac{\partial E_0}{\partial\rho} + \frac{v_s^2}{2} = 0. \tag{7}$$

The last of the equations (7) is nonhydrodynamic, since it describes rapid rotation around the vector $\mathbf{l}$ with a frequency equal to the chemical potential. This formal deficiency is easily removed as follows. It is known (Ref. 12) that the dynamics of any nonrelativistic particle system can be made invariant to a time-dependent gauge transformation (for which $\boldsymbol{\Delta} \to \boldsymbol{\Delta}e^{i\chi(t)}$) by adding to the energy density the term $-(\hbar\rho/2m)A_0(t)$, where $A_0(t)$ is a "scalar potential" subject to the gauge transformation $A_0 \to A_0 - \dot{\chi}$. The corresponding gauge invariant form of the last of the equations (7) is the following:

$$-\frac{\hbar}{2m}\left\{\mathbf{l}\frac{\delta\theta}{dt} + A_0(t)\right\} + \frac{\partial E_0}{\partial\rho} + \frac{v_s^2}{2} = 0. \tag{8}$$

By choosing the arbitrary function $A_0(t)$ one can "stop" the rotation at an arbitrary point in space. In view of the arbitrariness of $A_0(t)$ the physical content of Eq. (8) is the same as that obtained from Eq. (8) by taking the gradient

$$\dot{v}_{si} + \partial_i\left(\frac{\partial E_0}{\partial\rho} + \frac{v_s^2}{2}\right) = \frac{\hbar}{2m}\partial_i\mathbf{l}\cdot[\mathbf{l}\times\dot{\mathbf{l}}]. \tag{9}$$

This equation has a completely hydrodynamic form.

2. THE FERMI PART OF THE ACTION

A characteristic peculiarity of ^{3}He-A is the fact that in addition to the usual ground-state degeneracy related to spontaneous breaking of continuous symmetries, there exists an additional degeneracy related to the vanishing of the quasiparticle energy at two points of the Fermi surface. It turns out that the states

$$a_1^+|0\rangle, \quad a_2^+|0\rangle,$$

(where $|0\rangle$ is the quasiparticle vacuum, and a_1^+ and a_2^+ are the creation operators for quasiparticles with momenta $p_F\mathbf{l}$ and $-p_F\mathbf{l}$, respectively) have ground state energy. Similarly to the way in which the usual degeneracy leads in the hydrodynamic description to the appearance of Bose fields which vary slowly in space and time, this additional degeneracy leads to the appearance of "Fermi goldstones," i.e., slowly varying anticommuting fields $a_1(x)$, $a_2(x)$, $a_1^*(x)$, $a_2^*(x)$, where $x = (\mathbf{r},t)$. In fact, it is more convenient in this case to make use of certain linear combinations of these fields. The reason for this is that in systems with Cooper pairing the quantities $a_1, \ldots$ are subject to complicated gauge transformation laws. We introduce their linear combinations $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_1^*(x)$, $\varphi_2^*(x)$, so that they satisfy the same anticommutation relations

$$\{\varphi_1^*, \varphi_1\} = \{\varphi_1, \varphi_2\} = \{\varphi_2^*, \varphi_1\} = \ldots = 0$$

as before, but under the gauge transformations they transform as

$$\varphi_{1,2} \to \varphi_{1,2}e^{i\chi/2}, \qquad \dot{\varphi}_{1,2} \to \dot{\varphi}_{1,2}e^{-i\chi/2}.$$

We note that on account of the known properties of the mentioned linear (Bogolyubov) transformations for spatially homogeneous systems, the subscripts 1, 2 refer, as before, to states with momenta $p_F\mathbf{l}$ and $-p_F\mathbf{l}$.

The presence of the additional degeneracy of the ground state and the related Goldstone character of the fields φ is due to the symmetry properties of the A-phase. We shall convince ourselves below of this independently, by determining the general form of the fermionic part of a Lagrangian satisfying all the necessary symmetry requirements.

We call attention to the following important circumstance. Hydrodynamics deals with slowly varying quantities corresponding to low statistical volume near certain points of momentum space. For fermionic variables (in contradistinction to bosonic ones) this leads automatically to a small spatial fermion density. In the Lagrangian we can therefore limit ourselves to the consideration of terms which are quadratic in φ, φ^*.

The Lagrangian L_F of the Fermi subsystem, which together with L_B forms the total Lagrangian of superhydrodynamics, must be hermitean, invariant to rotations and gauge transformations, as well as with respect to the reflections $z\to -z$, $t\to t$, where the z axis is directed along the vector **l**. Moreover, on account of momentum conservation, the Lagrangian L_F must contain the products $\varphi_1^*\varphi_1$ $\varphi_2^*\varphi_2$, $\varphi_1, \varphi_2, \ldots$ but not $\varphi_1^*\varphi_2$, $\varphi_2^*\varphi_1 \ldots$. The fields φ, φ^* behave as scalars under rotations. Under reflections they have the transformation properties:

$$\varphi_1{}^z=\varphi_2,\quad \varphi_2{}^z=\varphi_1,\quad \dot{\varphi}_{1,2}^{T}=\dot{\varphi}_{1,2},\quad (\dot{\varphi}_{1,2})^T=\varphi_{1,2}.$$

Here the superscripts z and T denote respectively the operations $z\to -z$ and $t\to -t$. The operation T is accompanied, as always, by a reversal of the order of factors.

There is a unique expression not containing derivatives and satisfying the enumerated requirements:

$$g(\varphi_1{}^*\varphi_1+\varphi_2{}^*\varphi_2).$$

The coefficient g appearing here is in reality a function of the magnitude of the momentum, a function that vanishes at some point $p=p_F$. One may assume that the term under discussion is absent from the Lagangian, since the equation $g(p_F)=0$ is in fact a definition of the excitation momentum p_F. The existence of a zero in the function $g(p)$ is that "topological" property of ^{3}He-A which, together with the vector character of the order parameter **Δ**, is responsible for the gapless nature of the fields φ.

There exists a unique hermitean invariant involving the time derivatives:

$$\frac{i}{2}(\varphi_1{}^*\dot{\varphi}_1+\varphi_2{}^*\dot{\varphi}_2-\dot{\varphi}_1{}^*\varphi_1-\dot{\varphi}_2{}^*\varphi_2).$$

There are two invariants which are linear in the spatial derivatives. One of them contains the vector **l**. Owing to the conditions $\mathbf{l}^z=\mathbf{l}$, $\mathbf{l}^T=-\mathbf{l}$ it has the form

$$i\mathbf{l}(\varphi_1{}^*\nabla\varphi_1-\varphi_2{}^*\nabla\varphi_2)-i\mathbf{l}(\nabla\varphi_1{}^*\varphi_1-\nabla\varphi_2{}^*\varphi_2).$$

The second invariant contains **Δ** and on account of the transformation properties $\mathbf{\Delta}^z=\mathbf{\Delta}$, $\mathbf{\Delta}^T=-\mathbf{\Delta}^*$ it equals

$$i\mathbf{\Delta}(\varphi_1{}^*\nabla\varphi_2{}^*+\varphi_2{}^*\nabla\varphi_1{}^*)-i\mathbf{\Delta}^*(\nabla\varphi_1\varphi_2+\nabla\varphi_2\varphi_1).$$

Thus,

$$L_F=\frac{i}{2}(\varphi_1{}^*\dot{\varphi}_1+\varphi_2{}^*\dot{\varphi}_2-\dot{\varphi}_1{}^*\varphi_1-\dot{\varphi}_2{}^*\varphi_2)$$
$$+i\frac{v_l}{2}\mathbf{l}(\varphi_1{}^*\nabla\varphi_1-\varphi_2{}^*\nabla\varphi_2-\nabla\varphi_1{}^*\varphi_1+\nabla\varphi_2{}^*\varphi_2)$$
$$+i\frac{v_t}{2}\{\mathbf{\Delta}(\varphi_1{}^*\nabla\varphi_2{}^*+\varphi_2{}^*\nabla\varphi_1{}^*)-\mathbf{\Delta}^*(\nabla\varphi_1\varphi_2+\nabla\varphi_2\varphi_1)\},\tag{10}$$

where v_l and v_t are functions of the density. In real ^{3}He-A the condition $v_t\sim(T_c/\varepsilon_F)v_l\ll v_l\sim v_F$ is satisfied.

The Lagrangian (10) refers to the spatially homogeneous case, when $\mathbf{l}=\text{const}$, $\mathbf{\Delta}=\text{const}$. To treat the spatially nonhomogeneous case it is necessary to note the following. Since the states 1 and 2 have a finite momentum $\pm p_F\mathbf{l}$, the "genuine" fields $\psi_{1,2}$ are related to the slowly varying fields $\varphi_{1,2}$ by the equations

$$\psi_1=\varphi_1 e^{ip_F\mathbf{lr}},\quad \psi_2=\varphi_2 e^{ip_F\mathbf{rl}}.\tag{11}$$

For curl $\mathbf{l}\neq 0$ transformations of the form (11) do not exist. It is necessary to make use of the fields ψ and expand not only in terms of gradients, but in terms of the combinations $\nabla\pm ip_F\mathbf{l}$. The corresponding Lagrangian is obtained from the expression in (10) by means of the substitution $\varphi\to\psi$ and $\nabla\to\nabla\pm ip_F\mathbf{l}$.

Up to total derivatives (which appear when the differentiations are transposed either completely to ψ_1 or completely to ψ_2^*) we have then

$$L_F=\psi_1{}^*\left(i\frac{\partial}{\partial t}+i\frac{v_l}{2}\{\mathbf{l},(\nabla-ip_F\mathbf{l})\}\right)\psi_1$$
$$+\psi_2\left(i\frac{\partial}{\partial t}-i\frac{v_l}{2}\{\mathbf{l},(\nabla-ip_F\mathbf{l})\}\right)\psi_2{}^*$$
$$+i\frac{v_t}{2}(\psi_1{}^*\{\mathbf{\Delta},(\nabla-ip_F\mathbf{l})\}\psi_2{}^*+\psi_2\{\mathbf{\Delta}^*,(\nabla-ip_F\mathbf{l})\}\psi_1,$$

where the curly brackets stand, as before, for anticommutators.

In addition, it is necessary to add to the Lagangian independent invariant terms which contain explicitly the spatial derivatives of **Δ**. The time derivatives may be omitted, since according to Eqs. (7) they are quadratic in the spatial derivatives.

The invariants which are linear in the spatial derivatives $\partial\Delta_i/\partial x_k$ and are at the same time of zero order in the derivatives of the fields ψ can be of two types. They may contain expressions obtained by means of contractions with δ_{ik} or ε_{ikl}, or terms of the form

$$\underbrace{\Delta_{l_1}^*\Delta_{l_2}^*\ldots}_{n}\underbrace{\Delta_{m_1}\Delta_{m_2}\ldots}_{n}\frac{\partial\Delta_i}{\partial x_k}$$

and their complex conjugates, or terms of the form

$$\underbrace{\Delta_{l_1}^*\Delta_{l_2}^*\ldots}_{n}\underbrace{\Delta_{m_1}\Delta_{m_2}\ldots}_{n-1}\frac{\partial\Delta_i}{\partial x_k}.$$

All terms of the first type are obviously genuine scalars. They are therefore invariant with respect to the transformation $z\to -z$ and consequently they must be multiplied by the combinations of the ψ-fields $\psi_1^*\psi_2^*+\psi_2^*\psi_1^*$ and $\psi_1\psi_2+\psi_2\psi_1$, which vanish on account of the anticommutation relations. There exist only three independent expressions of the second type which are invariant with respect to rotations. These are div **l**, **l**·curl **l** and $\mathbf{v}_s\cdot\mathbf{l}$. These are all pseudoscalars, and therefore must be multiplied by $\psi_1^*\psi_1-\psi_2^*\psi_2$. Only the last two are invariant with respect to the transformation $t\to -t$. Taking all this into account we have

$$L_F=\psi^*\Lambda\psi,\tag{12}$$

where

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2{}^*\end{pmatrix},\quad \psi^*=(\psi_1{}^*\psi_2),$$

$$\Lambda=i\frac{\partial}{\partial t}+\begin{pmatrix}iv_l\mathbf{l}(\nabla-ip_F\mathbf{l}) & iv_t\mathbf{\Delta}\nabla\\ iv_t\mathbf{\Delta}^*\nabla & -iv_l\mathbf{l}(\nabla-ip_F\mathbf{l})\end{pmatrix}$$
$$+\begin{pmatrix}i\frac{v_l}{2}\operatorname{div}\mathbf{l} & i\frac{v_t}{2}\operatorname{div}\mathbf{\Delta}\\ i\frac{v_t}{2}\operatorname{div}\mathbf{\Delta}^* & -i\frac{v_l}{2}\operatorname{div}\mathbf{l}\end{pmatrix}+a(\mathbf{l}\operatorname{rot}\mathbf{l})+b(\mathbf{v}_s\mathbf{l}),\tag{13}$$

and a and b are functions of the density. The function $b(\rho)$ is determined from the requirement of Galilean invariance of the Lagrangian. Under a Galilean transformation $\mathbf{r} = \mathbf{r}' + \mathbf{V}t$, we have $\mathbf{v}_s = \mathbf{v}_s' + \mathbf{V}$, and (see Ref. 13):

$$\psi_{1,2}=\psi_{1,2}'\exp\left\{im\mathbf{V}\left(\mathbf{r}'+\frac{\mathbf{V}t}{2}\right)\right\},$$

$$\Delta=\Delta'\exp\left\{2im\mathbf{V}\left(\mathbf{r}'+\frac{\mathbf{V}t}{2}\right)\right\}.$$

From this it is clear that, acting on quantities which transform like $\psi_{1,2}$, the invariant operators are the expressions

$$i\frac{\partial}{\partial t}+i\mathbf{v}_s\nabla+\frac{mv_s^2}{2},\qquad \nabla-im\mathbf{v}_s$$

and, correspondingly, acting on the quantities which transform like $\psi_{1,2}^*$—their complex conjugate operators. Since in the adopted approximation one should neglect terms involving v_s^2, the Galilei-invariant Lagrangian is obtained from the expression for $\mathbf{v}_s = 0$ by means of the substitution

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}+\mathbf{v}_s\nabla,\qquad \nabla\to\nabla\mp im\mathbf{v}_s,$$

where the upper sign refers to operators acting on quantities transforming like $\psi_{1,2}$, and the lower sign—to quantities transforming like $\psi_{1,2}^*$. As a result of this we find $b = mv_l - p_F$. In the weak coupling approximation we have $v_l = v_F$ and $b = 0$. In this approximation the Lagrangian L_F corresponds to the known (Refs. 1, 2) Bogolyuov equations for the A-phase, linearized in $\nabla \pm ip_F\mathbf{l}$.

3. THE EFFECTIVE BOSE ACTION

We apply the results obtained above to a calculation of the effective action of the bosonic subsystem. For this it is necessary to eliminate the fermionic subsystem by calculating the functional integral over the Fermi fields. To facilitate the calculations we go over to the Euclidean formulation by substituting $-\partial/\partial\tau$ for $i\partial/\partial t$, and setting $\tau = it$. The effective action is $S_B + \Delta S_B$, where

$$S_B=\int d^4x\,L_B,$$

$x = (\tau,\mathbf{r})$, L_B is defined in Eq. (4), and (see, e.g., Ref. 14)

$$\Delta S_B=\ln\int D\psi D\psi^*\exp\left(\int d^4x\,\psi^*\Lambda\psi\right)=\ln\mathrm{Det}(\Lambda\Lambda_0^{-1})$$
$$=\mathrm{Tr}\ln\Lambda\Lambda_0^{-1}=\mathrm{Tr}\left(F_1-\frac{1}{2}F_2+\ldots\right).\qquad(14)$$

Here Λ is defined in Eq. (13), Λ_0 is a normalizing operator, $F_1 = \delta\Lambda\Lambda_0^{-1}$, $F_2 = (\delta\Lambda\Lambda_0^{-1})^2$, $\delta\Lambda = \Lambda - \Lambda_0$. The operator Λ_0 is usually chosen equal to the operator Λ in the unperturbed equilibrium state. In accord with the hydrodynamic character of the theory we are developing we choose Λ_0 in the following manner.

In the spatially homogeneous case the operator $\Lambda^{-1}\equiv G$, an operator whose matrix elements $G(x_1,x_2)$ coincide with the fermionic Green's function, can be determined easily by solving the equation $\Lambda G = \delta(x_1 - x_2)$. We have

$$G(x)=-\frac{e^{ip_F\mathbf{lr}}}{2\pi^2v_lv_t^2}[\tau^2+v_t^{-2}|\Delta\mathbf{r}|^2$$
$$+v_l^{-2}(\mathbf{lr})^2]^{-2}\begin{pmatrix}\tau+iv_l^{-1}\mathbf{lr} & iv_t^{-1}\Delta\mathbf{r}\\ iv_t^{-1}\Delta^*\mathbf{r} & \tau-iv_l^{-1}\mathbf{lr}\end{pmatrix},\qquad(15)$$

where $x = x_1 - x_2$.

In the general case, when the quantities $\mathbf{l}, \Delta, p_F(\rho), \ldots$ are slowly varying functions of the coordinates and time, we introduce in place of x_1 and x_2 the coordinates $X = (x_1 + x_2)$ and $x = x_1 - x_2$, and the "local-equilibrium" Green's function $G(X,x)$ which is obtained from (15) by setting $(\mathbf{l}) = \mathbf{l}(X)$, $\Delta = \Delta(X)$, $p_F = p_F(X), \ldots$. We consider as a definition of the operator Λ_0 the requirement that the matrix elements $(\Lambda_0^{-1})_{x_1x_2}$ should be equal to the functions $G(X,x)$.

The product ΛG of Λ and any other operator G defined by its matrix elements $G(x_1,x_2)$ has matrix elements which are obtained from $G(x_1,x_2)$ by applying the operator (13), where all differential operators act on the first argument x_1 and the arguments $\mathbf{l}(x_1), \Delta(x_1), p_F(x_1), \ldots$ contain also x_1. The action of the operator Λ_0 inverse to $G(X,x)$ is obviously defined by the first two terms in (13), where the differentiations must be with respect to x_1, and X must appear in the arguments $\mathbf{l}$, Δ, p_F, $\ldots$. Making use of the equalities $x_1 = X + (x/2)$, $\partial/\partial x_1 = \partial/\partial x + (1/2)\partial/\partial X$ and expanding in terms of the gradients of the slowly varying functions, it is easy to calculate the operator $\delta\Lambda$ in Eq. (14). In doing this it should be kept in mind that in the theory under discussion only hydrodynamic asymptotic behavior is meaningful, i.e., the asymptotics for large $|x|$, accordingly one has to retain only the leading terms for $|x|\to\infty$. Moreover, since the original action S_B contains the density itself, but the quantities Δ and $\mathbf{l}$ enter only via derivatives, in the action ΔS_B too, the zero order terms in the derivatives will by definition be absent, the density may be considered constant, and one must take into account only the spatial derivatives of Δ, and $\mathbf{l}$ (as noted above, the consideration of the time derivatives would lead to terms of higher order of smallness).

We write the formula (14) for ΔS_B in terms of the matrix elements $F_1(X,x)$ of the operator F_1 in the (X,x) representation

$$\Delta S_B=\int d^4X\lim_{x\to0}\mathrm{tr}\left\{F_1(X,x)\right.$$
$$\left.+\int d^4x'\,F_1(X,x-x')F_1(X,x')+\ldots\right\},\qquad(16)$$

where, in distinction from Tr, tr is to be understood as a matrix rather than a complete operator.

We are interested in the part of ΔS_B which consists of terms of the lowest (second) order in the spatial derivatives $(\partial/\partial X)$. Such terms arise from the first term in the curly brackets in Eq. (16) and in this case they are proportional to $1/|x|$. In addition, the matrix elements F_1 contain also terms which are linear in the derivatives $\partial/\partial X$ and are proportional to $1/|x|^2$. On account of the second term in (16), which is given by the integral in the curly bracket, they contribute to the curly bracket expressions proportional to $(\partial/\partial X)^2$ and $\ln(\lambda/|x|)$, i.e., it is exactly these terms which determine the hydrodynamic asymptotics. Thus one may restrict one's attention to the integral term in Eq. (16). The resulting expression for F_1, linear in the derivatives $\partial/\partial X$, in a reference frame in which the coordinate axes for a given X are respectively along $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{l}$, has the form

$$F_1=-\frac{p_F}{4\pi^2v_lv_t}\frac{e^{ip_Fz}}{(\tau^2+\tau_n^2)^2}\sigma_i(\tau\tau_k\omega_{ik}-\varepsilon_{imk}\tau_k\tau_l\omega_{ml}),\qquad(17)$$

where $\tilde{\sigma}_i$ are the transposed Pauli matrices, $\tau_i = (x/v_t, y/v_t, z/v_l)$, $T_i = (X/v_t, Y/v_t, Z/v_l)$, and

$$\omega_{ik} = \frac{\partial l_i}{\partial T_k} - \frac{\partial l_k}{\partial T_i}.$$

Substitution into Eq. (16) yields

$$\Delta S_B = -\int d^4X \frac{p_F^2}{32\pi^2 v_l^2} \omega_{ik}^2 \int \frac{d|x'|}{|x'|}. \quad (18)$$

The logarithmically diverging integral must be cut off at the upper limit, at the wavelength λ of the motion, and at the lower limit at the length l. In the chosen coordinate system, in view of $dl_z = 0$, one can represent the quantity ω_{ik}^2 in the form

$$\omega_{ik}^2 = 2v_l^2 \left(\frac{\partial l_\alpha}{\partial Z}\right)^2 + v_t^2 \left(\frac{\partial l_\alpha}{\partial X_\beta} - \frac{\partial l_\beta}{\partial X_\alpha}\right)^2,$$

where $\alpha, \beta = 1, 2$.

Therefore Eq. (18) corresponds to the following invariant expression for the variation ΔL_B of the bosonic Lagrangian:

$$\Delta L_B = -\frac{p_F^2 v_l}{16\pi^2} \left\{ [\mathbf{l} \times \mathrm{curl}\, \mathbf{l}]^2 + \frac{v_t^2}{v_l^2} (\mathbf{l} \cdot \mathrm{curl}\, \mathbf{l})^2 \right\} \ln \frac{\lambda}{l}.$$

The elimination of the Fermi degrees of freedom thus leads to the appearance in L_B of nonlocal (albeit weakly logarithmically divergent) terms, similar to the known (Refs. 15, 16) terms in the energy of ^{3}He-A in the region $\xi \ll \lambda \ll l$.

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