Quantum Brownian particle in a periodic potential

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We investigate the dynamics of a quantum particle located in a "washboard" potential and interacting with a heat bath at temperature T. We show that the diagonal elements of the density matrix for such a particle in the position and quasimomentum representations are the duals of one another, for practically any t and over a wide range in T and the external force F. We study the breakdown of coherence, diffusion, and localization in quasimomentum (quasicharge) space and the analogous dual phenomena in position space, and we also calculate the mobility of a quantum particle in x- and p-space. We investigate the effect of quantum fluctuations on the current-voltage characteristics of Josephson junctions, and we show that the range of existence of Bloch oscillations in such junctions is bounded by the condition $R > R_Q = \pi/2e^2$ (R is the shunt resistance), whereas Josephson oscillations occur when $R < R_Q$.

1. INTRODUCTION AND BASIC RELATIONS

The occurrence of dissipation leads to a significant restructuring of the behavior of a quantum system located in a potential with several minima closely spaced in energy.[1,2] The wave function loses its coherence in such systems. In the limit $T\to 0$, as the effective viscosity parameter η increases, a phase transition takes place: the particle is localized near one of the minima of the potential and its static mobility vanishes abruptly.[1–5] The temporal evolution of the density matrix of a dissipative quantum system in a two-well potential was investigated in Refs. 6–9, as well as in a number of other papers (see the review in Ref. 7). References 8–12 are devoted to a study of the motion of a quantum particle in a "washboard" potential for various special cases.

In the present paper we investigate the dynamics of a quantum system with linear (resistive) dissipation, at a finite temperature T and with finite external force F. Such a dissipation occurs, for example, when a Josephson junction is shunted by a normally conducting metal. In the limit of sufficiently weak tunneling between different minima of the potential, we obtain general expressions describing the temporal evolution of the probability of finding the particle near those minima. The recent spate of attention with regard to the quantum behavior of very small Josephson junctions has resulted in a description of the dynamics of the system in the quasimomentum (quasicharge, see below) space. For a periodic potential, we derive expressions for the diagonal elements of the density matrix in the quasimomentum representation, and prove that these are dual to the expressions for the corresponding quantities in the position representation. Our method also makes it possible to generalize the duality relations for mobility which were previously derived in another way by Schmid[3] in the limit $T\to 0$, $F\to 0$. We study the breakdown of coherence, "spreading" of probability, and diffusion and localization in quasimomentum space, i.e., those phenomena which are "dual" to the corresponding phenomena in position space, and we also calculate the mobility (in both x- and p-space) of a quantum particle in a "washboard" for various T and F. Our results make it possible to determine the form of the current-voltage characteristic (IVC) of superconducting junctions with macroscopic quantum effects taken into account.

To calculate $\rho(q, q')$, the density matrix of a dissipative quantum system in the position representation averaged over the states of a heat bath, we make use of a well known formalism[13,14] and introduce the quantity

$$J(q,q';q_iq_i')=\int_{q_i}^{q} Dq_1\int_{q_i'}^{q'} Dq_2 \exp\{iS_1-iS_2+\Phi\}, \tag{1}$$

where

$$S_{1,2}=\int_0^t dt'\left\{\frac{m}{2}(\dot q_{1,2})^2-V(q_{1,2})+Fq_{1,2}\right\}, \tag{2}$$

m is the mass of the particle, $V(q)$ is the potential, and F is the external force. We are largely interested here in the case of a periodic potential. We assume for definiteness that

$$V(q)=-V_0\cos(2\pi q/q_0). \tag{3}$$

Potentials of the form (3) are of considerable interest for specific physical realizations of systems to be examined. Note, however, that the results obtained here do not depend in fact on the exact form of the periodic potential $V(q)$. The quantity Φ appears when the exact density matrix of the system consisting of the particle plus heat bath is averaged over the degrees of freedom of the heat bath.[13,14] It can be represented in the form

$$\Phi=\int_0^t dt'\int_0^t dt''\, q_l(t')f_{lm}(t',t'')q_m(t''),\quad l(m)=1,2. \tag{4}$$

The heat bath is usually modeled as a large number of harmonic oscillators in thermodynamic equilibrium, interacting linearly with the qth degree of freedom. Φ has been calculated for this simple case in Refs. 15 and 16. In our notation, we have for the Fourier transform $f_{l,m}$

$$f_{l,m}(\varepsilon)=\frac{\eta}{2}(-1)^{l+1}\varepsilon\left(1-(-1)^m\operatorname{ctg}\frac{\varepsilon}{2T}\right), \tag{5}$$

which corresponds to linear, nondispersive friction. In superconducting junctions, the applicability of (5) is limited by the condition $|\varepsilon|\ll\Delta_g$, which is unimportant in what follows here, where Δ_g is the energy gap. The density matrix is related to the quantity J of (1) by the obvious relation

$$\rho(q,q')=\int dq_i\,dq_i'\,J(q,q';q_i,q_i')\rho_i(q_i,q_i'), \tag{6}$$

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where ρ_i is the initial density matrix. Equations (1)–(6) describe completely the motion of a quantum particle in a periodic potential of the type (3), interacting with a quantum heat bath, which (for an appropriate choice of interaction) produces linear friction in the system.

2. THE STRONG COUPLING LIMIT

We first consider the limit of large V_0. In that case, at sufficiently low temperature, the particle is situated in the immediate neighborhood of a minimum of the potential (3) at $q_k = kq_0$, $k = 0, \pm 1, \ldots$, and the density matrix for the system becomes quasi-discrete: $\rho(q,q') \to \rho(kq_0, k'q_0)$. The diagonal elements $\rho(kq_0, kq_0) \equiv W_k(t)$ of this matrix determine the probability of finding the particle in the neighborhood of the k th minimum. We shall assume that at the initial instant of time the particle is located near one of the minima of the potential (3), i.e., $W_k(0) = \delta_{k0}$. If

$$\omega_m \equiv \max\{\sigma, T\} \ll \omega_0 \ll V_0 \tag{7}$$

($\omega_0 \equiv \tau_0^{-1}$ is the frequency of small oscillations about the potential minimum, $\sigma \equiv Fq_0$ is the energy difference of adjacent minima), particle motion at subsequent times proceeds solely by direct tunneling between wells. The present authors have proposed a general method[17] for studying the evolution of the density matrix in the presence of such sub-barrier motion, and applied this method to the determination of $W_k(t)$ for two-well and periodic potentials with $\sigma = 0$ and $T = 0$.[8] The method can be directly generalized to $\omega_m \neq 0$. We parametrize the time-dependence of $q_{1,2}$ by

$$q_j(t') = \tilde{q}_j(\tau_j(t')), \quad d\tau_j/dt' = (-1)^{j+1}i, \quad j = 1, 2. \tag{8}$$

We can isolate trajectories which determine, with exponential accuracy, the probability of jumps between wells. In the space of the parameter τ, such trajectories (instantons) are well known, and are found by setting $\delta S/\delta\tilde{q}(\tau) = 0$. Subsequent calculations are customarily broken up into integrations over small deviations from saddle-point trajectories and the isolation of so-called null modes of the collective instanton coordinates. Keeping in mind the instanton configurations on a τ-contour, which describe all possible transitions between wells, we have

$$W_k(t) = \sum_{n=k}^{\infty} \left(\frac{\Delta}{2}\right)^{2n} \hat{T}_\tau \sum_{\{e_{1i}, e_{2i}\}} \int \prod_{i=1}^{m} d\tau_{1i} \int \prod_{i=m+1}^{2n} d\tau_{2i}$$
$$\times \exp\Big\{2\alpha\Big[\sum_{i,j=1}^{m} e_{1i}e_{1j}\Lambda(\tau_{1i}-\tau_{1j}) + \sum_{\substack{i,j=m+1\\ i>j}}^{2n} e_{2i}e_{2j}\Lambda(\tau_{2i}-\tau_{2j})$$
$$+ \sum_{i=1}^{m}\sum_{j=m+1}^{2n} e_{1i}e_{2j}\Lambda(\tau_{1i}-\tau_{2j})\Big] + \sigma\Big(\sum_{i=1}^{m} e_{1i}\tau_{1i} + \sum_{i=m+1}^{2n} e_{2i}\tau_{2i}\Big)\Big\},$$
$$\Lambda(\tau) = \ln(\sin \pi\tau T/\pi\tau_0 T). \tag{9}$$

Here $\alpha = \eta q_0^2/2\pi$, and Δ is double the tunneling amplitude between wells: $\Delta \sim \omega_0 A^{1/2} e^{-A}$, $A \sim W_0/\omega_0 \gg 1$; $e_{1i} = \pm 1$ and $e_{2i} = \pm 1$ are topological charges of instantons on the "upper" ($\tau \equiv \tau_1$) and "lower" ($\tau \equiv \tau_2$) portions of the τ-contour. The ordering operator $\hat{T}_\tau$ places the "instants of time" τ_{1i} and τ_{2i} on the τ-contour in order of decreasing i. All "instants of time" τ_{1i} on the τ_1-contour are "later" than any "instant of time" τ_{2i} on the τ_2-contour. It can be seen from the boundary conditions on $\tilde{q}(\tau)$ that the summation in (9) must be carried out over charge configurations satisfying

$$\sum_{i=1}^{m} e_{1i} + \sum_{i=m+1}^{2n} e_{2i} = 0, \quad \sum_{i=1}^{m} e_{1i} - \sum_{i=m+1}^{2n} e_{2i} = 2k. \tag{10}$$

To obtain the final expressions for $W_k(t)$, it remains for us to integrate over the collective variables on the real time contour. It is then necessary, for every $\{e_{1i}, e_{2i}\}$ configuration, to take into account all possible combinations of successive positions of the collective variables t_i, $i = 1, 2, \ldots, 2n$, in the order in which they arise. Making use of (8), the result we obtain from (9) is

$$W_k(t) = \sum_{n=k}^{\infty} \left(\frac{\Delta}{2}\right)^{2n} (-1)^{n+k} \sum_{\{e_{1i}, e_{2i}; t_i\}} \int \prod_{i=1}^{2n} dt_i$$
$$\times \exp\Big\{2\alpha \sum_{\substack{i,j=1\\ i>j}}^{2n} e_i e_j \ln\left|\frac{\operatorname{sh}\pi(t_i - t_j)T}{\pi T\tau_0}\right|$$
$$+ i\sigma \sum_{i=1}^{2n} e_i t_i + i\beta\{e_{1i}, e_{2i}; t_i\}\Big\}. \tag{11}$$

We have omitted the first subscript from the quantities e_{1i} and e_{2i} in the exponent in (11), since the sign of the "interaction" between instantons i and j is determined solely by the sign of the product $e_i e_j$, regardless of whether these instantons belong to contours τ_1 or τ_2. It can be seen from (11) that for a given $\{e_{1i}, e_{2i}\}$ configuration only the phase factor β can take on a variety of values for different combinations $\{t_i\}$. We shall find it convenient to designate the various configurations $\{e_{1i}, e_{2i}; t_i\}$ as in Fig. 1. Charges situated in the upper row belong to contour τ_1, and those in the lower row, to τ_2. The horizontal arrangement of charges corresponds to the temporal sequence of the related collective coordinates. Employing a direct transformation from the variables τ_i of (9) to the t_i of (11) and taking (8) into account, we can establish the following rules for calculating the phase factor β for all possible instanton configurations:

1) the factor $\beta = \pi\alpha n$ if all $2n$ charges e_i lie on the contour τ_1;

2) the remaining configurations can be obtained by moving the charges from cell $(1, 2n)$ to cell $(2, 2n)$. During each such move, β remains fixed;

3) for every change in relative position of charges e_{1i} and e_{2i} from $t_i < t_j$ to $t_i > t_j$, the phase factor β changes by $e_{1i}e_{2j}2\pi\alpha$.

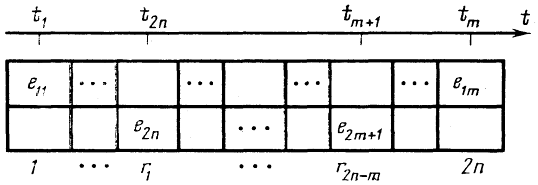

FIG. 1. Different configurations of topological charges of instantons on contours $\tau(t)$ and $\tau_2(t)$, and the arrangement of their "coordinates" in real time.

In general (Fig. 1), we have

$$\beta\{e_{1i}, e_{2i}; t_i\}=\pi\alpha n+2\pi\alpha\Big(\sum_{\substack{i,j=m+1\\ i>j}}^{2n} e_{2i}e_{2j} + e_{2n}\sum_{i=r_1+1}^{m} e_{1i}+\ldots+e_{2m+1}\sum_{i=r_{2n-m}+1} e_{1i}\Big). \qquad (12)$$

The foregoing considerations and Eq. (11) are valid for a potential with an arbitrary number of minima. Cases having different numbers of such minima differ only by the set of instanton configurations over which the summation in Eq. (11) takes place. In the case of interest to us here, that of a periodic potential, all configurations satisfying the conditions of (10) are found to be important. According to Refs. 8 and 9, the number of such configurations of $2n$ instantons to be summed over in Eq. (11) for $W_k(t)$ is

$$M=(2n!)^2/(n!)^2(n-k)!(n+k)!; \qquad (13)$$

the total number of such configurations can easily be shown to equal $4^n(2n!)/(n!)^2$.

3. DYNAMICS AND DUALITY

In the preceding section we obtained a general expression for the diagonal elements of the density matrix of a quantum particle, averaged over states of a heat bath, for a high potential barrier $V_0(7)$ between minima. Here we consider the more general case of arbitrary V_0. To determine the density matrix and calculate the mean values, we will find it convenient to introduce the generating function

$$\chi(y,\xi)=\sum_x \rho(x+y/2, x-y/2)e^{-i\xi x}. \qquad (14)$$

We also introduce the Fourier representation of the density matrix,

$$\rho_{pp'}=\sum_{qq'}\rho(q,q')\exp(-ipq+ip'q'). \qquad (15)$$

It is easy to verify that

$$\chi(y,0)=\sum_p \rho_{pp}e^{ipy}. \qquad (16)$$

We put in (6) $q=x+y/2$ and $q'=x-y/2$, multiply the equation by $\exp(-i\xi x)$, and integrate over x. With the definition (14), this gives

$$\chi(y,\xi)=\int dx\int dq_i\,dq_i'\,J(x+y/2, x-y/2; q_i, q_i')\times e^{-i\xi x}\rho_i(q_i,q_i'). \qquad (17)$$

We then carry out the shift $q_j\to q_j+Ft'/\eta$ in the functional integral (1) for J, and carry out the parametrization (8). Assuming that

$$\tilde q_j(\tau_j(t'))=q_j(t')+y/2(-1)^j,\quad j=1,2,$$

we have

$$\chi(y,\xi)=\int dq_i\,dq_i'\,\rho_i(q_i,q_i')\exp\Big[i\frac{mF}{\eta}(y-q_i+q_i')-i\xi Ft/\eta\Big]\int D\tilde q(\tau)\exp\{-S_E\},$$

$$S_E=\frac{1}{2}\int d\tau\,d\tau'\,K(\tau,\tau')(\tilde q(\tau)+\tilde y(\tau))(\tilde q(\tau')+\tilde y(\tau'))+\int d\tau[V(\tilde q(\tau)+\tilde y(\tau)+Ft'(\tau)/\eta)+i\tilde q(\tau)\xi\delta(\tau)], \qquad (18)$$

$$K(\tau,\tau')=m\delta(\tau-\tau')\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'}+\eta T\sum_{\omega_n}|\omega_n|e^{i\omega_n(\tau-\tau')},$$

$$\omega_n=2\pi Tn,\quad n=0,\pm1,\ldots;\quad \tilde y(\tau\equiv\tau_j)=(-1)^{j+1}y/2,\quad j=1,2,$$

where

$$\int D\tilde q(\ldots)\equiv\int dx\int D\tilde q_1 D\tilde q_2(\ldots).$$

To calculate the functional integral (18) with the potential (3), we make use of the well known relation

$$\exp\Big(V_0\int_0^\tau d\tau'\cos(2\pi q/q_0)\Big)=\sum_{m=0}^{\infty}\Big(\frac{V_0}{2}\Big)^m\sum_{\{e_{1i}\}}\int_0^\tau d\tau_m\int_0^{\tau_m}d\tau_{m-1}\ldots\int_0^{\tau_2}d\tau_1\exp\Big\{-i\int_0^\tau d\tau'\,\rho(\tau')q(\tau')\Big\}. \qquad (19)$$

The second summation on the right-hand side of (23) is over all possible "charge" configurations $e_{1i}=\pm1$, while the corresponding "charge" density is given by (compare Refs. 3, 4, 12)

$$\rho_1(\tau)=\frac{2\pi}{a}\sum_{i=1}^{m}e_{1i}\delta(\tau-\tau_{1i}). \qquad (20a)$$

In a completely analogous manner, we can introduce the "charge" density on contour τ_2:

$$\rho_2(\tau)=\frac{2\pi}{q_0}\sum_{i=m+1}^{s}e_{2i}\delta(\tau-\tau_{2i}),\quad e_{2i}=\pm1. \qquad (20b)$$

The advantage of choosing for "charge" in (19) and (20) the same notations as for the topological charge of instantons in the preceding section will become apparent shortly. Substitution of Eqs. (19) and (20) into the functional integral (18) produces a sum of Gaussian integrals, each of which is easily calculated. As a result, we have

$$\chi(y,\xi)=\exp\Big(i\frac{mF}{\eta}y-iFt\xi/\eta\Big)\sum_{\substack{m=0\\ s-m=0}}^{\infty}\Big(\frac{V_0}{2}\Big)^s\sum_{\{e_{1i},e_{2i}\}}\hat T_\tau\times\int\prod_{i=1}^{m}d\tau_{1i}\prod_{i=m+1}^{s}d\tau_{2i}\exp\Big\{-\int d\tau\Big[i\Big(\tilde y(\tau)+\frac{Ft'(\tau)}{\eta}\Big)\rho_\xi(\tau)+\frac{1}{2}\rho_\xi(\tau)\int d\tau'\,K^{-1}(\tau,\tau')\rho_\xi(\tau')\Big]\Big\},$$

$$\rho_\xi(\tau)=\rho_1(\tau)+\rho_2(\tau)+\xi\delta(\tau). \qquad (21)$$

Let us first take $\xi=0$. It is not hard to show that (just as in Ref. 3), only neutral configurations with $\rho_1+\rho_2=0$, i.e., $s=2n$, contribute to Eq. (21). With this in mind, we obtain

$$\chi(y,0)=\exp\Big(i\frac{mFy}{\eta}\Big)\sum_{k=-\infty}^{\infty}w_k(t)\exp(ip_0ky), \qquad (22)$$

where $p_0 = 2\pi/q_0$ is the reciprocal lattice constant, and under the duality transformations

$$\Delta \leftrightarrow V_0, \quad \tau_0 \leftrightarrow \tilde{\tau}_0 \sim m/\eta, \quad \alpha \leftrightarrow \tilde{\alpha} = 1/\alpha, \quad q_0 \leftrightarrow q_0/\alpha, \tag{23}$$

w_k and W_k of (9) transform into one another. The change to integration over the t_i on the real time contour is carried out exactly the same way as was done in the previous section to find the expressions for the $W_k(t)$ (11). We finally obtain

$$w_k(t) = \sum_{n=k}^{\infty} \left(\frac{V_0}{2}\right)^{2n} (-1)^{n+k} \sum_{\{e_{1i}, e_{2i}; t_i\}} \int \prod_{i=1}^{2n} dt_i \cdot \times \exp\left\{2\alpha^{-1} \sum_{\substack{i,j=1 \\ i>j}}^{2n} e_i e_j \ln\left|\frac{\operatorname{sh} \pi T(t_i - t_j)}{\pi T m/\eta}\right| + \frac{i\sigma}{\alpha} \sum_{i=1}^{2n} e_i t_i + i\beta\{e_{1i}, e_{2i}; t_i\}\right\}. \tag{24}$$

In (24), as in (11), the summation is over configurations which satisfy (10).

Comparing (16) and (22), we necessarily have

$$\rho_{pp}(t) = w_k(t), \quad p = p_0 k + mF/\eta. \tag{25}$$

We have thus shown that the duality[1)] between the expressions for the diagonal elements of the density matrix in the position (in the limit of large V_0) and quasimomentum representations (11) and (24), (25), with the transformations (23), is valid at practically any instant (with the possible exception of a very few) in time. The physical meaning of (24), (25) is quite simple: the probability that the quasimomentum has changed by k reciprocal lattice constants by time t is given directly by $w_k(t)$. The state of the particle does not change during this process, since its quasimomentum in a periodic potential is defined, as is well known, accurate to p_0, and the heat bath takes up to momentum kp_0. This makes the $w_k(t)$ measurable in principle. Note that in deriving (24), we have made no assumptions about the magnitude of V_0.

We now establish the relations for the mobility of a quantum particle in the x- and p-spaces:

$$\mu_x = \langle \dot{x} \rangle / F, \quad \mu_p = \langle \dot{p} \rangle / F \equiv F^{-1} \sum_p p \dot{\rho}_{pp}(t). \tag{26}$$

Assuming that $y = 0$ in (21), making use of (10), and integrating over the collective coordinates t_i as before, we obtain

$$\langle x \rangle = i \frac{\partial \chi(0, \xi)}{\partial \xi}\bigg|_{\xi=0} = \frac{Ft}{\eta} - \sum_k k p_0 w_k(t), \tag{27}$$

whereupon

$$\eta \mu_x(\alpha, \sigma, T) = 1 - \eta \tilde{\mu}(\alpha, \sigma, T), \tag{28}$$

$$\tilde{\mu}(\alpha, \sigma, T) = \frac{q_0}{\alpha F} \sum_k k \dot{w}_k(t). \tag{29}$$

The duality between $W_k(t)$ and $w_k(t)$ makes it possible to relate the quantity $\tilde{\mu}$ in (29) to mobility in the position space as calculated with $V_0 \gg \omega_0$. Note that when the replacements $w_k(t) \to W_k(t)$ and $q_0/\alpha \to q_0$ are made, the right-hand side of (29) goes into the expression for the mobility $\mu_x(\alpha, \sigma, T)$ of a quantum particle in the limit $\Delta\tau_0 \ll 1$ (or $V_0 \gg \omega_0$), i.e., with the substitutions in (23),

$$\mu_x(\alpha, \sigma, T) \leftrightarrow \tilde{\mu}(\alpha, \sigma, T). \tag{30}$$

Thus, to find the mobility μ_x both in regions of small and large potential barrier height V_0, it is sufficient to calculate this quantity in only one of these limiting cases ($V_0 \gg \omega_0$ or $V_0 \ll \eta/m$) and then use Eqs. (28) and (30).

In the special case $T \to 0$, $F \to 0$, Eqs. (28) and (30) were derived in Ref. 3 in another way. Note that a relation of the form (28) was also discussed in Ref. 12 for $T \neq 0$, $F \neq 0$. As we show in Appendix 1, however, the derivation in Ref. 12 relied on a somewhat flawed calculation of the functional integral for the system density matrix.

In Refs. 3, 4, and 12, $\tilde{\mu}$ was treated solely as an auxiliary quantity, governing the mobility of a fictitious particle tunneling between the nodes of a one-dimensional lattice having period q_0/α. Equations (25) of the present paper, however, indicate that $\tilde{\mu}$ is physically meaningful in its own right, and in principle can be measured independently. In fact, making use of (25), (26), and (29), it is easy to find an expression relating $\tilde{\mu}$ to the mobility μ_p of (26) in the quasimomentum space: $\eta\tilde{\mu} = \mu_p$, i.e.,

$$\eta \mu_x(\alpha, \sigma, T) = 1 - \mu_p(\alpha, \sigma, T). \tag{31}$$

Equations (28) and (31) make it possible to calculate μ_p easily when μ_x is known, and we shall also make use of this in what follows.

4. DIFFUSION, LOCALIZATION, AND THE BREAKDOWN OF COHERENCE

If we are to describe the properties of interest in the system we are considering, it is obviously necessary to determine the quantities (11) and (24). The exact calculation of these quantities is of course a very difficult problem. In a number of special cases, however, the problem simplifies considerably. The simplest of these is the diffusion approximation, in which one only takes into accoiunt those trajectories which with (7) describe tunneling between different states $\rho(kq_0, k'q_0)$ with $|k - k'| \leqslant 1$. The contribution of such configurations can then be calculated by making the approximation of non-interacting bi-instanton "molecules." It was demonstrated in Refs. 6 and 7 for a two-well potential that the interaction between "blips," which formally corresponds to an interaction between such "molecules," can be neglected when $\alpha > 1$ or $\alpha < 1$ with sufficiently large ω_m. For a periodic potential, the calculation is similar, differing only in the number of realizations for which $|k - k'| \leqslant 1$, which for given k and number of instantons $2n$ is equal to

$$N(k, n) = \sum_{\substack{p=k \\ n+k \geqslant 2p}}^{n} \frac{2^{k-2p} n!}{p!(p-k)!(n+k-2p)!}. \tag{32}$$

Exploiting the duality of (11) and (24), we have

$$w_k(t) = \exp\left(\frac{k\sigma}{2\alpha T}\right) \sum_{n=k}^{\infty} \frac{(-1)^{n+k} N(k, n) (\Gamma t)^n}{n! (\operatorname{ch}(\sigma/2\alpha T))^{2p-k}},$$

$$\Gamma = \frac{V_0^2 m}{2\eta} \left(\frac{\eta}{2\pi T m}\right)^{(\alpha-2)/2} \frac{|\Gamma(\alpha^{-1} + i\sigma/2\pi T\alpha)|^2}{\Gamma(2/\alpha)} \operatorname{ch}\left(\frac{\sigma}{2\alpha T}\right) \tag{33}$$

with analogous expressions for $W_k(t)$. Note, however, that the validity of (32) and (33) is not at all obvious *a priori*, in contrast to the situation for the two-well potential, since for sufficiently large n, the number of discarded "charged" configurations $M(k,n) - N(k,n)$ is much greater than the number $N(k,n)$ in (32). A detailed analysis of the contribution of the neglected correlation functions is given in Appendix. 2.

The sums in (32) and (33) are easily evaluated. When $\max\{V_0,\sigma/\alpha,T\} \ll \eta/m$, we obtain

$$w_k(t) = I_k\left(\frac{\tilde{\Gamma} t}{\mathrm{ch}(\sigma/2\alpha T)}\right)\exp\left(\frac{k\sigma}{2\alpha T} - \tilde{\Gamma} t\right), \tag{34}$$

where $I_k(x)$ is a Bessel function of imaginary argument. From here on, we assume that $\tilde{\tau}_0 = m/\eta$. Equation (34) holds both for $\alpha < 1$ and for $\alpha > 1$ with $\max\{\sigma/\alpha,T\} \gg \tilde{\Gamma}$. Thus, under these circumstances, we have diffusive "spreading" of probability in the quasi-momentum space: the probability $w_k(t)$ of transfer processes varies with time according to (34). Using (34), it is straightforward to calculate the mean values $\langle p(t)\rangle$ and $\langle p^2(t)\rangle$. The first of these is a linear function of time. For the mobility $\mu_p(\alpha,\sigma,T)$ of (26), we obtain

$$\mu_p = \frac{2\pi}{\sigma}\tilde{\Gamma}\,\mathrm{th}\,\frac{\sigma}{2\alpha T} = \begin{cases} \dfrac{\pi^2}{\alpha\Gamma(2/\alpha)}\left(\dfrac{V_0 m}{\eta}\right)^2\left(\dfrac{\sigma m}{\alpha\eta}\right)^{2(1-\alpha)/\alpha}, & \sigma/\alpha \gg T, \\ \dfrac{\pi^{5/2}\Gamma(\alpha^{-1})}{2\alpha\Gamma(1/2+1/\alpha)}\left(\dfrac{V_0 m}{\eta}\right)^2\left(\dfrac{\pi T m}{\eta}\right)^{2(1-\alpha)/\alpha}, & \sigma/\alpha \ll T. \end{cases} \tag{35}$$

The mean squared quasi-momentum is of the form

$$\langle p^2(t)\rangle = p_0^2\tilde{\Gamma} t(\tilde{\Gamma} t\,\mathrm{th}^2(\sigma/2\alpha T) + 1). \tag{36}$$

When $\sigma/\alpha \gg T$, the second term is negligible. In the opposite limiting case $\sigma/\alpha \ll T$, up to $t \lesssim 4\alpha^2 T^2/\tilde{\Gamma}\sigma^2$,

$$\langle p^2(t)\rangle = D_p t, \quad D_p = 2T\mu_p(\alpha,\sigma,T) \propto (Tm/\eta)^{(2-\alpha)/\alpha}. \tag{37}$$

Equation (37) implies that when $\alpha > 2$, the diffusion coefficient D_p increases with decreasing temperature and when $\alpha < 2$, it decreases.

When condition (7) holds, the duality transformations (23) enable one to obtain formulas similar to (34) for the diagonal matrix elements W_k. In this limiting case, such formulas were derived in Ref. 10 using simple balance equations for $W_k(t)$. Here we will be interested in the magnitude of the mobility μ_x. When (7) holds and $\sigma \ll T$, we have

$$\mu_x = \frac{\pi^{3/2}\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\left(\frac{\Delta q_0}{2\omega_0}\right)^2\left(\frac{\pi T}{\omega_0}\right)^{2\alpha-2}. \tag{38a}$$

When $\sigma \gg T$,

$$\mu_x = \frac{\pi}{2\Gamma(2\alpha)}\left(\frac{\Delta q_0}{\omega_0}\right)^2\left(\frac{\sigma}{\omega_0}\right)^{2\alpha-2}. \tag{38b}$$

A similar formula was also derived in Ref. 10 for $T \to 0$ and $\alpha > 1$.

We see from Eqs. (38a) and (38b) that in the limit $T \to 0$, $F \to 0$, the mobility of a quantum particle in a periodic potential with $\alpha > 1$ tends to zero,[3] corresponding to localization of the particle near one of the minima of the potential (3). A similar phenomenon also occurs in the quasimomentum space: for $T = 0$, $F = 0$, and $\alpha < 1$, the probability of a transfer process is zero at any instant of time: $w_k(t) = \delta_{k0}$. In other words, in the absence of an external force or thermal fluctuations, transfer processes due to quantum fluctuations are only possible for sufficiently high interactive coupling with the heat bath, $\alpha > 1$. The diffusion approximation is not suitable for a description of such processes in the region $\max\{\sigma/\alpha,T\} \lesssim \tilde{\Gamma}, \alpha > 1$, and consequently, the behavior of $w_k(t)$ is not described by Eq. (34). Here we consider the case $F = 0$, $T = 0$. It can be shown (see Ref. 8 and Appendix 2) that in that case, to calculate the quantities $w_k(t)$ for $V_0 m/\eta \ll 1$, a good approximation is

$$w_k(t) = \sum_{n=k}^{\infty} \frac{(-1)^{n+k}(V_r t)^{2n(\alpha-1)/\alpha} M(k,n)}{4^n\Gamma(2n(\alpha-1)/\alpha+1)},$$
$$V_r(\alpha) = V_0(V_0 m/\eta)^{1/(\alpha-1)}[\cos(\pi/\alpha)\Gamma(1-2/\alpha)]^{\alpha/2(1-\alpha)}. \tag{39}$$

Taking the sum in (39), for $k \ll (V_r t)^{(\alpha-1)/\alpha}$ we obtain (compare Refs. 8, 9)

$$w_k(t) = \frac{(V_r t)^{(1-\alpha)/\alpha}}{\pi\Gamma(1/\alpha)}\left[\frac{\alpha-1}{\alpha}(\ln(V_r t) - \psi(\alpha^{-1})) + \psi(1) + \ln 2 - \psi(k+1/2)\right] + w_k^{osc}(t), \quad \psi(x) = \Gamma'(x)/\Gamma(x), \tag{40}$$

where $w_k^{\mathrm{osc}}(t) = 0$ for $\alpha < 2$, and becomes important only when $\alpha \gg 1$:

$$w_k^{osc}(t) = \frac{(-1)^k}{\pi V_r t}\sin(2V_r t)\exp\left(-\frac{\pi V_r t}{\alpha}\right). \tag{41}$$

In the limit $\alpha \to \infty$, (24) and (39) become identical, and give

$$w_k(t) = J_k^2(V_0 t), \tag{42}$$

where $J_k(x)$ is a Bessel function. When $(V_r t)^{(\alpha-1)/\alpha} \ll k$, the quantities $w_k(t)$ are exponentially small. Thus, for $\alpha > 1$, propagation of a "wave packet" in the quasimomentum space is governed by the term $(V_r t)^{(\alpha-1)/\alpha}$. The existence of the oscillatory term w_k^{osc} (41) in the expression (40) for w_k reflects the presence of coherence between states with different quasimomentum when $\alpha > 2$. We see from (41) that such coherence is disrupted in a characteristic time $\sim \alpha V_r^1$. This is exactly the dual of the breakdown of quantum coherence of different states W_k in position space, which occurs when $\alpha < 1/2$. Corresponding expressions for $W_k(t)$ have been studied in Refs. 8 and 9. Making use of (39), it is not difficult to determine $\langle p^2(t)\rangle$. Using the corresponding result for $\langle x^2(t)\rangle$ obtained in Refs. 8, 9, 11 for $\alpha < 1$ and with (7) taken into account, we find for $\alpha > 1$, $F = 0$, $T = 0$ that

$$\langle p^2(t)\rangle = \left(\frac{2\pi}{q_0}\right)^2\frac{(V_r t)^{2(\alpha-1)/\alpha}}{\Gamma(3-2\alpha)}, \tag{43}$$

i.e., the "wave packet" spreads "faster" for $\alpha > 2$, and "slower" for $\alpha < 2$ than for diffusive behavior, which corresponds to packet localization in the limit $\alpha \to 1$. We draw attention to the fact that when $\alpha = 2$, Eqs. (37) and (43), which were derived to differing degrees of approximation, coincide in different special cases.

Equations (29), (31), and (35) enable one easily to determine the mobility of a quantum particle for sufficiently low values of the potential, with $V_0 m/\eta \ll 1$ and $\max\{\sigma/\alpha, T\} \ll \eta/m$. We have (see also Ref. 12)

$$\mu_x = \eta^{-1} - \frac{\pi^{3/2}\Gamma(\alpha^{-1})}{\Gamma(1/2+1/\alpha)} \left(\frac{V_0 q_0 m}{2\alpha\eta}\right)^2 \left(\frac{\pi T m}{\eta}\right)^{2(1-\alpha)/\alpha}, \quad \sigma/\alpha \ll T,$$
$$\mu_x = \eta^{-1} - \frac{\pi}{2\Gamma(2/\alpha)} \left(\frac{V_0 q_0 m}{\alpha\eta}\right)^2 \left(\frac{\sigma m}{\alpha\eta}\right)^{2(1-\alpha)/\alpha}, \quad \sigma/\alpha \gg T. \qquad (44)$$

These expressions, along with (35), are valid for $\alpha < 1$ or for $\alpha > 1$ and $\max\{\sigma/\alpha, T\} \gg \tilde{\Gamma}$. We can similarly use (29), (31), and (38) to calculate the mobility μ_p when (7) holds:

$$\mu_p = 1 - \frac{\pi^{5/2}\alpha\Gamma(\alpha)}{2\Gamma(\alpha+1/2)} \left(\frac{\Delta}{\omega_0}\right)^2 \left(\frac{\pi T}{\omega_0}\right)^{2\alpha-2}, \quad \sigma \ll T,$$
$$\mu_p = 1 - \frac{\pi^2\alpha}{\Gamma(2\alpha)} \left(\frac{\Delta}{\omega_0}\right)^2 \left(\frac{\sigma}{\omega_0}\right)^{2\alpha-2}, \quad \sigma \gg T. \qquad (45)$$

The two equations in (45) are correct when $\alpha > 1$ or when $\alpha < 1$ and $\max\{\sigma/\alpha, T\} \gg \Gamma$. Note that by using the duality relations for mobility, we have managed to determine μ_x of (44) for small V_0 and μ_p of (45) for large V_0 without calculating the probability distribution in these limiting cases. For $\alpha < 1$ and sufficiently small σ and T, the mobility μ_x is not given by Eq. (44). Calculations of this quantity by renormalization-group methods in the low-frequency limit, when (7) holds and $\sigma \to 0$, $T \to 0$, give[2)] $\eta\mu_x(\alpha < 1) = 1$ and $\eta\mu_x(\alpha > 1) = 0$ (Refs. 3–5), whereupon with (29) and (31) taken into consideration we obtain under these same conditions

$$\mu_p(\alpha < 1) = 0, \quad \mu_p(\alpha > 1) = 1. \qquad (46)$$

Thus, in all of the cases considered, the expressions for μ_x and μ_p go into one another under the duality transformations (23). This occurs not only for Eqs. (35) and (38) [which is a consequence of the duality of (11) and (24)], but for (44) and (45) as well, which were obtained for the limiting cases in which the diagonal elements of the density matrices are not dual, and moreover cannot be calculated directly.

5. DISCUSSION OF RESULTS. QUANTUM PROPERTIES OF SUPERCONDUCTING JUNCTIONS

We now examine the phase diagram of a quantum system with ohmic dissipation[3-5,12] (Fig. 2). The duality of the diagonal elements of the density matrix in position and quasimomentum space which we have established in the present paper enables us to use this diagram to take a closer look at system properties. When $T \to 0$ and $F \to 0$, phase A corresponds to position delocalization ($\eta\mu_x = 1$) and quasimomentum localization ($\mu_p = 0$), while the opposite is true of phase B: $\mu_p = 1$, $\eta\mu_x = 0$. Our considerations also enable us to distinguish between regions A_1 and A_2 in the "upper" part of the diagram, and regions B_1 and B_2 at small V_0. In region $A_1(\alpha < 1/2)$, the interaction with the heat bath is still too weak to completely destroy quantum coherence in position space. In region B_2, on the other hand ($\alpha > 2$), this interaction is quite strong and is responsible for the partial coher-

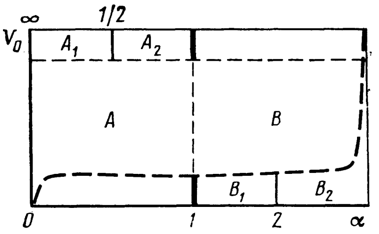

FIG. 2. Phase diagram of a quantum system with resistive dissipation. At $T = 0$, phase $A(B)$ corresponds to localization (delocalization) of quasimomentum and delocalization (localization) of position. Wave-packet spreading in the position and quasi-momentum space at $T = 0$ occurs faster for regions A_1 and B_2, and slower for regions A_2 and B_1, than for diffusive behavior. When T is high enough, the diffusion coefficients D_x in A_1 and D_p in B_2 decrease with increasing temperature.

ence of states at different quasimomenta. As a result, as $T \to 0$, wave-packet spreading in x-space (A_1) and p-space (B_2) proceeds more slowly than for total coherence, but more quickly than for diffusive behavior. A decrease in the diffusion coefficients D_x and D_p in these regions with increasing temperature (at temperatures T which are not too low) is associated with the destruction of coherence by thermal fluctuations of the heat bath. In the regions of intermediate interaction A_2 ($1/2 < \alpha < 1$) and $B_1(1 < \alpha < 2)$, there is no coherence whatsover, and thermal fluctuations faciliate an increase in D_x and D_p with increasing T. In these regions, wave-packet spreading as $T \to 0$ goes more slowly than for diffusion, corresponding to the transition to a localized state as $\alpha \to 1$. It is interesting to note that according to the duality transformations (23), as $\alpha \to \infty$ in region B_2, the system can be described (in the sense of being able to calculate means) by a "wave function" $\tilde{\psi}(p)$ which satisfies a "Schrödinger equation," with the roles of the "position" and "momentum" operators played by p and $i\partial/\partial p$ respectively.

As we have already stated, all of the results obtained here can be used directly to describe the quantum behavior of Josephson junctions, where then $q = \varphi/2$ (φ is the phase difference across the junction), $q_0 = \pi$, $m = C/e^2$, $\eta^{-1} = Re^2$ (C is the (renormalized) capacitance of the junction, R is the effective resistance of a normal shunt or metallic short-circuit, and e is the charge of the electron). The quantity α is the ratio of the fundamental "quantum" resistance $R_Q = \pi/2e^2 \approx 6.5$ kΩ to R: $\alpha = R_Q/R$. For superconducting junctions, the parameter V_0 defines the Josephson energy, $V_0 \equiv E_J = I_c/2e$ (I_c is the critical current of the junction), $F = I/e$ (I is the external current), $\omega_0 = (E_J E_Q)^{1/2}$, $E_Q = e^2/2C$, and the quantity p in (25) is the quasicharge Q on the capacitor[3)] C in units of e. With this in mind, the physical meaning of $w_k(t)$ is also clear: it determines the probability that prior to time t, the quasicharge (i.e., actually the number of Cooper pairs at the two superconducting sides of the junction has changed by $\Delta Q = kp_0 e \equiv 2ke$, or in other words, that k Cooper pairs have passed from one side of the Josephson junction to the other. Of course, due to the presence of a normal shunt and/or external circuit, the charges on the two sides of the superconducting junction will equalize in a time $\sim RC$, and when transition mechanisms between quasiparticles and the superconducting condensate are taken into account (for example, by the well known Andreev inversion mechanism at the normal-superconducting boundary), the preceding change

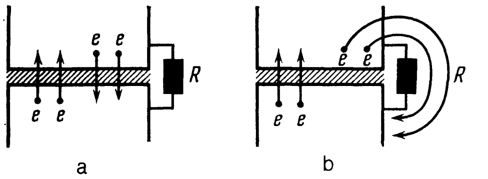

FIG. 3. a) Virtual transfers of Cooper pairs between superconductors with no change in the magnitude of the quasicharge. b) "Uncompensated" jump of a Cooper pair, changing the quasicharge by $2e$.

in the number of Cooper pairs can also be compensated. Nevertheless, the states before and after such a process, i.e., states characterized by different values of k, are *physically different*, and the probability distribution $w_k(t)$ can in principle be found experimentally. Let the exchange of Cooper pairs betweeen the superconductors somehow be prevented prior to the instant $t = 0$, and then let the Josephson effect be "turned on." Our results imply that virtual transfers of Cooper pairs between the superconductors (Fig. 3a) lead to renormalization of I_c, where for $T = 0, I = 0$, and $R > R_Q$, the renormalized value of the critical current tends to zero at sufficiently large t. Quasicharge is then localized ($w_k(t) = \delta_{k0}$), and the junction voltage vanishes. The presence of a fairly weak external current I "shifts" the mean junction voltage V: to a first approximation $\langle V \rangle = RI$, and the quasicharge mobility μ_p of (35) is small. When the current exceeds a certain value I_B

$$I_B(R) \sim (\Delta/Re)(\Delta^2/E_J E_Q)^{1/2(R/R_Q-1)} \qquad (47)$$
$$(E_J \gg E_Q, \quad R > R_Q)$$

the quasicharge starts to "move," corresponding to the onset of oscillation. The theory of these so-called Bloch oscillations for $R \to \infty$ was worked out in Reference 18. Our thory enables one to describe such an effect for $E_J \gg E_Q, I \gg I_B$, but over a much wider range in R. Under these conditions, we have from (38b) and (45) that

$$\langle \dot{Q} \rangle = I, \qquad (48a)$$

$$\langle V \rangle = \frac{\pi}{2\Gamma(2R_Q/R)} \frac{\Delta^2}{I} \left[\frac{(2\pi I)^2 C}{I_c e^3} \right]^{R_Q/R},$$

$$\Delta \sim E_J^{3/4} E_Q^{1/4} e^{-A}, \quad A \sim \left(\frac{E_J}{E_Q} \right)^{1/2}. \qquad (48b)$$

What Eq. (48a) in fact means is that junction voltage oscillations occur at a fundamental frequency $\omega_B = \pi/e$, and for such oscillations to take place as $T \to 0$ it is necessary that $R > R_Q$ (but not $R \gg R_Q$). Equation (48a) specifies the voltage-current characteristic of a Josephson junction under conditions for which Bloch oscillations exist. Using this relation, it is also easy to establish the condition for applicability of the theory of Ref. 18, which takes the form $R \gg R_Q \ln[I_c e^3/C(2\pi I)^2]$. When this condition holds, the current-voltage characteristic of (48b) is the same as that found[4)] in Ref. 18 with $E_J \gg E_Q, I \gg I_B$.

In contrast to the situation for $R > R_Q$, when $R < R_Q$, renormalized high-frequency quantum fluctuations of I_c are no longer zero. Consequently, the probability of "uncompensated" jumps of Cooper pairs (Fig. 3b) is also nonzero. With such a process, there will be no charge in the final state of a junction, just like the initial state. As we have already noted, however, this state can in principle be differentiated experimentally from the case shown in Fig. 3a using the state of the "heat bath," since the probability of a charge $2ek$ flowing through the shunt prior to time t will just be $w_k(t)$. Thus, by measuring the current or voltage at the shunt, or connecting a large capacitance in series with the shunt,[5)] it is possible to study experimentally the probability distribution $w_k(t)$ of quasicharge "spreading" for different values of $2ek$, as well as the quantities $\langle Q(t) \rangle$ and $\langle Q^2(t) \rangle$. When $R < R_Q$, no matter how weak the current, $\langle Q(t) \rangle$ will be given by Eq. (48a), which in the present instance is not a simple consequence of charge conservation. Naturally, charge conservation holds for each event w_k, but it obviously does not determine the actual values of $w_k(t)$ (and consequently $\langle Q(t) \rangle$), which depend only on the nature of the quantum fluctuations in the system. It must be emphasized that when $R < R_Q$, regardless of whether (48a) holds, there are in fact no Bloch oscillations (in the sense that their probability is small for sufficiently large t), in view of the strong fluctuations of quasicharge.

When $R < R_Q$, the phase difference φ is a "good" quantum number. For sufficiently small I, the mobility in φ-space is small, i.e., $\langle V \rangle \approx 0$ (stationary Josephson effect). When $E_Q \gg E_J$, we have from (44) for the voltage-current characteristic

$$\langle V \rangle = IR - \frac{\pi}{8\Gamma(2R/R_Q)} \frac{I_c^2}{Ie^2} \left(\frac{\pi R^2 IC}{eR_Q} \right)^{2R/R_Q}. \qquad (49)$$

With $R \ll R_Q [2\ln(eR_Q/\pi R^2 I_c)]^{-1}$, Eq. (49) turns into the well known expression for the voltage-current characteristic obtained when $I \gg I_c$ and Josephson oscillations are taken into account within the scope of the simple resistive model of Aslamazov and Larkin.[19] Oscillations obviously also occur for all $R < R_Q$, and Eq. (49) is a generalization of the corresponding theoretical result of Ref. 19 to the case in which

$$R \gtrsim R_Q [2\ln(eR_Q/\pi R^2 I_c)]^{-1}.$$

Finally, the results obtained here make it possible to account for the quantum behavior of superconducting junctions when thermal fluctuations are present as well. Specifically, when $E_J \gg E_Q$ and $T \gg I/e$, the current-voltage characteristic follows Ohm's law with an effective resistance, when $R_{\text{eff}} \ll R$, governed by Eq. (38a):

$$R_{\text{eff}} = \frac{\pi^{3/2}\Gamma(R_Q/R)}{4\Gamma(1/2 + R_Q/R)} \left(\frac{\Delta}{eT} \right)^2 \left(\frac{4\pi TC}{eI_c} \right)^{2R_Q/R}.$$

We note in conclusion that the simple model we have considered does not take into account the discrete nature of the tunneling of "normal" electrons, which, in addition to Bloch oscillations, leads to the occurrence of so-called single-electron oscillations,[20] as well as alters the system phase diagram.[21]

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APPENDIX 1

Following a number of simple manipulations (see Ref. 12), we can rewrite the expression for a diagonal element of the density matrix of Eqs. (1)–(6), averaged over states of the heat bath, in the form

$$\rho(x,x)=H(x,t)/N, \quad N=\int dx\, H(x,t), \tag{A1}$$

$$H(x,t)=\sum_{\substack{m,m'=0\\ m+m'=2n}}^{\infty}\left(\frac{iV_0}{2}\right)^m\left(-\frac{iV_0}{2}\right)^{m'}\sum_{\{\varkappa_{1i},\varkappa_{2j}\}}\int\prod_{i=1}^{m}dt_{1i}$$

$$\times\int\prod_{j=1}^{m'}dt_{2j}\int dx_i\, dy_i\, \rho_i\left(x_i+\frac{y_i}{2},x_i-\frac{y_i}{2}\right)G,$$

where is an easily calculated Gaussian double integral[12]:

$$G=\frac{\eta}{2\pi d(t)}\exp\Big\{im(x\dot{y}-x_i\dot{y}_i)$$
$$+i\int_0^t dt'\left[F-\frac{\nu_1+\nu_2}{2}\right]y^*(t')-S_2(y^*)\Big\}, \tag{A2}$$

where $d(t)=1-\exp(-\eta t/m)$, $S_2(y^*)$ has been defined in Ref. 12, and the function $y^*(t')$ satisfies the equation

$$m\ddot{y}^*-\eta\dot{y}^*=\nu_2(t)-\nu_1(t) \tag{A3}$$

with boundary conditions $y(0)=y_i, y(t)=0$; $\nu_{1,2}$ gives the "charge" density for $\varkappa=\pm1$ on real time curves

$$\nu_1(t)=\frac{2\pi}{q_0}\sum_{i=1}^{m}\varkappa_{1i}\delta(t-t_{1i}), \quad \nu_2(t)=\frac{2\pi}{q_0}\sum_{j=1}^{m'}\varkappa_{2j}\delta(t-t_{2j}). \tag{A4}$$

Note that the "charge" configurations $\{\varkappa_{1i},\varkappa_{2j}\}$ of (A4) do not reduce to the configurations $\{e_{1i},e_{2j}\}$ of (20). The only difference between Eqs. (A1)–(A4), which determine the temporal evolution of the probability density $\rho(x,x)$ and the corresponding equations (3.19)–(3.25) of Ref. 12 is the normalization factor N appearing in (A1). It is not difficult to show that $N\neq1$. As in Ref. 12, we represent $y^*(t')$ in the form

$$y^*(t')=y_h(t')+y_p(t'), \quad y_h(t')=\frac{y_i}{d(t)}\left\{1-\exp\left[\frac{(t'-t)\eta}{m}\right]\right\}. \tag{A5}$$

The particular solution $y_p(t')$ of (A3) which satisfies the boundary conditions $y_p(0)=y_p(t)=0$ is of the form

$$y_p(t')=\frac{q_0}{\alpha}\left[\sum_{i=1}^{m}\varkappa_{1i}\theta(t'-t_{1i})-\sum_{j=1}^{m'}\varkappa_{2j}\theta(t'-t_{2j})\right], \tag{A6}$$

where $\theta(t>0)=1$, $\theta(t<0)=0$, and $\theta(0)=1/2$. Substituting (A5) and (A6) into (A1) and (A2) and expanding N as a power series in V_0^2, we find

$$N(t)=1+\sum_{n=1}^{\infty}V_0^{2n}a_n(t),$$

where the coefficients $a_n(t)\neq0$ are independent of V_0. Calculating the mean values using the expression obtained for $\rho(x,x)$ is fairly simple only for sufficiently small times, using perturbation theory. Such calculations for the mobility give results consistent with those we have obtained and those of Ref. 12. In the more interesting long-time domain, Eqs. (A1)–(A6) are unsuitable to determine the mobility and the problem becomes extremely complex. We have circumvented this difficulty in the present paper by selecting a set of "charges" (20) different from (A4), enabling us to obtain the relations (28), (29). The normalization factor does not appear in Ref. 12, and the relation $\langle 1\rangle=1$ can be made to hold (at least in the diffusion approximation) by choosing a particular solution of (A3) which does not satisfy the vanishing boundary conditions. In fact, the function obtained by making the substitutions

$$\theta(t'-t_{i(j)})\to\theta(t'-t_{i(j)})+C_{i(j)}\theta(t_{i(j)}-t')\exp\left[(t'-t_{i(j)})\eta/m\right],$$
$$t_{i(j)}\equiv t_{1i}(t_{2j})$$

in (A6) will also be a solution of (A3) for any $C_{i(j)}$, with $y_p(0)$ differing but little from zero when $C_{i(j)}\ll\exp\{|t'-t_{i(j)}|\eta/m\}$. As we see from (A2), however, the values of $y_p(t)$ at $t=t_{i(j)}$ are important to the determination of $\rho(x,x)$.

APPENDIX 2

We shall analyze the approximations we have used by calculating the probability $W_0(t)$ appearing in Eq. (11) (the calculation of $W_k(t)$ for $k\neq0$ proceeds similarly). In determining this quantity in the diffusion approximation, we have taken into account only those configurations consisting of neutral charge-pairs. For the sake of definiteness, we assume that $\sigma\ll T$. Then the contribution of each such configuration of $2n$ instantons for $\alpha>1$, or for $\alpha<1$ and $T\gg\Gamma$, is $(-\Gamma t)^n/n!$, and the number of such configurations $N(0,n)$ is given by Eq. (32). It is not hard to show that the contribution of any of the discarded configurations is small compared with the indicated contribution of any one of the neutral-pair configurations. As we have already pointed out, however, the number of discarded configurations $M(0,n)-N(0,n)\gg N(0,n)$ for large enough n. Let us consider configurations consisting of $n-2$ neutral charge-pairs and two pairs with charge $+2$ and -2. The contribution of such configurations is of the form $C_{n1}(-\Gamma t)^n/Tt(n-1)!$, with $C_{n1}\sim1$, and their number, equal to $N(0,n-2)(n-1)/2$, $n\geqslant2$, may be determined by direct combinatorial calculation. The remaining configurations can be grouped in like manner. As a result, we have

$$W_0(t)=1-\Gamma t+\sum_{n=2}^{\infty}\frac{(-\Gamma t)^n}{n!}N(0,n)\left[1+\sum_{p=1}^{n-1}N_{np}c_{np}(Tt)^{-p}\right], \tag{A7}$$

where $C_{np}\sim1$. It is straightforward to demonstrate that $N_{np}<1$ for all n and p (for example, $N_{n1}=n(n-1)N(0,n-2)/2N(0,n), n\geqslant2$), and consequently, for $t\gg T^{-1}$, (A7) is in fact the same as the equation for $W_0(t)$ obtained in the diffusion approximation. The latter approximation is not applicable for $t\lesssim T^{-1}$, although for $\alpha>1$, or $\alpha<1$ and $T\gg\Gamma$, the probability $W_0(t)$ is close to unity for this range of t. It can be shown in a completely analogous way that when $T\ll\sigma$, the diffusion approximation is applicable for $\alpha>1$ or for $\alpha<1$ and $\sigma\gg\Gamma$.

Now let $\sigma=0$, $T=0$, and $\alpha<1$. The approximation used in Ref. 8 to determine the probability $W(t)$ (see also Eq. (39) of the present paper) was that the contribution of any instanton configuration was taken to be equal to the contribution of a configuration with alternating charges. The correction to this approximation is due to the difference between the phase factors for a number of configurations and $\pm i\pi\alpha n$. In particular, for $W_0(t)$, we have

$$W_0(t)=1-\frac{\cos(\pi\alpha)\Delta^2 t^{2(1-\alpha)}}{2\omega_0^{2\alpha}(1-\alpha)(1-2\alpha)}+\sum_{n=2}^{\infty}\left(\frac{\Delta}{2}\right)^{2n}(-1)^n\frac{t^{2n(1-\alpha)}}{\omega_0^{2\alpha}}$$
$$\times\frac{(\Gamma(1-2\alpha))^n M(0,n)}{\Gamma(2n(1-\alpha)+1)}\left[\cos^n\pi\alpha(1-M_{1n})+\sum_p^{3n-4}M_{pn}\cos p\pi\alpha\right], \tag{A8}$$

where

$$\sum_p^{3n-4}M_{pn}-M_{1n}=0,\quad \begin{array}{ll} p=2,4,\ldots 3n-4, & n=2l, \\ p=3,5,\ldots,3n-4, & n=2l+1. \end{array} \tag{A9}$$

The coefficients M_{pn} are small (for $n=2$, $M_{22}=1/9$; for $n=3, M_{53}=0.03, M_{33}=0.06$), and tend to zero for large n. For sufficiently large $t\gg\Delta(\Delta/\omega_0)^{\alpha/(1-\alpha)}$, the expression in square brackets in (A8) is close to $(\cos\pi\alpha)^n$ (except possibly for a very narrow region near $\alpha=1/2$) due to the smallness of M_{pn} and the rapid oscillations in $\cos p\pi\alpha$ at large p.

[1)]Schmid[3] was the first to indicate the existence of duality transformations like (25), relating different representations of the statistical sum for the system in question with $T\to 0$, $F\to 0$.

[2)]This result can in fact be demonstrated without resorting to renormalization group calculations. To do so, it is sufficient to make use of Eqs. (28)–(30) and (38), as well as the fact that when $\eta\gg m\omega_0$, it is possible simultaneously to satisfy (7) and $V_0 m/\eta\ll 1$ (see Fig. 2).

[3)]The mean value of the real physical charge in the junction is $C\langle\dot\varphi\rangle/e$, and this goes to zero in the absence of external current. The quasicharge concept was introduced in Ref. 18.

[4)]It is not difficult to show that the parameter $\delta^{(0)}$ of Ref. 18 is given by $\delta^{(0)}=2\Delta$.

[5)]It is interesting to note that the latter system is an exact analog of a high-inductance SQUID. The physical quantity which is the dual of the quantum of magnetic flux is then the Cooper-pair charge $2e$.

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