Point-defect scattering of bound quasiparticles as a soliton problem (one-dimensional case)

Yu. S. Kivshar', A. M. Kosevich, and O. A. Chubykalo

A. M. Gorki State University, Kharkov
(Submitted 17 December 1986)
Zh. Eksp. Teor. Fiz. **93**, 968–977 (September 1987)

The scattering of a dynamical soliton by a point defect is considered in a one-dimensional system whose excitations are described by the nonlinear Schrödinger equation. The dynamical soliton can be interpreted as a bound state of quasiparticles, and its amplitude is proportional to the number of bound quasiparticles. The soliton-scattering intensity is found and compared with the total intensity of the scattering of the equivalent number of free quasiparticles. It is shown that the soliton-scattering intensity is smaller than the free-quasiparticle scattering intensity and, in particular, tends rapidly to zero with increase of the binding energy of the quasiparticles. In the scattering process the soliton radiates free quasiparticles (linear harmonic waves) in both the forward and backward directions. In the weakly nonlinear limit in the scattering of a fast soliton free quasiparticles are radiated only in the backward direction. The scattering of a slow soliton consisting of a large number of bound particles has no analog in the scattering of free quasiparticles. The problem of the radiative capture of a slow soliton by an attractive inhomogeneity is also considered.

1. INTRODUCTION

Many kinetic characteristics of crystals are determined by the character of the scattering of the quasiparticle excitations (phonons, spin waves, etc.). Under intense external perturbations high-density fluxes of quasiparticles can arise in the crystal, and the interaction of the quasiparticles being scattered then becomes important. Allowance for this interaction is especially important when the interaction is attractive and can lead to the formation of a bound state of quasiparticles. If collisions of quasiparticles with impurities occur sufficiently infrequently, then between successive scatterings the quasiparticles have time to form a bound state. They are then scattered, by some inhomogeneity, not one by one but as a single formation or "cluster." It is natural to pose the question as to how the intensity of scattering of a single cluster of N quasiparticles compares with the total intensity of the scattering of N noninteracting quasiparticles. To answer this question it is necessary to analyze the role of the interaction of the quasiparticles in the scattering processes. This analysis is simplest in the one-dimensional situation, in which many particle-interaction problems have exact solutions. It is known that the dynamics of a system of interacting quasiparticles is described in the mean-field approximation by a certain nonlinear equation. We suppose that in such a nonlinear one-dimensional system there is a point center capable of scattering elementary excitations. In studying the scattering by such a center we suppose that the incident flux of quasiparticles is sufficiently dense for a bound state of them to be formed before the scattering event. A bound state of a large number of quasiparticles in a nonlinear system is equivalent to a dynamical soliton, and its amplitude is proportional to the number of bound quasiparticles (see, e.g., Refs. 1 and 2). We shall calculate the intensity of scattering of a soliton consisting of N quasiparticles, and compare it with the total intensity of the scattering of N free quasiparticles.

As a concrete realization of this problem we consider the scattering of a soliton by a localized (point) inhomogeneity in the framework of the nonlinear Schrödinger equation. As is well known, this equation arises in the description of the nonlinear dynamics of quasiparticles in many problems in the physics of condensed matter. For example, the excitations of a weakly nonideal Bose gas in the self-consistent field method can be described by this equation (see, e.g., Refs. 1 and 2). The analysis of the dynamics of nonlinear magnetization waves in a ferromagnet with anisotropy of the "easy axis" type,[3] like the problem of the motion of solitons along a quasi-one-dimensional molecular chain, reduces to the analysis of the soliton solutions of the nonlinear Schrödinger equation. Finally, this equation describes the dynamics of the envelope of the phonon excitations in intense pulsed action on a crystal,[4,5] and also the dynamics of surface nonlinear waves in dielectrics or semiconductors.[6]

The nonlinear Schrödinger equation, with a specially chosen scale of coordinate has the form

$$i\psi_t-\omega_0\psi+\psi_{xx}+2g|\psi|^2\psi=0, \tag{1}$$

where the parameter ω_0 determines the gap in the spectrum of the linear excitations (i.e., in the dependence of the frequency ω on the wave number k):

$$\omega=\omega_0+k^2,$$

and the parameter g is a characteristic of the pair-interaction potential of the quasiparticles.

In the following we shall consider a point impurity that locally changes the characteristic frequency ω_0. Then we must introduce into the right-hand side of (1) an additional delta-function term of the type $\delta(x)\psi$, the amplitude of which characterizes the intensity of the action of the impurity. We represent the solution of Eq. (1) in the form

$$\psi(x,\ t)=g^{-1/2}u(x,\ t)e^{-i\omega_0 t}$$

and obtain the following equation for the function $u(x, t)$:

$$iu_t+u_{xx}+2|u|^2u=\varepsilon\delta(x)u, \tag{2}$$

 0038-5646/87/090545-06$04.00 © 1988 American Institute of Physics

where the parameter ε characterizing the perturbation is assumed to be small ($\varepsilon \ll 1$).

Equation (2) with any value of the real parameter ε admits the integral of motion

$$N=\int_{-\infty}^{\infty} |u|^2\,dx, \tag{3}$$

which has the meaning of the total "number of particles."

In the linear limit ($u \ll 1$), when the third term in the left-hand side of (2) can be neglected, the problem of the scattering of a wave excitation by a point inhomogeneity can be solved trivially. This scattering is characterized by a single quantity—the wave-reflection coefficient R_0:

$$R_0=\varepsilon^2/(\varepsilon^2+4k^2), \tag{4}$$

where k is the wave number of the incident wave. The wave-transmission coefficient T_0, by definition, is equal to

$$T_0=1-R_0=4k^2/(\varepsilon^2+4k^2).$$

In the scattering of N equivalent free quasiparticles by the impurity, R_0N of them are reflected and T_0N pass through.

When the interaction of the quasiparticles, i.e., the nonlinear term in (2), is taken into account, the situation becomes more complicated, since the scattering of a cluster is characterized by a larger number of parameters. First of all, we note that in a homogeneous nonlinear system, i.e., with $\varepsilon = 0$, there exists a bound state of quasiparticles that is describable by an exact solution (of the nonlinear Schrödinger equations) in the form of a two-parameter dynamical soliton[1)].

$$u_s(x,t)=2i\eta\frac{\exp[-2i\xi x-i\Delta(t)]}{\mathrm{ch}[2\eta(x-x_0(t))]}, \tag{5}$$

$$\Delta(t)=4(\xi^2-\eta^2)t,\quad x_0(t)=-4\xi t, \tag{6}$$

where 2η is the soliton amplitude, characterizing the number of quasiparticles bound in the soliton,

$$N=4\eta, \tag{7}$$

and in this notation the soliton velocity $v = -4\xi$. The energy of the soliton (5) is equal to

$$E_s=4\eta\omega_0+16(\xi^2\eta-{}^1/_3\eta^3) \tag{8}$$

and is made up of the sum of the energies of the free particles

$$4\eta\omega_0+16\xi^2\eta=N(\omega_0+4\xi^2)=N(\omega_0+v^2/4)$$

and the energy of their interaction (attraction):

$$E_{int}=-{}^{16}/_3\eta^3=-{}^1/_{12}N^3. \tag{9}$$

The form of the energy of attraction (9) agrees with the well known property of a weakly nonideal Bose gas of particles with attraction—that it collapses as $N\to\infty$. This unphysical property of the real system can be "corrected," e.g., by including a weak three-particle repulsion between the particles.[2]

With the aid of (9) it is easy to determine the binding energy of the quasiparticles in the soliton as the minimum work needed to "extract" one particle from the soliton and carry it over into the free state. The binding energy is equal to

$$|\partial E_{int}/\partial N|={}^1/_4N^2=4\eta^2.$$

For $\eta\to 0$ the soliton (5) is delocalized and corresponds to a linear wave with wave number k, where $k^2 = 4\xi^2$.

It is well known that the unperturbed nonlinear Schrödinger equation, i.e., Eq. (2) with $\varepsilon = 0$, is exactly integrable by the method of the inverse scattering problem.[7] Therefore, the investigation of the scattering process for small ε is carried out naturally in the framework of perturbation theory based on the method of the inverse problem.[8,9]

2. RADIATION FROM A FAST SOLITON

First of all, we shall consider the scattering of a sufficiently fast soliton by an impurity (the scheme of this scattering is shown in Fig. 1 and corresponds to $\xi > 0$). By "sufficiently fast" we understand a soliton whose change in velocity upon interaction with the inhomogeneity is small and can be disregarded. The condition corresponding to this case has the form

$$\xi^2\gg|\varepsilon|\eta. \tag{10}$$

The spectral density $n(\lambda)$ of the quasiparticles generated by the soliton being scattered can be calculated using the formula of the method of the inverse problem[7] ($|b(\lambda)|^2 \ll 1$)

$$n(\lambda)=\pi^{-1}|b(\lambda)|^2,$$

where $b(\lambda)$ is the scattering coefficient (the so-called Jost coefficient) used in the method of the inverse problem, and λ is a real spectral parameter, related in a simple manner to the wave number $k(\lambda)$ and frequency $\omega(\lambda)$ of the linear waves generated[7]:

$$\omega(\lambda)=k^2(\lambda)=4\lambda^2.$$

The perturbation-induced change of the Jost coefficient $b(\lambda)$ in the case $\varepsilon\neq 0$ is determined by the well known equation[8,9]:

$$\frac{\partial b(\lambda,t)}{\partial t}=4i\lambda^2 b(\lambda,t)+\varepsilon\int_{-\infty}^{\infty}dx\,P[u]\varphi_1(x,t;\lambda)\psi_2^*(x,t;\lambda)$$
$$-\varepsilon^*\int_{-\infty}^{\infty}dx\,P^*[u]\varphi_2(x,t;\lambda)\psi_1^*(x,t;\lambda), \tag{11}$$

where, in the case under consideration, $\varphi_{1,2}(x,t;\lambda)$ and

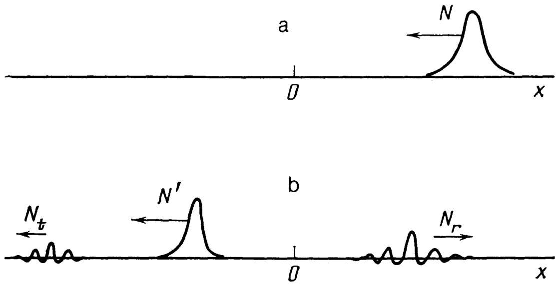

FIG. 1. Scattering of a fast soliton (the impurity is located at the point $x = 0$): a) before the scattering, b) after the scattering; N_t and N_r are the numbers of particles radiated forward and backward, respectively, and N' is the number of particles that remain bound after the scattering.

$\psi_{1,2}(x, t; \lambda)$ are the one-soliton Jost functions[7] (written out explicitly in, e.g., Ref. 9), and $\varepsilon P[u]$ is the right-hand side of Eq. (2), i.e., the perturbation, which in our case is equal to

$$\varepsilon P[u]=\varepsilon\delta(x)u.$$

We shall assume that prior to the scattering ($t\to-\infty$) all the quasiparticles formed a bound state. This implies that the initial condition for Eq. (11) should be chosen in the form $b(\lambda, t=-\infty)=0$. Then the density $n_{\rm rad}(\lambda)$ of the total number of quasiparticles radiated by the soliton in the scattering is determined by the value of the Jost coefficient $b(\lambda, t)$ for $t\to+\infty$, and is equal to

$$n_{rad}(\lambda)=\pi^{-1}|b(\lambda, t=+\infty)|^2. \tag{12}$$

The solution of Eq. (11) can be obtained most simply for a fast soliton, when the change $4|\delta\xi|$ of its velocity after scattering is small ($(\delta\xi)^2\sim\varepsilon\ll\xi^2$). Then the expression for $n_{\rm rad}(\lambda)$ acquires the form

$$n_{rad}(\lambda)=\frac{\pi\varepsilon^2}{4(2\xi)^6}\frac{[(\lambda-\xi)^2+\eta^2]^2}{\mathrm{ch}^2[\pi(\lambda^2-\xi^2+\eta^2)/4\eta\xi]}. \tag{13}$$

The spectral distribution $n_{\rm rad}(\lambda)$ is presented in Fig. 2. As follows from (13), for $\eta\ll\xi$ it has two sharply pronounced maxima (the width of which is $\sim\eta$) at the points

$$\lambda=\pm\lambda_m,\ \lambda_m=(\xi^2-\eta^2)^{1/2}\sim\xi,$$

and the estimates for the maximum values of the density are ($\eta\ll\xi$)

$$n_1\equiv n_{rad}(-\lambda_m)\sim\varepsilon^2/\xi^2,\ n_2\equiv n_{rad}(\lambda_m)\sim\varepsilon^2\eta^4/\xi^6\ll n_1.$$

For $\lambda=0$ it follows from (13) that

$$n_{rad}(0)\sim n_1\,\mathrm{sech}^2(\pi\xi/2\eta)\ll n_{1,2}.$$

Away from the points $\lambda=\pm\lambda_m$ the function $n_{\rm rad}(\lambda)$ falls off sharply, since for $|\lambda\pm\lambda_m|\gg\eta$ this function has the asymptotic form

$$n_{rad}(\lambda)\sim\frac{\varepsilon^2(\lambda^2+\xi^2)^2}{\xi^6}\exp\left[-\frac{\pi(\lambda^2-\lambda_m^2)}{2\eta\xi}\right].$$

We obtain the numbers of particles emitted by the soliton in the forward direction (N_t) and in the backward direction (N_r) (see Fig. 1) by integrating the density (13) over the corresponding range of variation of the spectral parameter:

$$N_t=\int_0^\infty n_{rad}(\lambda)\,d\lambda,\quad N_r=\int_{-\infty}^0 n_{rad}(\lambda)\,d\lambda. \tag{14}$$

Since the quantity (3) is conserved, the number of particles that remain in the bound state is equal to

$$N'=N-N_r-N_t.$$

We define the reflection coefficient R of the soliton as the ratio of the number N_r of reflected quasiparticles to the total number $N=4\eta$ of quasiparticles. From (12)–(14) we have

$$R=\frac{N_r}{N}=\frac{\pi\varepsilon^2}{2^{10}\alpha\xi^2}\int_0^\infty dx\frac{[(x+1)^2+\alpha^2]^2}{\mathrm{ch}^2[\pi(x^2+\alpha^2-1)/4\alpha]},\quad \alpha=\frac{\eta}{\xi}. \tag{15}$$

We compare (15) with the reflection coefficient (4) of the linear waves. Since in the derivation of (15) we used the inequality (10), it is necessary to simplify (4) by writing $R_0\cong\varepsilon^2/16\xi^2$. In the limit $\eta\ll\xi$ (a "light" soliton), the reflection coefficient R first increases slowly to the value $1.004R_0$ in accordance with the law

$$R\approx R_0(1+2\eta^4/15\xi^4), \tag{16}$$

but begins to decrease at $\alpha>\alpha_c\cong0.178$. In the case $\xi\ll\eta$ (a "heavy" soliton) we obtain from (15) the asymptotic formula

$$R=\frac{\pi}{16\sqrt{2}}R_0\alpha^{7/2}e^{-\pi\alpha/2},\quad \frac{\xi}{\varepsilon}\gg\alpha\gg1. \tag{17}$$

The general form of the function defined in (15),

$$R/R_0\approx(16\xi^2/\varepsilon^2)R(\eta/\xi)$$

is presented in Fig. 3. A characteristic feature of this dependence is the relatively rapid decrease of the reflection coefficient with increase of the parameter η/ξ, i.e., with increase of the relative number of quasiparticles and of their binding energy in the soliton. As $\eta/\xi\to0$ the reflection coefficient tends, as we should expect, to its linear limit (4).

We now calculate the number of quasiparticles radiated by the soliton in the forward direction. We denote by T^+ the relative number of such particles:

$$T^+=N_t/N.$$

It is not difficult to verify that the quantity T^+ is given by formula (15), in which under the integral we must make the replacement $(x+1)^2\to(x-1)^2$. The qualitative form of the dependence of the relative number of quasiparticles radiated in the forward direction is presented in Fig. 4. For large values of the parameter $\alpha=\eta/\xi$ the asymptotic form of the function $T^+(\alpha)$ coincides with the asymptotic form of the function $R(\alpha)$ and has the form (17). For $\eta\ll\xi$ the number of particles radiated in the forward direction is substantially smaller ($\sim R_0\alpha^4$):

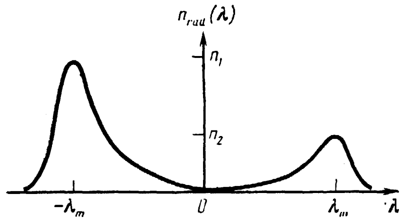

FIG. 2. Spectral distribution of the particles radiated by a fast soliton (the case $\eta\ll\xi$).

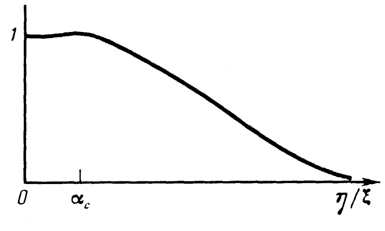

FIG. 3. Dependence of the relative number of reflected particles on the parameter η/ξ for $|\varepsilon|\eta\ll\xi^2$; the case $\eta/\xi\to0$ corresponds to the scattering of a linear wave.

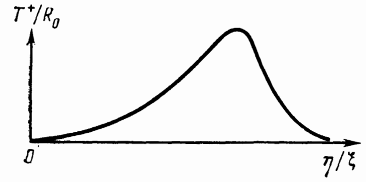

FIG. 4. Dependence of the relative number of particles radiated in the forward direction on the parameter η/ξ.

$$T^+=\varepsilon^2\eta^4/120\xi^6, \quad \eta\ll\xi.$$

We shall discuss the results briefly. First of all, in the limit $\alpha\ll 1$, when the soliton of the nonlinear Schrödinger equation is locally close to a linear wave (a "light" soliton), we have the following situation: In the principal approximation (to within terms $\sim\alpha^2$) the fast soliton radiates linear waves only in the backward direction, i.e., all the quasiparticles that have passed through the impurity remain in a bound state. Thus, the transmitted wave corresponds to a "pure" soliton.

A characteristic feature of the scattering of a "heavy" soliton ($\eta\gg\xi$) is the exponentially small radiation intensity (and exponentially small coefficients R and T^+) for large values of the parameter α. This result can be understood without difficulty from simple physical considerations: in this limit the binding energy of all the quasiparticles in the soliton ($\sim\eta^3$) is substantially greater than the kinetic energy of the soliton ($\sim\xi^2\eta$), and therefore for $\eta\gg\xi$ it is much more difficult to "tear" a quasiparticle away from the soliton than it is in the opposite case. This circumstance is manifested in the decrease of the quantities R and T^+.

3. EVOLUTION OF THE SOLITON PARAMETERS

In our treatment of the scattering of a fast soliton it was assumed that the parameters of the soliton were specified and did not depend on the perturbation This could be done in accordance with the condition (10). If we disregarded this inequality, it becomes necessary to analyze the change of the parameters of the soliton in the interaction of the soliton with the impurity. It is well known that the change induced in soliton parameters by the action of a perturbation (in our case, an inhomogeneity) can be found in the adiabatic approximation. The adiabatic approximation is based on the fact that the shape of the soliton is described as before by the expression (5) but its parameters ξ, x_0, Δ, and η are functions of the "slow" time. In our case the condition for applicability of the adiabatic approximation is the requirement that the intensity of radiation of particles during nonsteady motion of the soliton be small. One of these conditions has in fact been used in the form of the inequality (10). The other will be formulated below in the form of the inequality (27).

The following is a very simple derivation of the equations determining the time dependence of the soliton parameters. We write the Hamilton function for the perturbed system (2) under consideration:

$$H=\int_{-\infty}^{\infty}(|u_x|^2-|u|^4+\varepsilon\delta(x)|u|^2)\,dx,$$

and take as the function $u(x, t)$ the expression (5) with parameters that are as yet undetermined. A simple calculation gives

$$H=16(\xi^2\eta-{}^1\!/_3\eta^3)+4\varepsilon\eta^2/\mathrm{ch}^2(2\eta x_0). \tag{18}$$

We now assume that the relations between the canonically conjugate variables that correspond to the soliton in the unperturbed system remain unchanged after the perturbation is switched on. These canonically conjugate variables can be introduced naturally in the framework of the method of the inverse scattering problem[7]:

$$p_1=-4\xi, \quad z_1=2\eta x_0, \qquad p_2=4\eta, \quad z_2=\Delta. \tag{19}$$

If we now represent the Hamilton function (18) as a function of the canonically conjugate variables (19) and write the Hamilton equations for these variables, i.e., $dp_k/dt = -\partial H/\partial z_k$, $dz_k/dt=\partial H/\partial p_k$ ($k=1, 2$), then as a result we obtain the following equations for the initial soliton parameters:

$$d\eta/dt=0, \tag{20a}$$

$$dx_0/dt=-4\xi, \tag{20b}$$

$$d\xi/dt=-2\eta^2\varepsilon\,\mathrm{th}(2\eta x_0)\,\mathrm{sech}^2(2\eta x_0), \tag{20c}$$

$$d\Delta/dt=4(\xi^2-\eta^2)+2\eta\varepsilon\,\mathrm{sech}^2(2\eta x_0). \tag{20d}$$

If we do not impose restrictions on the amplitude ε of the perturbation, Eqs. (20) can be regarded as a consequence of the variational principle based on the Hamiltonian (18). However, it is necessary to keep in mind that Eqs. (20) are the first step in a systematic perturbation theory for solitons.[8,9]

As follows from (20a), the amplitude of the soliton in the adiabatic approximation is not changed: $\eta=\eta_0=\mathrm{const}$. The equations (20b) and (20c) are decoupled from Eq. (20d) and reduce to equations of classical mechanics, i.e., they describe the motion of a particle of unit mass with coordinate $\rho=2\eta x_0$ in the effective potential

$$u(\rho)=8\eta^3\varepsilon/\mathrm{ch}^2\rho. \tag{21}$$

We note that the strength of the effective potential (21) depends in an essential way on the amplitude of the soliton.

From the system (20b), (20c) we obtain the following laws of variation of the coordinate $x_0(t)$ in the scattering of a soliton by a defect: a) In the case $4\xi^2<\eta\varepsilon$ for $\varepsilon>0$ (reflection of the soliton from the impurity)

$$x_0(t)=-\frac{1}{2\eta}\,\mathrm{arsh}\left[\left(\frac{\eta\varepsilon}{4\xi_0^2}-1\right)^{1/2}\mathrm{ch}(8\eta\xi_0 t)\right], \tag{22}$$

b) In the case $4\xi^2>\eta\varepsilon$ for $\varepsilon>0$ and the case of any ξ^2 for $\varepsilon<0$ ("flight" of the soliton over the impurity)

$$x_0(t)=-\frac{1}{2\eta}\,\mathrm{arsh}\left[\left(1-\frac{\varepsilon\eta}{4\xi_0^2}\right)^{1/2}\mathrm{sh}(8\eta\xi_0 t)\right], \tag{23}$$

where $\xi_0\equiv\xi(t=-\infty)$. The soliton velocity $v=-4\xi(t)$ and soliton phase $\Delta(t)$ for a known dependence $x_0(t)$ are determined by the relations (20b) and (20d). For $\varepsilon<0$, oscillatory motion of the soliton about the attractive inhomogeneity is also possible (a static bound state of the soliton with the impurity is equivalent in the linear limit to so-called local oscillations[10]). However, we do not consider such solutions here, as we are interested only in processes of the scattering type.

The evolution of the soliton parameters that follows either from the dependence (22) or from (23) should be taken into account in the calculation of the radiation from the soliton in the cases when the condition $|\varepsilon|\eta \ll \xi^2$ is not fulfilled and the soliton is not fast.

4. RADIATION FROM A SLOW SOLITON

We shall calculate the radiation generated by a soliton for $\xi^2 \ll |\varepsilon|\eta$, when the kinetic energy of the soliton is comparable to the energy of its interaction with the impurity. Since the condition $|\varepsilon|\omega \ll 1$ is assumed to be fulfilled, we carry out the calculation by perturbation theory, starting from the general formula (12), where $b(\lambda, +\infty)$ represents the limiting value of the solution of Eq. (11) with the condition that $b(\lambda, -\infty) = 0$. Taking into account the dependence of the soliton parameters on the time, we can obtain the following general expression for $|b(\lambda, +\infty)|$:

$$|b(\lambda,+\infty)|=2\eta\left|\varepsilon\int_{-\infty}^{\infty} dt\frac{\exp\{-4i\lambda^2t+i\Delta(t)\}}{\mathrm{ch}[2\eta x_0(t)]}\times\left\{1+2\eta\frac{i[\lambda-\xi(t)]-\eta\,\mathrm{th}[2\eta x_0(t)]}{[\lambda-\xi(t)]^2+\eta^2}\mathrm{th}[2\eta x_0(t)]\right\}\right|. \quad (24)$$

Now, in place of $x_0(t)$, $\xi(t)$, and $\Delta(t)$ we must substitute the solutions of Eqs. (20b), (20c), and (20d).

Before considering the radiation from a slow soliton, we note that the use of Eqs. (20) for calculating the reflection coefficient R of a fast soliton with allowance for the condition (10) gives a correction of order ε^4/ξ^4 to the result (15). We have verified that in the limit $\eta/\xi \to 0$ this correction coincides with the next (not taken into account previously) term of the expansion of the result (4) of the linear approximation.

We turn now to the study of the process of the scattering of a slow soliton. But first we discuss the conditions for applicability of our perturbation theory in the case when the soliton parameters become comparable to the small quantity $|\varepsilon|$. The energy (18) of a soliton interacting with a point impurity can be represented in the form

$$E=E_s+U_{int}, \quad U_{int}=4\varepsilon\eta^2/\mathrm{ch}^2(2\eta x_0), \quad (25)$$

where $U_{\rm int}$ is the energy of the interaction of the soliton with the inhomogeneity: $U_{\rm int} \sim \varepsilon\eta^2$, and the first term gives the energy (8) of the soliton before the scattering. The conditions for applicability of the perturbation theory can be obtained either from the requirement that the kinetic energy of the soliton be much greater than the energy of its interaction with the impurity,

$$\xi_0^2 \gg |\varepsilon|\eta, \quad (26)$$

or from the requirement that the binding energy is greater than the interaction energy:

$$\eta \gg |\varepsilon|. \quad (27)$$

The condition (26) in the form (10) was used by us in Sec. 2 in the calculation of the radiation from a fast soliton. The second condition, i.e., the condition (27), becomes important in the study of the scattering of slow solitons, for which $\xi_0^2 \ll |\varepsilon|\eta$.

Unlike the scattering of free quasiparticles, the process of the interaction of a bound state of quasiparticles with an impurity depends in an essential way on the type of local inhomogeneity, i.e., on the sign of the parameter ε [compare (4) with (22) and (23)]. We first consider the case $\varepsilon > 0$ (a repulsive impurity), when the adiabatic dynamics of the soliton is characterized by the law of motion (22), describing the reflection of the soliton from the impurity. This means that the reflected wave will contain not only free quasiparticles but also a bound state of a certain number of quasiparticles (a reflected soliton). For the case (27), using the formulas (24), (22), (12), and (14), we obtain

$$R=\frac{N'+N_r}{N}\approx 1-\frac{\pi\varepsilon}{2\eta}\left(\frac{\xi_0}{2\eta}\right)^{1/2}\exp\left(-\frac{\pi\eta}{2\xi_0}\right). \quad (28)$$

The result (28) differs in an essential way from (15) and, in particular, contains the small parameter ε to the first power. For $\eta \gg \xi_0$, as in the case of the fast soliton (see Sec. 2), the radiation turns out to be exponentially small.

The interaction of a bound state of quasiparticles with an attractive ($\varepsilon < 0$) inhomogeneity occurs in a different manner. In this case, in particular, capture of the slow soliton by the impurity can occur. We shall consider the limiting case $\xi_0 \to 0$. According to (23), for $\xi_0 = 0$ the change of the soliton coordinate x_0 is determined by the expression

$$\mathrm{sh}\,[2\eta x_0(t)]=-4\eta(\eta|\varepsilon|)^{1/2}t. \quad (29)$$

Calculating $|b(\lambda, +\infty)|$ from the general formula (24) with the aid of (29), we obtain the reflection coefficient R in the form

$$R=\frac{\pi^{1/2}}{16\cdot 2^{1/2}}\left(\frac{|\varepsilon|}{\eta}\right)^{3/4}\exp\left\{-\frac{2\eta^{1/2}}{|\varepsilon|^{1/2}}\right\}. \quad (30)$$

The result (30) has been written for the case $|\varepsilon| \ll \eta$.

It is not difficult to verify by direct calculation that the number N_r of free particles radiated from the soliton in the backward direction and the number N_t of particles radiated in the forward direction coincide in this approximation.

The energy carried away by the free quasiparticles, after subtraction of their rest energy, can be obtained if we make use of the well-known formula of the method of the inverse problem[7]:

$$E_{rad}\approx\frac{4}{\pi}\int_{-\infty}^{\infty}\lambda^2|b(\lambda,t=+\infty)|^2d\lambda. \quad (31)$$

Substituting $|b(\lambda, +\infty)|$ from (24) into (31) and taking (29) and (30) into account, we obtain the following asymptotic dependence for $|\varepsilon| \ll \eta$:

$$E_{rad}\approx 2RN\eta^2(|\varepsilon|/\eta)^{1/2}, \quad (32)$$

where R is given in (30). It follows from (32) that the energy carried away by one quasiparticle is equal to $\eta^2(|\varepsilon|/\eta)^{1/2} \ll \eta^2$, i.e., is considerably smaller than the binding energy of the quasiparticle. In other words, the energy carried away by free particles is considerably smaller than the work expended on extracting them from the soliton.

The results (30) and (32) make it possible to go over to the analysis of the problem of the radiative capture of a soliton (see the similar problem for a kink of the sine-Gordon equation in Ref. 11). We suppose that a soliton with low velocity ($\xi = \xi_0 \ll 1$ at $t = -\infty$) is incident on an impurity. We find that limiting value ξ_0^l for which the soliton completely loses its kinetic energy as $t \to +\infty$ as a result of the radiation of free quasiparticles ($\xi(+\infty) = 0$). We use the laws of conservation of the energy and number of particles. The energy of the system as $t \to +\infty$ contains, besides the energy of the bound state (the soliton) at rest, the contribution of the radiated energy carried away by the free quasiparticles; i.e., we can write

$$16(\xi_0^2\eta - {}^1/_3\eta^3) = E_{rad} - {}^{16}/_3(\eta+\delta\eta)^3, \tag{33}$$

where $\delta\eta$ is the change in the amplitude in the soliton on account of the radiative losses. From the condition of conservation of the number of particles it follows that

$$4\eta = 4(\eta+\delta\eta) + N_{rad}, \tag{34}$$

where

$$N_{rad} = N_r + N_t \approx 2N_r = 2RN.$$

Since the kinetic energy of the radiated particle is considerably smaller than its binding energy in the soliton, we can omit the quantity E_{rad} in (33). Subsequent simplifications of the relations (33) and (34) and an elementary calculation lead to the following expression for the limiting value ξ_0^l:

$$(\xi_0^l)^2 = 2R\eta^2. \tag{35}$$

For $\xi_0 < \xi_0^l$ the soliton is captured by the attractive point inhomogeneity, i.e., localization of the bound state of quasiparticles occurs. The result (35) has a simple physical meaning: If we multiply (35) by $4N$, where N is the number of quasiparticles bound in the soliton, then in the left-hand side we obtain the kinetic energy of the incident soliton, and in the right-hand side we obtain the work necessary for the detachment of $2RN$ particles (we recall that $N_t \cong N_r$).

The authors are grateful to V. M. Agranovich and A. S. Kovalev for discussion of the results of the paper.

[1)] We use the notation adopted in the monograph Ref. 7.

[1]J. B. McGuire, J. Math. Phys. **5**, 622 (1964); F. Calogero and A. Degasperis, Phys. Rev. A **11**, 265 (1975).
[2]A. S. Kovalev and A. M. Kosevich, Fiz. Nizk. Temp. **2**, 913 (1976) [Sov. J. Low Temp. Phys. **2**, 449 (1976)].
[3]A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, Nonlinear Magnetization Waves. Dynamical and Topological Solitons [in Russian], Naukova Dumka, Kiev (1983).
[4]V. Narayanamurti and C. M. Varma, Phys. Rev. Lett. **25**, 1105 (1970).
[5]F. D. Tappert and C. M. Varma, Phys. Rev. Lett. **25**, 1108 (1970).
[6]T. Sakuma and Y. Kawanami, Phys. Rev. B **29**, 869, 880 (1984).
[7]S. P. Novikov, S. V. Manakov, L. P. Pitaevskiĭ, and V. E. Zakharov, *Theory of Solitons. The Inverse Scattering Method*, Consultants Bureau, New York (1984) [Russ. original, Nauka, Moscow (1980)].
[8]V. I. Karpman and E. M. Maslov, Zh. Eksp. Teor. Fiz. **73**, 537 (1977); **75**, 504 (1978) [Sov. Phys. JETP **46**, 281 (1977); **48**, 252 (1978)].
[9]V. I. Karpman, Phys. Scr. **20**, 462 (1979).
[10]A. M. Kosevich, Principles of Crystal-Lattice Mechanics [in Russian], Nauka, Moscow (1972).
[11]B. A. Malomed, Physica D **15**, 385 (1985).

Translated by P. J. Shepherd