Langevin description of mesoscopic fluctuations in disordered media

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A procedure, analogous to the Langevin scheme of calculating thermodynamic fluctuations, is proposed for measuring mesoscopic fluctuations of the resistance of disordered metallic samples and of the fluctuations of the optical transparency of elastically scattering media. It is shown that the magnitude of the fluctuations of the conductance of small metallic samples, measured by a four-probe procedure, depends on the sizes of the measurement contacts, and when these sizes are decreased the fluctuations can greatly exceed $e^2/\hbar$. The relative fluctuations of the transparency of mesoscopic samples greatly exceeds the relative fluctuations of the conductance.

I. INTRODUCTION

This paper is devoted to the study of the current-density spatial fluctuations produced in disordered metals by an electric field and due to random arrangement of the impurities in the sample (mesoscopic fluctuations), and also to the analogous light-energy flux fluctuations produced in inelastically scattering media irradiated by coherent light.

These fluctuations are calculated by a scheme reminiscent of the Langevin scheme of calculating thermodynamic fluctuations.[1–4] This scheme can be used because the spatial fluctuations of the current density in disordered metals (or of the energy-light flux energy in turbid media) can be divided into microscopic (over scales shorter than the mean free path l) and diffusional (over scales longer than l). The microscopic current-density fluctuations, averaged over a spatial scale of order l, play the role of extraneous currents and diffusion equations, which describe fluctuations over diffusional scales.

The difference between the description of mesoscopic fluctuations of the conductivity in metals from that of optical transparency of elastically scattering media is due to the different definitions of their extraneous-current correlators.

In the description of mesoscopic conductivity fluctuations in disordered metals, the extraneous currents are proportional to the gradient of the electrochemical potential. In this case, the use of the Langevin scheme makes the calculations easier than in the usual diagram technique in those situations in which the electric field is not uniform (e.g., in point contacts), and in samples of complicated geometry. In particular, such a scheme permits construction of a theory of the measurement of mesoscopic fluctuations in disordered metallic systems, i.e., identification of the quantity measured in the experiment.

It is shown in Refs. 5 and 6 that conductance fluctuations in weakly disordered ($p_F l \gg \hbar$ metallic samples are of the order of

$$\langle(\delta G)^2\rangle = \langle(G-\langle G\rangle)^2\rangle \approx e^4/\hbar^2.$$

Here $\langle\ldots\rangle$ denotes averaging over the realizations of the random potential (i.e., over the samples), and p_F is the Fermi momentum in the metal. The conductance G was measured here by a two-probe method.

In experiment,[7,8] however, the conductance is most frequently measured by a four-probe method, with two contacts used to set the current flowing through the sample, and the electrochemical-potential difference picked off the other two contacts (Fig. 1).

We shall show that in this case the fluctuations of the electrochemical potential across the measuring contacts is determined by gigantic macroscopic fluctuations averaged over the contact dimensions. The quantity $\langle(\delta\tilde{G})^2\rangle$ measured by the four-probe method should therefore depend strongly on the sizes of the measuring contacts, so that when these sizes are decreased the fluctuations of G can greatly exceed $e^2/\hbar$. Finally, it is possible to measure in experiment the conductances G_1 and G_2 of the point contacts C_1 and C_2 themselves (see Fig. 1). It will be shown below that there a correlation exists between the quantities δG_1 and δG_2 and falls off as a power law with increasing distance between the contacts.

Another possible use of the Langevin scheme proposed here is for investigations of mesoscopic fluctuations of the optical transparency of elastically scattering media.

If a coherent light beam is incident on an elastically scattering medium, the intensity of the scattered light fluctuates as a function of the location of the scatterers if the scattering angle is given, or as a function of the scattering angle if the scatterer locations are given. The interference pattern resulting from the scattering is called a speckle. A fairly complete theory was developed for speckles produced as a result of single scattering.[9]

We develop in the present paper a theory for speckles resulting from the passage of light through a multiply scattering medium. The extraneous currents are in this case proportional to the average density of the light energy at the given point (and not to the gradient of the electrochemical potential as in the case of metal-conductance fluctuations). As a result, the relative fluctuations of the optical transparency of mesoscopic samples turns out to be much larger than the relative fluctuations of the conductance of metallic samples. This demonstrates the difference between the mechanisms that give rise to extraneous currents in the two cases discussed above.

II. TRANSPARENCY FLUCTUATIONS OF DISORDERED MEDIA

a. We begin with a study of stationary spatial fluctuations of the density and of the light-flux energy, which are produced when a stationary coherent electromagnetic wave having an energy flux density $\mathbf{J}_0$ and a wavelength λ is incident on the interface of an elastically scattering medium

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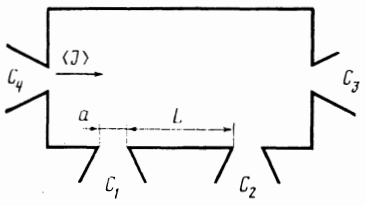


FIG. 1.

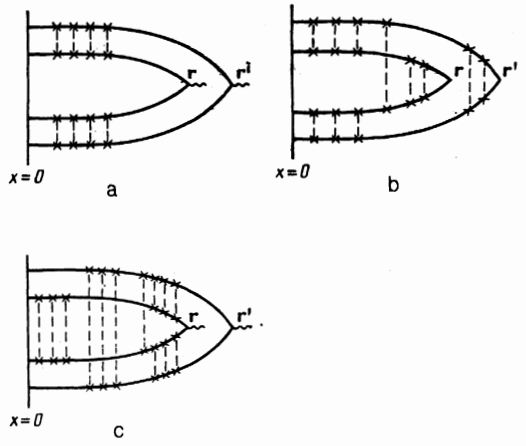


FIG. 2. The plane $x = 0$ corresponds to the boundary of the medium on which the light is incident.

with vacuum. The geometric dimensions of the medium are assumed much larger than the mean free path l. In the case $l \gg \lambda$ we can find the density and flux of the light energy in the medium, averaged over the realizations of a random potential, by using the kinetic equation, whose solution shows that the energy flux $\langle \mathbf{J} \rangle$ and the energy density $\langle n(\mathbf{r}) \rangle$ averaged over the realizations of the random potential vary smoothly as functions of the coordinates over scales exceeding l. If $l \gg \lambda$ the density and flux of the light energy in the medium, both averaged over the realizations of the random potential can be found by using the kinetic equation, whose solution shows that the energy flux $\langle \mathbf{J} \rangle$ and energy density $n\langle \mathbf{r} \rangle$ vary smoothly with the coordinates over scales exceeding l. To describe the spatial fluctuations of the energy flux in the scattering medium, averaged over scales larger than the mean free path l, we use a scheme similar to Langevin's scheme of calculating thermodynamic fluctuations.[2,3] The fluctuations over small scales assume here the role of random extraneous fluxes $\mathbf{J}_{ext}(\mathbf{r})$ in the diffusion equations

$$\operatorname{div} \mathbf{J} = 0, \tag{1}$$

$$\mathbf{J} = -D\nabla n + \mathbf{J}_{ext}(\mathbf{r}). \tag{2}$$

Here $D = lc/3$ is the diffusion coefficient and c the speed of light.

Equations (1) and (2) must be supplemented by the usual boundary conditions for the diffusion equations; they describe then the spatial fluctuations of the current and coincide with the equations that describe thermodynamic fluctuations.[1-4] The difference lies in the definition of the microscopic random currents $\mathbf{J}_{ext}$. In our case the correlator of the random extraneous currents is obtained by summing the diagrams of Fig. 2a, and at $|\mathbf{r} - \mathbf{r}'| \gg l$ we have

$$\langle J_{ext\,i}(\mathbf{r}) J_{ext\,j}(\mathbf{r}') \rangle = {}^2/_3 \pi l \lambda^2 c^2 \langle n(\mathbf{r}) \rangle^2 \delta_{ij} \delta(\mathbf{r} - \mathbf{r}'). \tag{3}$$

Equations (1)–(3) are valid at an arbitrary sample geometry, and $\langle n(\mathbf{r}) \rangle$ is the solution of the diffusion equations (1) and (2) without extraneous currents $\mathbf{J}_{ext}(\mathbf{r})$.

The phenomenological scheme (1)–(3) is based on the assumption that spatial fluctuations of the energy fluxes and densities resulting from random interference of the waves scattered by the randomly distributed scatterers can be grouped into microscopic fluctuations $|\mathbf{r} - \mathbf{r}'| < l$ and diffusional ones $|\mathbf{r} - \mathbf{r}'| > l$. To check on this assumption we obtained all the results of this study by using the much more complicated diagram-summation procedure. To determine the transparency of an elastically scattering medium it was necessary to sum the diagrams shown in Fig. 2b, which describe electron diffusion at $|\mathbf{r} - \mathbf{r}'| > l$.

The qualitative interpretation of Eq. (3) is the following. The light-energy density at some point $\mathbf{r}$ inside the medium is the result of random interference of waves scattered near the point $\mathbf{r}$ within a volume on the order of l^3. The result of this random interference are spatial fluctuations of the density and flux of the light energy of the medium.

The fluctuations of the microscopic density and flux energy from point to point are of the order of the values of these quantities:

$$\delta n \sim \langle n \rangle, \quad |\delta \mathbf{J}| \sim c\delta n,$$

where the direction on the vector $\delta \mathbf{J}(\mathbf{r})$ is random.

The correlation functions of these fluctuation functions decrease with distance over a characteristic scale λ all the way to $|\mathbf{r} - \mathbf{r}'| < l$:

$$\langle \delta n(\mathbf{r}) \delta n(\mathbf{r}') \rangle = \frac{\langle n(\mathbf{r}) \rangle^2}{2} \left(\frac{\lambda}{|\mathbf{r} - \mathbf{r}'|} \right)^2, \quad \lambda < |\mathbf{r} - \mathbf{r}'| < l, \tag{4}$$

$$\langle \delta J_i(\mathbf{r}) \delta J_j(\mathbf{r}') \rangle = c^2 \langle n(\mathbf{r}) \rangle^2 \frac{\lambda^2 (\mathbf{r} - \mathbf{r}')_i (\mathbf{r} - \mathbf{r}')_j}{(\mathbf{r} - \mathbf{r}')^4}, \quad \lambda < |\mathbf{r} - \mathbf{r}'| < l. \tag{5}$$

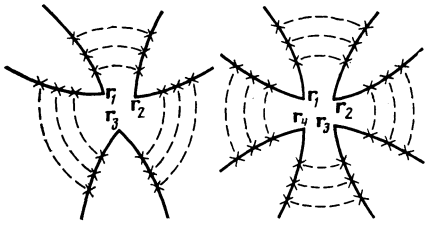
Equations (4) and (5) are obtained by averaging the diagrams of Fig. 2a. It is the averaging of (5) over a spatial scale of order l which leads to Eq. (3).

The use of the Langevin scheme (1) and (2) greatly facilitates also the calculation of the higher correlators of the quantities δn and $\delta \mathbf{J}$.

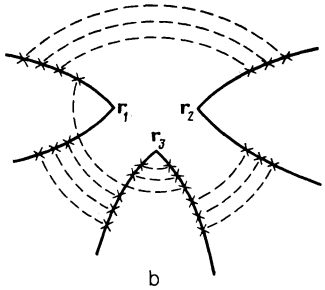
To this end it is necessary to specify higher correlations of the extraneous fluxes. The irreducible parts (cumulants) of these correlators are used by summing diagrams of the type of Fig. 3a and averaging over a scale of order l:

$$\langle J_{ext\,i_1}(\mathbf{r}_1) \dots J_{ext\,in}(\mathbf{r}_n) \rangle_c = -\frac{1}{2\pi l \lambda^2} [4\pi c \langle n(\mathbf{r}_1) \rangle l \lambda^2]^n \times \delta(\mathbf{r}_1 - \mathbf{r}_2) \dots \delta(\mathbf{r}_{n-1} - \mathbf{r}_n) \times \frac{\partial^{2n}}{\partial z^n \partial \omega_{i1} \dots \partial \omega_{in}} z^{1/2} \frac{\operatorname{sh}|\omega|}{|\omega|} \Big|_{\omega=0,\, z=1}. \tag{6}$$

Figure 3b shows by way of example the diagrams that must



a



b

FIG. 3.

be summed to calculate $\langle J_{i1}(\mathbf{r}_1)\ldots J_{in}(\mathbf{r}_n)\rangle$.

The expression obtained agrees with the known fact[9] that, in a speckle, the Poisson distribution functions of the microscopic density (F) and the light energy (P) are given by

$$F(n)=\langle n(\mathbf{r})\rangle^{-1}\exp\{-n/\langle n(\mathbf{r})\rangle\}, \tag{7}$$

$$P(\mathbf{J})=\{4\pi\mathcal{J}^2|\mathbf{J}-\langle\mathbf{J}\rangle|\}^{-1}\exp\{-|\mathbf{J}-\langle\mathbf{J}\rangle|/\mathcal{J}\},$$

$$\mathcal{J}=c\langle n(\mathbf{r})\rangle/2\cdot 3^{1/2}. \tag{7a}$$

b. We consider now the transparency of a scattering medium comprising a right parallelogram with dimensions $L_x\times L_y\times L_z$ ($L_x, L_y, L_z\gg l$). We assume the faces parallel to the x axis to be strongly reflecting. The light wave is incident on the medium in the x direction. The transparency $\langle I\rangle$ averaged over the realizations of the random potential (i.e., over the set of samples) is, in the zeroth approximation in the parameter $\lambda/l\ll 1$

$$\langle I\rangle=l/L_x. \tag{8}$$

To calculate the transparency fluctuations we can use Eqs. (1)–(3) with the following boundary conditions: the normal components of the energy fluxes on the faces parallel to the x axis are zero, and δn is zero on the faces that are perpendicular to the axis and through which the current flows. Using these boundary conditions, it is easy to show that the total fluctuational current through the cross section L_yL_z of the parallelepiped is

$$\delta i_x=L_x^{-1}\int_v J_{\text{ext}\,x}(\mathbf{r})\,d\mathbf{r}. \tag{9}$$

The integration in (9) is over the volume $v=L_xL_yL_z$ of the parallelepiped. This yields an equation for the transparency fluctuations

$$\langle(\delta I)^2\rangle=\langle(I-\langle I\rangle)^2\rangle=\frac{1}{J_0^2v^2}\int_v\langle J'_{\text{ext}\,x}(\mathbf{r})J_{\text{ext}\,x}(\mathbf{r}')\rangle\,d\mathbf{r}\,d\mathbf{r}'. \tag{10}$$

Recognizing that

$$\langle n\rangle=\frac{J_0}{c}\frac{L_x-x}{L_x},$$

we obtain from (3)

$$\langle J_{\text{ext}\,i}(\mathbf{r})J_{\text{ext}\,j}(\mathbf{r}')\rangle=6\pi J_0^2l\lambda^2\left(\frac{L_x-x}{L_x}\right)^2\delta(\mathbf{r}-\mathbf{r}')\delta_{ij}. \tag{11}$$

Substituting (11) in (10) we arrive at an expression for the fluctuations of a medium that is totally transparent

$$\left\langle\left(\frac{\delta I}{\langle I\rangle}\right)^2\right\rangle=\frac{2\pi\lambda^2L_x}{lL_yL_z}. \tag{12}$$

Using (6), we can show that in the principal order in the parameter $\lambda/l\ll 1$ the distribution of the total optical transparency of a mesoscopic sample is Gaussian as long as $\delta I\ll\langle I\rangle$.

If the wave incident on the sample is of duration $\tau\ll\tau_0=L_x^2/D$, then

$$\langle J_{\text{ext}\,i}(\mathbf{r},t)J_{\text{ext}\,j}(\mathbf{r}',t')\rangle=\frac{\pi l}{3}(c\lambda)^2\langle n(\mathbf{r},t)\rangle^2\tau\delta(\mathbf{r}-\mathbf{r}')\delta_{ij}\delta(t-t'),$$

$$\frac{\langle(\delta I_\tau)^2\rangle}{\langle I_\tau\rangle^2}\approx\frac{\lambda^2L_x}{lL_yL_z}\frac{\tau}{\tau_0}. \tag{13}$$

Here I_τ is the time-averaged sample transparency.

The interpretation of the foregoing results is similar to that in Refs. 10 and 11.[1)] The transparency of a sample under stationary illumination is determined by energy levels located in an energy band of order $\hbar/\tau_0$ near the frequency of the incident light. Their number in the sample is of the order of $N\sim\nu v\hbar/\tau_0$, where ν is the density of states. The contribution of each level to I fluctuates from realization to realization by roughly a factor of two. In this case

$$\langle(\delta I)^2\rangle/\langle I\rangle^2\approx N^{-1}$$

in accord with (12). If $\tau\ll\tau_0$ the sample transparency is determined by the quantum levels in an energy band $\hbar/\tau$, and the energy sublevels of width $\hbar/\tau_0$ make independent contributions to δI, so that

$$\langle(\delta I)^2\rangle/\langle I\rangle^2\approx N^{-1}\tau/\tau_0$$

in accord with (13).

c. The fluctuations considered above can be observed, in analogy with Refs. 12 and 13, by varying the incidence angle θ of the light on the sample. A plot of $\langle\delta I(\theta)\delta I(\theta')\rangle$ where $\delta=\theta-\theta'$, is shown in Fig. 4. The correlator $\langle J_{\text{ext}\,i}(\mathbf{r},\theta)J_{\text{ext}\,j}(\mathbf{r}',\theta')\rangle$ needed in this case is given by the diagrams 2a. The Green's functions for the outer and inner loops correspond to the incidence angles θ and θ', respectively. At $\delta\gg\lambda/l$ the value of $\langle J_{\text{ext}\,i}(\mathbf{r},\theta)J_{\text{ext}\,j}(\mathbf{r}',\theta')\rangle$ is determined by the diagram 2c, which is the analog of the diagrams that give the conductance fluctuations of mesoscopic samples.[5,6]

A dependence similar to that plotted in Fig. 4 is obtained if $I(\theta)$ is taken to mean the intensity, per unit incidence-light intensity, of the light scattered into an angle θ, given the incidence angle.

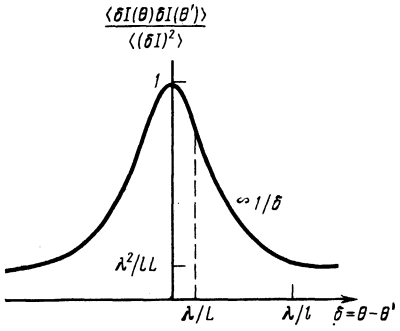


FIG. 4.

Finally, the diffusion of the scatterers in the sample leads to random oscillations of I with time. It is shown in Refs. 14 and 15 that if a typical scatterer is displaced a distance of order λ in a time of order τ, a complete change of the realization takes place within a time $\tau_2 = (l^2/L_x^2)\tau_1 \ll \tau_1$, meaning that the speckle has an anomalous sensitivity to changes in the scattering positions.

To conclude this section we emphasize that the discussed fluctuations of the optical transparency differ in character from the mesoscopic conductance fluctuations considered in Refs. 5 and 6. In the latter was obtained at $T = 0$ the result $\langle (\delta G)^2 \rangle = e^4/\hbar^2$, i.e.

$$\langle (\delta G/\langle G \rangle)^2 \rangle \approx 1/N^2 \ll \langle (\delta I/\langle I \rangle)^2 \rangle \approx 1/N. \tag{14}$$

It is seen from (14) that the relative transparency fluctuations are much larger than the relative conductance fluctuations. This result could be deduced from the Landauer formula[16] that relates G with the transparency $I(\theta)$ averaged over the angles θ of electron incidence on the surface of a disordered medium. As shown in Sec. IIc, the correlator $\langle J_{\text{ext}\,i}(\mathbf{r},\theta) J_{\text{ext}\,j}(\mathbf{r}',\theta') \rangle$ is a rapidly decreasing function of $\delta = \theta - \theta'$. As a result, the contributions from diagrams 2a and 2b (i.e., of the extraneous currents proportional to $\langle n(\mathbf{r}) \rangle$) to $\langle (\delta G)^2 \rangle$ turn out to be small compared with whose of diagrams 2c in terms of the parameter $\lambda/l \ll 1$. The contributions of these diagrams are independent of θ and are, roughly speaking, proportional to $(\nabla \langle n(\mathbf{r}) \rangle)^2$. It is just the diagrams 2c which are the analogs of the diagrams summed in study (Refs. 5 and 6) of mesoscopic fluctuations of a linear response to an external electric field.

III. FLUCTUATIONS OF THE RESISTANCE OF MESOSCOPIC SAMPLES

a. The sizes of the current-density fluctuations produced in disordered metallic samples by an external electric field can also be obtained from Eqs. (1) and (2). In this case $n(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$ have the meaning of the electron density and of the current density, while the correlator of the extraneous current is determined by averaging the microscopic currents over spatial scales of order l:

$$\langle J_{\text{ext}\,i}(\mathbf{r}) J_{\text{ext}\,j}(\mathbf{r}') \rangle = K_1 + K_2 + K_3. \tag{15}$$

We have resolved the extraneous-current correlator into three terms corresponding to different physical mechanisms and represented by diagrams a, b, and c of Fig. 5:

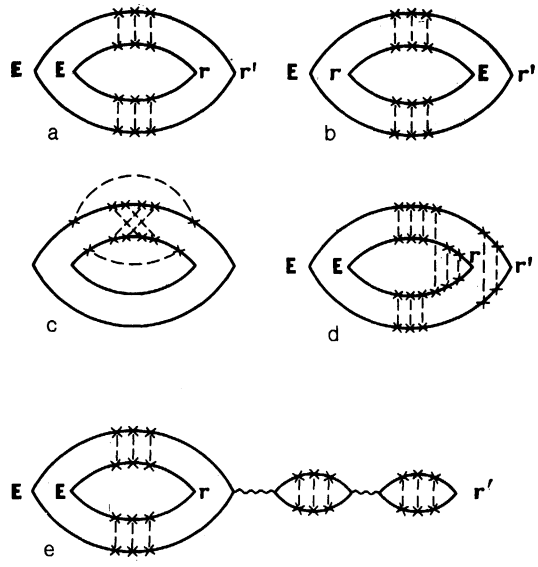


FIG. 5.

$$K_1 = (4e^2D)^2 \delta_{ij} \delta(\mathbf{r}-\mathbf{r}') \int d\mathbf{r}_1 \Gamma(\mathbf{r}, \mathbf{r}_1) \mathbf{E}^2(\mathbf{r}_1),$$

$$\Gamma(\mathbf{r}, \mathbf{r}_1) = \int \frac{d\varepsilon_1\, d\varepsilon_2}{(2\pi)^2} f'(\varepsilon_1) f'(\varepsilon_2) D_{\varepsilon_{12}}(\mathbf{r}, \mathbf{r}_1) D_{\varepsilon_{21}}(\mathbf{r}_1, \mathbf{r}). \tag{16}$$

Here $D_\varepsilon(\mathbf{r},\mathbf{r}_1)$ is the Green's function of the equation

$$(i\varepsilon - D\nabla^2) D_\varepsilon(\mathbf{r}, \mathbf{r}_1) = \delta(\mathbf{r}-\mathbf{r}_1),$$

$f(\varepsilon) = [\exp(\varepsilon/T) + 1]^{-1}$ is the Fermi function, and $\varepsilon_{12} = \varepsilon_1 - \varepsilon_2$. Expression (16) was obtained by summing diagrams 5a and is valid at arbitrary temperature and arbitrary spatial distribution of the electric field intensity $\mathbf{E}(\mathbf{r})$, if the variation scales of $\mathbf{E}(\mathbf{r})$ turn out to be larger than l. In the case of a homogeneous electric field we have

$$K_1 = \delta_{ij} \delta(\mathbf{r}-\mathbf{r}') (eE/\pi^2\hbar)^2 L_0, \tag{17}$$

and the values of L_0 for different limiting cases are

$$L_0 = \begin{cases} \sim L_x, & L_x \approx L_y \approx L_z \ll L_T \\ \sim L_x^2/L_z, & L_z \ll L_x \approx L_y \ll L_T \\ L_T, & L_T \ll L_z, L_y, L_x \\ \sim \dfrac{L_T^2}{L_z} \ln \dfrac{L_\varphi}{L_T}, & L_z \ll L_T \ll L_y, L_x \end{cases} \tag{18}$$

Here $L_T = (D/T)^{1/2}$ is the coherence length of the normal metal, and $L_\varphi = (D/\tau_\varphi)^{1/2}$ is the electron phase-loss length. Attention is called to the gigantic values of these currents, which are proportional to L_0.

The physical cause of these fluctuations is that the response of the fluctuational current at a point $\mathbf{r}$ to an electric field at a point $\mathbf{r}'$ has at $T = 0$ a long-range character[17]:

$$J_i(\mathbf{r}) = \int \sigma_{ik}(\mathbf{r}, \mathbf{r}') E_k(\mathbf{r}')\, d\mathbf{r}', \tag{19}$$

whereas

$$\langle \sigma_{ik}(\mathbf{r}, \mathbf{r}') \rangle = \frac{1}{4\pi} \frac{\sigma_D (\bar{\mathbf{r}}-\bar{\mathbf{r}}')_i (\mathbf{r}-\mathbf{r}')_k}{|\mathbf{r}-\mathbf{r}'|^4 l} \exp\left\{ -\frac{|\mathbf{r}-\mathbf{r}'|}{l} \right\}. \tag{20}$$

Here $\sigma_{ik}(\mathbf{r},\mathbf{r}')$ is the nonlocal conductivity, defined by the equation

$$J_i(\mathbf{r})=\int \sigma_{ik}(\mathbf{r},\mathbf{r}')E_k(\mathbf{r}')d\mathbf{r}'.$$

We emphasize that in this case the fluctuational currents turn out to be much larger than the average current $\langle\mathbf{J}\rangle=\sigma_D\mathbf{E}$. Here $\sigma_D=e^2D\nu$ is the Drude and Lorentz conductivity and ν is the density of states on the Fermi level.

It follows from (19) that in an infinite sample placed in a uniform electric field, integration with respect to r' in accordance with (17) makes the fluctuational response $\langle(\delta\mathbf{J})^2\rangle$ proportional to L_0 and causes it to tend to infinite as $T\to 0$. The connection between the existence of such currents and the ergodicity problem was discussed in Ref. 18.

From (1), (2) and (17) we obtain in an infinite system with constant $\mathbf{E}$, if $|\mathbf{r}-\mathbf{r}'|>l$,

$$\langle\delta n(\mathbf{r})\delta n(\mathbf{r}')\rangle=\nu^2\langle\delta\mu(\mathbf{r})\delta\mu(\mathbf{r}')\rangle=2\nu^2(eE\lambda)^2\frac{L_T\lambda^2}{l^2|\mathbf{r}-\mathbf{r}'|},\tag{21}$$

$$\langle\delta J_i(\mathbf{r})\delta J_j(\mathbf{r}')\rangle$$

$$=8\left(\frac{e^2E}{2\pi^2\hbar}\right)^2 L_T\left\{\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')-\frac{1}{4\pi}\frac{\partial^2}{\partial r_i\partial r_j}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\}.\tag{22}$$

Here $\delta\mu(\mathbf{r})$ are the fluctuations of the chemical potential in the metal.

The same expressions were obtained by us by summing diagrams a and d of Fig. 5.

Expressions (21) and (22) were obtained for a model in which the electrons interact with an external field but not with one another. The scheme described, however, easily admits of introduction of fluctuational electric fields that result from fluctuations of the electron density. Equation (2) must then be replaced by

$$\mathbf{J}(\mathbf{r})=\sigma_D\mathbf{E}(\mathbf{r})-eD\nabla n+\mathbf{J}_{c\tau}(\mathbf{r})=-eD\nu\nabla\eta+\mathbf{J}_{c\tau}(\mathbf{r}),\tag{23}$$

and the Poisson equation

$$\operatorname{div}\mathbf{E}=4\pi e\delta n\tag{24}$$

must be added to (1) and (23). We have used here the Einstein relation, $\eta=\mu+e\varphi$ is the electrochemical potential, and φ is the scalar electric potential. This fact can be proven by summing diagrams d and e of Fig. 5, where the wavy lines correspond to Coulomb propagators.

Expression (22) is not altered thereby, and the expression for $\langle\delta\eta(\mathbf{r})\delta\eta(\mathbf{r}')\rangle$ is obtained from (21) by the substitution $\delta\mu\to\delta\eta$. Using (21) we can show that so long as the inequality $(\lambda/l)^4L_T/l\ll 1$ holds it is possible to neglect in (16) the difference between the mean field and the applied one. This inequality holds in most experiments.

The fluctuations of the conductance of a mesoscopic sample $\langle(\delta G)^2\rangle\approx e^4/\hbar^2$, calculated for $p_Fl\gg h$ in Refs. 5 and 6, can also be obtained from Eqs. (1), (23), (16), and (17). To obtain this result from the diagrams it is necessary to sum the sequence shown in Figs. 5a, b, c, and d. It is interesting that even though the diagrams of Fig. 5d are not small, the final result is obtained by calculating the correlator of the fluctuational currents with the aid of diagrams a, b, and c of Fig. 5, followed by averaging of the x components of these currents over the sample volume.[5,6]

b. The calculation of the conductance G of a mesoscopic sample corresponds to the so-called two-probe scheme of measuring the resistance, wherein a voltage is applied to the investigated sample and the total current flowing through the sample is measured. Much more widely used, however, is the four-probe method, in which two contacts, C_3 and C_4, are used to pass a specified current through the sample, and the difference between the electrochemical potentials between the remaining two contacts C_1 and C_2 is measured (see Fig. 1).

In this case, the measured conductance fluctuations depend strongly on the sizes of the contacts. Actually, according to (21), application of a potential difference to the sample leads to the onset of fluctuations $\delta\eta(\mathbf{r})$ of the electrochemical potential. The amplitude of these fluctuations increases with decrease of the investigated spatial scale.[18]

We consider now a case in which the potential is measured by ideal point contacts, i.e., contacts with infinite conductivity. The boundary condition for Eqs. (1) and (23) on the interface of the contact and the sample is then $\eta(\mathbf{r})=\eta_c=\text{const}$. The quantity to be determined in this case is $\langle(\Delta\eta_c)^2\rangle=\langle[\delta\eta_c^{(1)}-\delta\eta_c^{(2)}]^2\rangle$. Here $\eta_c^{(1)}$ and $\eta_c^{(2)}$ are the electrochemical potentials at the first and second contacts, respectively.

Equations (1), (23), and (16) are easier to solve if $a\ll L_0$. Here a is the characteristic dimension of the contact. If this condition is met, one can substitute in (16) the value of $\mathbf{E}(\mathbf{r})$ not perturbed by the point contact, i.e., contributions are made to the integral by $|\mathbf{r}-\mathbf{r}'|\sim L_0$. To determine $\langle(\Delta\eta_c)^2\rangle$ we can solve Eqs. (1), (23), and (16) with the usual boundary conditions unperturbed by the point contacts, and then average the resultant solutions $\langle\delta\eta(\mathbf{r})\delta\eta(\mathbf{r}')\rangle$ for $\mathbf{r}$ and $\mathbf{r}'$ over a volume of order a^3. Apart from a numerical factor that depends on the shape of the point contact, we obtain then ($a<L_z$)

$$\langle(\Delta\eta_c)^2\rangle\approx\left(\frac{eE\lambda^2}{l}\right)^2\begin{cases}L_0/a, & a\gg l\\ lL_0/a^2, & \lambda\ll a\ll l\end{cases}.\tag{25}$$

Accurate to a numerical factor, the same result is obtained if the electrochemical potential is measured by a tunnel junction that does not perturb the solution of Eqs. (1) and (23) in the region of the contacts. It is necessary here to average over the contact areas. If the distance between the measurement contacts satisfies the condition $L\gg a$, the quantity $\langle\delta\eta_c^{(1)}\delta\eta_c^{(2)}\rangle$ can be neglected and the expression for the conductance fluctuations $\langle(\delta\tilde{G})^2\rangle$ measured by the four-probe method is

$$\langle(\delta\tilde{G})^2\rangle\approx 2G_0^2\frac{\langle(\Delta\eta_c)^2\rangle}{E^2L^2}=\frac{e^4}{\hbar^2}\frac{L_y^2L_z^2}{L^4}\begin{cases}L_0/a, & a\gg l\\ lL_0/a^2, & \lambda<a<l\end{cases}.\tag{26}$$

Here $G_0=\sigma_DL_yL_z/L$.

Thus, when account of (18) is taken and a is regarded as small, the value of $\langle(\delta\tilde{G})^2\rangle$ becomes much larger than $e^4/\hbar^2$, and the large value of the mesoscopic fluctuations is due not so much to the small size of the sample as to the small sizes of the measuring contacts.

In the derivation of (26) we have neglected the correla-

tion between the values of $\delta\eta_c^{(1)}$ and $\delta\eta_c^{(2)}$, measured with point contacts,

$$\langle \delta\eta_c^{(1)} \delta\eta_c^{(2)} \rangle \approx \langle (\Delta\eta_c)^2 \rangle a/L, \quad L < L_z. \tag{27}$$

We examine now the expressions for K_2 and K_3 in (15):

$$K_2 = (4e^2D)^2 E_i(\mathbf{r}) E_j(\mathbf{r}') \Gamma(\mathbf{r}, \mathbf{r}'), \tag{28}$$

$$K_3 = (4e^2D)^2 E_i(\mathbf{r}) E_j(\mathbf{r}') B(\mathbf{r}, \mathbf{r}'), \tag{29}$$

$$B(\mathbf{r}, \mathbf{r}') = \int \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi)^2} f'(\varepsilon_1) f'(\varepsilon_2) \operatorname{Re} D_{\varepsilon_1 \varepsilon_2}(\mathbf{r}, \mathbf{r}'), \tag{29a}$$

which are obtained by summing the diagrams shown in Figs. 5b and 5c.

The physical essence of these terms is the long-range character of the correlations of the extraneous currents (in contrast to the δ-correlated currents (16)). The amplitudes of these currents, however, are much smaller than in (16). As a result, the contributions of (28) and (29) to the conductance fluctuations $\langle (\delta G)^2 \rangle$ are the same as that of (16).

These diagrams play no significant role in the calculation of the fluctuations of the electron density or of the scalar electric potential in small volumes near a point contact of size $a \ll L_0$, i.e., in the calculation of $\langle (\delta\tilde{G})^2 \rangle$.

The term (29) that corresponds to the diagrams c of Fig. 5 describes the long-range correlation of local (on a scale of order l) conductivities at the points $\mathbf{r}$ and $\mathbf{r}'$. Such a phenomenon can be studied by measuring the correlation of the intrinsic conductances of the contacts C_1 and C_2:

$$\langle \delta G_1 \delta G_2 \rangle \approx \frac{e^4}{\hbar^2} \left(\frac{a}{L} \right)^2, \quad L < L_z, L_T, \tag{30}$$

where G_1 and G_2 are respectively the conductances of the point contacts C_1 and C_2. In accordance with Refs. 5 and 6 we have

$$\langle (\delta G_1)^2 \rangle \approx \langle (\delta G_2)^2 \rangle \approx e^4/\hbar^2.$$

The result obtained in Ref. 19

$$\langle (\delta G_{xx})^2 \rangle \approx \langle (\delta G_{xy})^2 \rangle$$

is obtained also from the foregoing analysis, since $\langle (\Delta\eta_c)^2 \rangle$ is independent of the location of the contacts.

c. Application of a magnetic field $\mathbf{H}$ alters the character of the interference between the electron wave functions and leads as a result to fluctuations of $G(\mathbf{H})$ as a function of $\mathbf{H}$.[20,6] Another method of studying mesoscopic fluctuations is to measure δG as a function of the level of the chemical potential,[6] of the electric field intensity,[19] or of the purity diffusion.[14,15] According to an ergodic hypothesis advances in Ref. 6, averaging of the moments of δG over different realizations of a random potential is equivalent to averaging, in a single sample, over different value of H. We shall use this hypothesis.

To calculate $\langle \delta G(\mathbf{H}) \delta G(\mathbf{H} + \Delta\mathbf{H}) \rangle$ we must know the correlator $\langle J_{\text{ext}\,i}(\mathbf{r}, \mathbf{H}) J_{\text{ext}\,j}(\mathbf{r}', \mathbf{H}') \rangle$:

$$\langle J_{\text{ext}\,i}(\mathbf{r}, \mathbf{H}) J_{\text{ext}\,j}(\mathbf{r}', \mathbf{H}') \rangle = \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3. \tag{31}$$

Only the first term of (31) is significant in the calculation of $\langle \delta\tilde{G}(\mathbf{H}) \delta\tilde{G}(\mathbf{H} + \Delta\mathbf{H}) \rangle$. A calculation of the conductance in Refs. 6 and 19 yielded the quantity $\tilde{K}_1(\mathbf{H}, \mathbf{H}')$, which depends on the experimental geometry and is given by

$$\tilde{K}_1(\mathbf{H}, \mathbf{H}') = \left(\frac{e^2 E}{\pi\hbar} \right)^2 \frac{L_{|\mathbf{H}-\mathbf{H}'|}}{\pi} \delta_{ij} \delta(\mathbf{r} - \mathbf{r}'), \quad L_{|\mathbf{H}-\mathbf{H}'|} \ll L_0, \tag{32}$$

where $L_H = (c/4eH)^{1/2}$ is the magnetic length.

It turns out here that $\tilde{G}$ fluctuates when H varies in the same characteristic scale $\Delta H_c \approx \Phi_0/L_0^2$ (Ref. 20) as G, but the amplitude $\langle (\delta\tilde{G}) \rangle$ of the fluctuations turns out to be of the order of (26). Experiment[8] revealed in certain cases conductance fluctuations that exceeded $e^4/\hbar^2$ as H was varied. This can possibly be explained by the result (26) above.

Note that the conductances G_1 and G_2 of the point contacts C_1 and C_2 also undergo fluctuations when the magnetic field is changed by an amount $\Delta H_{c1} \approx \Phi_0/a^2$. At the same time, the correlator $\langle \delta G_1(\mathbf{H}) \delta G_2(\mathbf{H} + \Delta\mathbf{H}) \rangle$ decreases as a function of ΔH at $\Delta H \gtrsim \Delta H_{c2} \approx \Phi_0/L^2$ and decreases by a factor of two at $\Delta H = 0$, $H > \Delta H_{c2}$ ($\Delta H_{c2} \ll \Delta H_{c1}$). This means that the correlation of G_1 and G_2 is destroyed by weak magnetic fields that change the values of G_1 and G_2 only little.

These phenomena can apparently be investigated by experiments on MIS structures, with the averaging over the realizations $\langle \ldots \rangle$ effected by varying the gate voltage, in analogy with Ref. 8.

We discuss now the applicability of the Onsager relation to the conductance of mesoscopic samples in an external magnetic field. The standard derivation of these relations[21] pertains to the case when the conductance is measured by the two-probe method. It was shown in Ref. 19 that to prove the relation

$$\langle [\delta G(\mathbf{H}) - \delta G(-\mathbf{H})]^2 \rangle = 0.$$

It is necessary to take into account in (31) all three terms corresponding to diagrams a, b, and c of Fig. 5. The quantity $\tilde{K}_3(\mathbf{H}, -\mathbf{H})$ is an even function of $\mathbf{H}$. The quantities $\tilde{K}_1(\mathbf{H}, -\mathbf{H})$ and $\tilde{K}_2(\mathbf{H}, -\mathbf{H})$, depend generally speaking on the sign of $\mathbf{H}$ and lead to the Onsager relation only if summed and following an appropriate integration.

In Ref. 7, the conductances of mesoscopic metallic samples were measured by the four-probe method. The Onsager relation was not satisfied in this case. It was shown in a theoretical paper,[22] with the aid of the Landauer relation between I and G, that the Onsager relation should not be satisfied in four-probe experiments. In the Langevin scheme described above, this circumstance is manifested by the fact that at $a \ll L_0$ the second and third terms in (15) can be neglected, and the value of $\langle [\delta\tilde{G}(\mathbf{H}) - \delta\tilde{G}(-\mathbf{H})]^2 \rangle$ turns out to be of order of $\langle (\delta\tilde{G})^2 \rangle$ or $H > \Delta H_c$. The measured Hall effect of mesoscopic samples also turns out to be of the same order.

IV. CONCLUSION

Let us summarize our results. The mesoscopic fluctuations of the optical transparency of elastically scattering media can be calculated by using the Langevin scheme for solving diffusion equations with random extraneous currents $\mathbf{J}_{\text{ext}}$. These extraneous currents result from random interference of waves reaching a given point along different diffusion trajectories, and are proportional to the density of

the light energy at the given point of the medium. As a result, the transparency of mesoscopic samples fluctuates from sample to sample.

These fluctuations can be investigated by varying the angle of incidence of the radiation on the sample, in analogy with Refs. 12 and 13, or by studying the diffusion of the scatterers in the sample and measuring the time variation of the transparency. The transparency is here anomalously sensitive to variation of the scatterer locations.[14,15]

Another experimental possibility is to study the transparency fluctuations produced in samples by changes of the external magnetic field. The waves traveling along different diffusion trajectories acquire different polarizations, owing to the Faraday rotation of the light propagation in the medium. This mechanism of the action of a magnetic field on localized changes of the average optical transparency of a sample was considered in Ref. 3. The estimated magnetic field that leads to transparency fluctuations of the order of (12) is $\Delta H_c \approx l/L_x^2\beta$, where β is the Verdet constant.

In calculations of the conductivity fluctuations, the quantity $J_{ext}(r)$ is determined by the long-range character of the fluctuational current response at a point $\mathbf{r}$ to an electric field applied to a point $\mathbf{r}'$. It turns out here that the measured mesoscopic fluctuations of the conductance depend strongly on the measurement method. If a two-probe method is used, the fluctuations of the conductance at $T=0$ are of the order of $e^4/\hbar^2$ (Refs. 5 and 6). In the four-probe method, which is most widely used in experiment, the measured fluctuations can greatly exceed this value. The reason for this behavior is that the measuring contacts average the gigantic microscopic mesoscopic fluctuations over the contact dimensions. It is possible that the fluctuations in several samples exceeded $e^4/\hbar^2$ in Ref. 8 as a result of this effect.

Mesoscopic fluctuations of the conductivity can be studied experimentally also by other methods. For example, it is possible to investigate the correlation of mesoscopic fluctuations of the resistance of two point contact by varying the external magnetic field. This correlation is given by Eq. (30).

The use of the Langevin scheme allows, naturally, for both the agreement between the Onsager relation and the value of $G(H)$ measured by a two-probe method, and the disagreement with the value of $G(\mathbf{H})$ in four-probe measurements.

The described Langvin scheme of calculating mesoscopic fluctuations can be used only to calculate correlators in first order in the parameter $(p_F l/\hbar)^{-1} \ll 1$. It is shown at the same time in Ref. 24 that high correlators of the fluctuations of the conductance G have a non-Gaussian character in approximations of higher order in the parameter $(p_F l/\hbar)^{-1}$. We hope to extend the described scheme to include also this case.

We note in conclusion that the mesoscopic-sample transparency fluctuations investigated by us differ in nature from mesoscopic conductivity fluctuations. In the former case $J_{ext} \sim \langle n(r)\rangle$, and in the latter $J_{ext} \sim \nabla\langle n(r)\rangle$. It is this which determines the relatively large fluctuations of the conductivity compared with the conductance fluctuations.

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