Magnetic properties of a charged Bose gas and the electrodynamics of a strongly coupled electron-phonon gas

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Strong electron-phonon interaction leads to formation of small bipolarons in crystals. It is shown that their properties, at all densities, are identical with the properties of a charged heavy interacting Bose gas. The lower (thermodynamic) and upper (critical) magnetic fields of a charged Bose gas with a short-range interaction potential between the particles are obtained.

INTRODUCTION

The electrons responsible for the superconductivity of superconducting d and f metals, their alloys, and crystalline compounds interact strongly as a rule with the lattice vibrations. Estimates of the corresponding coupling constant λ that determines the effective interelectron attraction, obtained from tunneling and other experiments and from approximate equations for T_c and the tunneling characteristics, based on the Éliashberg equations,[1] yield values $\lambda \sim 1$ and in a number of cases even considerably higher ones ($\lambda \sim 4$ to 5) (see, e.g., Ref. 2). One physical cause of such a strong electron-phonon interaction is the high density of states $N(0)$ in the d and f bands, since they are narrow (pseudopotential calculations yield for the width of the d or f band $\mathscr{D} \lesssim 1$ eV, so that $N(0) \geqslant 1$ states/eV·atom). Another cause of the large Fröhlich-interaction constant ($g^2 > 1$). The latter is due to the softness of the phonon spectrum in the case of the interaction with acoustic oscillations, and the presence in most considered substances of an ion-coupling admixture, i.e., of optical polarization modes that interact strongly with the electron subsystem. As a result,

$$\lambda \approx z g^2 \omega / \mathscr{D} > 1, \tag{1}$$

where z is the coordination number of the lattice and ω is the characteristic phonon frequency.

It was repeatedly noted (see, e.g., Ref. 3) that the values of λ calculated for these compounds do not agree with the experimental ones. The discrepancy amounts to several hundred percent at a band-calculation accuracy not worse than 20%. The d- and f-metal compounds have also many other anomalous properties that cannot be explained in the framework of the usual theory of electron-phonon interactions in metals. One of them is the anomalously high density of the electronic states, exceeding by a decade or more the calculated pseudopotential values.[3] The heat-capacity coefficient can exceed by several ten times (A-15 compounds) or by hundreds of times (systems with heavy photons) the corresponding values for simple metals.

In Refs. 4 (designated AR) one of us and Ranninger have indicated that a possible cause of the anomalies of the superconducting d- and f-metal compounds can be the polaron effect, a sufficient condition for the onset of which is the inequality (1). It is known[5] that in semiconductors and insulators this inequality, which means that the polaron potential well due to local lattice deformation has a depth, compared with the electron kinetic energy, such that the width of the electron band decreases rapidly and exponentially to

$$W = \mathscr{D} \exp(-g^2). \tag{2}$$

In a many-electron system, according to AR, the abrupt decrease of the electron kinetic energy (2), which was not accounted for by the usual theory of electron-phonon interation in metals, leads in the presence of even a small attraction Δ to formation of real polaron pairs spatially localized on one or several neighboring lattice sites (small-radius bipolarons) and are capable of tunneling through the lattice as a unit, with a gigantic but finite effective mass ($m^{**} \gg m_e$).

It was established in Refs. 4 and 6 that formation of bipolarons should cause the transport and thermodynamic properties of the electron-phonon system to be similar, in the close-coupling limit, to the properties of a charged heavy Bose gas.

We report here the first investigations of certain magnetic properties of a charged Bose gas. In contrast to a charged Fermi system, in which the long-range Coulomb interaction can be correctly taken into account in the limit of high density or in the gas limit, allowance for the the interaction in a charged Bose gas is a much more complicated matter in view of the possible formation of a Bose condensate. For a Coulomb Bose gas, nonetheless, in the high-density limit, the excitation spectrum was obtained,[7] and the dielectric properties were investigated in the absence[8] and in the presence[9] of a magnetic field. On the other hand, to our knowledge, the question of the critical magnetic field of an interacting charged Bose gas, the vortex structure, an the thermodynamic properties of such a gas in a magnetic field, which are strongly dependent on the interaction, has not been investigated. For example, as first noted by Schafroth,[10] an ideal charged Bose gas in a constant magnetic field cannot be condensed because of the one-dimensional character of particle motion on the lower Landau level.

Interaction broadens the Landau levels, thereby eliminating the one-dimensional singularity of the density of states, and according to Ref. 6 an interacting charged Bose gas is capable of being condensed in fields lower than a certain critical value H_{c2}.

In the first section of the present paper, an electron-phonon system with a strong Fröhlich interaction is reduced, $\lambda > 1$, to an interacting charged Bose gas whose condensate is described by the known Ginzburg-Pitaevskiĭ equation generalized to the case of charged particles.[6] In the

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second and succeeding sections we obtain the lower (H_{c1}), and higher (H_{c2}) critical fields of a charged Bose gas. Analytic expressions were obtained for the case of a short-range interaction potential.

In real systems, where heavy bosons can be produced,[6] light fermions exist in wide bands overlapping the d or f band and screen the heavy particles.[11] As a result, the assumption of a short-range boson-interaction potential not only facilitate the calculation greatly, but is also the most realistic one.

1. HEAVY BOSONS IN A STRONGLY-COUPLED ELECTRON-PHONON SYSTEM

We assume as a start, for a strongly-coupled electron-phonon system the AR bipolar Hamiltonian[4]:

$$\hat{H}=\sum_{\mathbf{m}\neq\mathbf{m}'}\{v(\mathbf{m}-\mathbf{m}')b_{\mathbf{m}}^{+}b_{\mathbf{m}'}^{+}b_{\mathbf{m}}b_{\mathbf{m}'}-t(\mathbf{m}-\mathbf{m}')b_{\mathbf{m}}^{+}b_{\mathbf{m}'}\}, \tag{3}$$

where $\mathbf{m}$ is the number of the crystal cell, $t(\mathbf{m}-\mathbf{m}')$ is the bipolaron hopping integral, $v(\mathbf{m}-\mathbf{m}')$ is the effective interaction of the bipolarons, while $b_{\mathbf{m}}$ and $b_{\mathbf{m}}^{+}$ are the bipolaron annihilation and creation operators and satisfy the Pauli comutation relations

$$b_{\mathbf{m}}b_{\mathbf{m}}^{+}+b_{\mathbf{m}}^{+}b_{\mathbf{m}}=1,\quad b_{\mathbf{m}}b_{\mathbf{m}'}^{+}-b_{\mathbf{m}'}^{+}b_{\mathbf{m}}=0\quad(\mathbf{m}\neq\mathbf{m}'). \tag{4}$$

The Hamiltonian (3) is obtained from the usual Hamiltonian with a Fröhlich electron-phonon interaction:

$$\hat{H}_{e-ph}=\sum_{m,m'}T_{mm'}C_{m}^{+}C_{m'}+\sum_{m,\mathbf{q}}\{U_{m}(\mathbf{q})C_{m}^{+}C_{m}d_{\mathbf{q}}+\text{h.c.}\}+\sum_{m,m'}V_{mm'}^{mm'}C_{m}^{+}C_{m'}^{+}C_{m}C_{m'}+\sum_{\mathbf{q}}\omega_{\mathbf{q}}d_{\mathbf{q}}^{+}d_{\mathbf{q}} \tag{5}$$

for the case of a large coupling constant ($\lambda>1$) and $\Delta\gg W$ (Δ is the bipolaron binding energy.)

In Eq. (5), $T_{mm'}$ is the one-electron hopping integral ($T_{mm'}\sim\mathscr{D}$), $U_{m}(\mathbf{q})=U(\mathbf{q})e^{i\mathbf{qm}}$ is the Fourier component of the electron-phonon interaction, $\omega_{\mathbf{q}}$ is the phonon frequency, $m=(\mathbf{m},\alpha)$ (α is the spin index), $V_{mm'}^{mm'}$ is the electron-electron interaction, and C_{m} and $d_{\mathbf{q}}$ are respectively the electron and phonon operators.

The transition from (5) to (3) is via two canonical transformations (see, e.g., Refs. 4 and 6). The result of the first transformation, which leads to a change of the equilibrium positions of the sites as a result of the electron-phonon interaction, is a Hamiltonian that describes the motion and interaction of small-radius polarons. The second transformation, by annihilating the bipolaron-destroying terms, leads directly to the Hamiltonian (3). Using the results of Ref. 6, we write down an expression that connects $t(\mathbf{m}-\mathbf{m}')$ and $v(\mathbf{m}-\mathbf{m}')$ with the band electron and phonon spectra and with the matrix elements of the electron-phonon and Coulomb interactions.

The bipolaron hopping integral is defined as

$$t(\mathbf{m}-\mathbf{m}')=2i\int_{0}^{\infty}dt\exp[-(\delta+i\Delta)\tau]\langle\sigma_{mm'}(\tau)\sigma_{mm'}(0)\rangle,\quad\delta=+0. \tag{6}$$

Here

$$\langle\sigma_{mm'}(\tau)\sigma_{mm'}(0)\rangle=T_{mm'}^{2}\exp(-2g^{2})\times\exp\left\{-2\sum_{\mathbf{q}}|U(\mathbf{q})|^{2}[1-\cos(\mathbf{q}(\mathbf{m}-\mathbf{m}'))]\times\cos\left(\omega_{\mathbf{q}}\left(\tau+\frac{i}{2T}\right)\right)\times\left[\omega_{\mathbf{q}}^{2}\operatorname{sh}\left(\frac{\omega_{\mathbf{q}}}{2T}\right)\right]^{-1}\right\}, \tag{7}$$

where T is the temperature and g^{2} is given by

$$g^{2}=\sum_{\mathbf{q}}\omega_{\mathbf{q}}^{-2}\operatorname{cth}\left(\frac{\omega_{\mathbf{q}}}{2T}\right)|U(q)|^{2}[1-\cos(\mathbf{q}(\mathbf{m}-\mathbf{m}'))]$$

($\hbar=c=k_{B}=1$, where k_{B} is the Boltzmann constant).

The dynamic interaction of bipolarons located in different cells is defined by the expression

$$v(\mathbf{m}-\mathbf{m}')=4\tilde{v}(\mathbf{m}-\mathbf{m}')+2i\int_{0}^{\infty}d\tau\exp[-(\delta+i\Delta)\tau]\langle\sigma_{mm'}(\tau)\delta_{m'm}(0)\rangle. \tag{8}$$

Here $\tilde{v}(\mathbf{m}-\mathbf{m}')$ is the sum of the direct Coulomb repulsion and of the attraction due to the lattice deformation

$$\tilde{v}(\mathbf{m}-\mathbf{m}')=\frac{1}{4}\sum_{\alpha}V_{ii'}^{ii'}-\sum_{\mathbf{q}}\frac{|U(\mathbf{q})|^{2}}{\omega_{\mathbf{q}}}\exp[i\mathbf{q}(\mathbf{m}-\mathbf{m}')]. \tag{9}$$

The second term in (8) is the effective repulsion due to exchange of virtual polarons. The expression for $\langle\sigma_{mm'}(\tau)\sigma_{m'm}(0)\rangle$ differs from (7) in that the sign of the sum in the second exponential is reversed and that the square of the matrix element $T_{mm'}$ is replaced by the product $T_{mm'}T_{m'm}$. We assume hereafter that the interaction (8) corresponds to repulsion. In addition, a direct calculation for actual models of the phonon spectrum[12] leads as a rule to $v\gg t$ irrespective of the size of the Coulomb contribution.

We transform in the Hamiltonian (3) from Pauli bipolaron operators to Bose operators, in analogy with procedure used in exciton theory[13]:

$$b_{\mathbf{m}}=\sum_{\nu=0}^{\infty}\beta_{\nu}(a_{\mathbf{m}}^{+})^{\nu}(a_{\mathbf{m}})^{\nu+1},\qquad b_{\mathbf{m}}^{+}=\sum_{\nu=0}^{\infty}\beta_{\nu}(a_{\mathbf{m}}^{+})^{\nu+1}(a_{\mathbf{m}})^{\nu}, \tag{10}$$

where the operators $a_{\mathbf{m}}$ and $a_{\mathbf{m}}^{+}$ obey the Bose-Einstein statistics: $[a_{\mathbf{m}},a_{\mathbf{m}'}^{+}]=\delta_{\mathbf{m},\mathbf{m}'}$. We write down the first few coefficients β_{ν}, which are determined by substituting (10) in (4):

$$\beta_{0}=1,\quad\beta_{1}=-1,\quad\beta_{2}=\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right),\ldots.$$

We transform now to the operators of paticle creation and annihilaton on sites to field bipolaron and boson operators, which we define as follows:

$$\hat{\varphi}(\mathbf{r})=\frac{1}{N^{1/2}}\sum_{\mathbf{m}}\delta(\mathbf{r}-\mathbf{m})b_{\mathbf{m}},\qquad\hat{\varphi}^{+}(\mathbf{r})=\frac{1}{N^{1/2}}\sum_{\mathbf{m}}\delta(\mathbf{r}-\mathbf{m})b_{\mathbf{m}}^{+},$$

$$\hat{\psi}(r)=\frac{1}{N^{1/2}}\sum_{\mathbf{m}}\delta(\mathbf{r}-\mathbf{m})a_{\mathbf{m}},\qquad\hat{\psi}^{+}(\mathbf{r})=\frac{1}{N^{1/2}}\sum_{\mathbf{m}}\delta(\mathbf{r}-\mathbf{m})a_{\mathbf{m}}^{+}. \tag{11}$$

Here N is the number of unit cells, and $\delta(\mathbf{r}-\mathbf{m})$ is the eigenfunction of the operator $\hat{\mathbf{r}}$ in the coordinate representation.

The transformation (10) for the field operators takes the form

$$\hat{\varphi}(\mathbf{r})=\hat{\psi}(\mathbf{r})-\frac{1}{N}\hat{\psi}^{+}(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})$$

$$+\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\frac{1}{N^{2}}\hat{\psi}^{+}(\mathbf{r})\hat{\psi}^{+}(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})+\dots . \quad (12)$$

Substituting in (3) the operators $b_{\mathbf{m}}$ and $b_{\mathbf{m}}^{+}$ expressed with the aid of (11) and (12) in terms $\hat{\psi}(r)$ and $\hat{\psi}^{+}(r)$, we obtain the Hamiltonian of the interacting boson field

$$\hat{H}=\int d^{3}r\,d^{3}r'\,\hat{\psi}_{+}(\mathbf{r})t(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r})$$

$$+\frac{1}{2}\int d^{3}r\,d^{3}r'\Big\{v(\mathbf{r}-\mathbf{r}')\hat{\psi}^{+}(\mathbf{r})$$

$$\times\hat{\psi}^{+}(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')-\frac{2}{N}[\hat{\psi}^{+}(\mathbf{r})t(\mathbf{r}-\mathbf{r}')\hat{\psi}^{+}(\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r}')$$

$$+\hat{\psi}^{+}(\mathbf{r})\hat{\psi}^{+}(\mathbf{r})\hat{\psi}(\mathbf{r})t(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}')]\Big\}+\hat{H}_{\mathrm{up}}, \quad (13)$$

where

$$t(\mathbf{r}-\mathbf{r}')=-\frac{1}{N}\sum_{\mathbf{m}\neq\mathbf{m}'}t(\mathbf{m}-\mathbf{m}')\delta(\mathbf{r}-\mathbf{m})\delta(\mathbf{r}'-\mathbf{m}'),$$

$$v(\mathbf{r}-\mathbf{r}')=\frac{2}{N^{2}}\sum_{\mathbf{m}\neq\mathbf{m}'}v(\mathbf{m}-\mathbf{m}')\delta(\mathbf{r}-\mathbf{m})\delta(\mathbf{r}'-\mathbf{m}'),$$

and $\hat{H}_{\mathrm{UP}}$ includes terms with powers of $\hat{\psi}$ higher than the fourth (unpaired interaction). In the momentum representation we have

$$t(\mathbf{r}-\mathbf{r}')=-\sum_{\mathbf{k}}t_{\mathbf{k}}\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')],$$

$$v(\mathbf{r}-\mathbf{r}')=\frac{2}{N}\sum_{\mathbf{k}}v_{\mathbf{k}}\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')], \quad (14)$$

where

$$t_{\mathbf{k}}=\sum_{\mathbf{m}\neq 0}t(\mathbf{m})\exp(-i\mathbf{k}\mathbf{m}),\quad v_{\mathbf{k}}=\sum_{\mathbf{m}\neq 0}v(\mathbf{m})\exp(-i\mathbf{k}\mathbf{m}).$$

We obtain thus a Bose gas with a complicated unpaired interaction of the particles with one another, consisting of a dynamic contribution $v(\mathbf{r}-\mathbf{r}')$ and a kinetic one $t(\mathbf{r}-\mathbf{r}')$. The condition met in systems with a strong electron-phonon coupling is $v\gg t$, since t contains an additional small factor $\exp(-2g^{2})$.[12] This allows us to retain only the dynamic part of the interaction.

A quasi-classical allowance for the magnetic field is sufficient, since real fields are weak compared with atomic[6]: $eHa^{2}\ll 1$, where a is the crystal-lattice period.

If the variation of the vector potential is slow enough, the bipolaron hopping integral is renormalized as follows:

$$\tilde{t}(\mathbf{m},\mathbf{m}')=t(\mathbf{m}-\mathbf{m}')\exp[-2ie\mathbf{A}(\mathbf{m})(\mathbf{m}-\mathbf{m}')]. \quad (15)$$

The dynamic interaction $v(\mathbf{m}-\mathbf{m}')$ in a magnetic field remains unchanged,[6] and for $\tilde{t}(\mathbf{r}-\mathbf{r}')$ we obtain in lieu of (14), on account of the renormalization (15),

$$\tilde{t}(\mathbf{r}-\mathbf{r}')=-\frac{1}{N}\sum_{\mathbf{k}}t_{\mathbf{k}+2e\mathbf{A}}\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')].$$

Expanding $t_{\mathbf{k}+2e\mathbf{A}}$ near $\mathbf{k}=0$ and using the weak dependence of $\mathbf{A}$ on the coordinates, we have

$$\tilde{t}(\mathbf{r}-\mathbf{r}')=-\{t_{0}+[\nabla-2ie\mathbf{A}(\mathbf{r})]^{2}/2m^{**}\}\delta(\mathbf{r}-\mathbf{r}'),$$

where

$$1/m^{**}=-(\partial^{2}t_{\mathbf{k}}/\partial\mathbf{k}^{2})_{\mathbf{k}=0}.$$

According to AR (Ref. 4), spatially homogeneous state of the bipolaron Hamiltonian (3) is realized only in the limit of low atomic density of the particles:

$$n_{\mathrm{at}}<t/v. \quad (16)$$

In the inverse case there appears a bipolaron charge-density wave. We confine ourselves here to an analysis of the magnetic properties of a spatially homogeneous superconducting phase.[4]

To present the results in analytic form, we consider the region of small values of the gas parameter:

$$\eta=n^{1/3}l\ll 1, \quad (17)$$

where

$$l=\frac{m^{**}}{4\pi}\int d^{3}r\,v(\mathbf{r})$$

is the scattering length in the Born approximation and $n=n_{\mathrm{at}}/a^{3}$ is the particle density. The constraint (17) on the particle density is stronger than (16). It permits neglect of the unpaired interaction in specific calculations.

As a result, the Hamiltonian of a strongly coupled ($\lambda>1$) electron-phonon system in a lattice reduces to that of a heavy weakly nonideal charged Bose gas:

$$\hat{H}=-\mu\int d^{3}r\,\hat{\psi}^{+}(\mathbf{r})\hat{\psi}(\mathbf{r})-\frac{1}{2m^{**}}\int d^{3}r\,\hat{\psi}^{+}(\mathbf{r})[\nabla-2ie\mathbf{A}(\mathbf{r})]^{2}\hat{\psi}(\mathbf{r})$$

$$+\frac{1}{2}\int d^{3}r\,d^{3}r'\,v(\mathbf{r}-\mathbf{r}')\hat{\psi}^{+}(\mathbf{r})\hat{\psi}^{+}(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'), \quad (18)$$

where μ is the chemical potential.

For the reason mentioned in the Introduction, the causes of the interaction can be considered to be of short duration.

2. ELECTRODYNAMIC EQUATIONS OF A CHARGED BOSE GAS. THE FIELDS H_{c1} AND H_{c}

The evolution equation of the Heisenberg operator $\hat{\psi}(\mathbf{r},t)$ for the Hamiltonian (18) is of the form

$$i\frac{\partial\hat{\psi}(\mathbf{r},\tau)}{\partial\tau}=-\left\{\mu+\frac{[\nabla-2ie\mathbf{A}(\mathbf{r})]^{2}}{2m^{**}}\right\}\hat{\psi}(\mathbf{r},\tau)$$

$$+\int d^{3}r'\,\hat{\psi}^{+}(\mathbf{r}',\tau)v(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}',\tau)\hat{\psi}(\mathbf{r},\tau). \quad (19)$$

We represent the field operator in the form $\hat{\psi}=\psi_{0}+\tilde{\psi}$, where ψ_{0} is a c-number having the meaning of a macroscopic wave function normalized to the particle density n_{0} in the condensate. The function ψ_{0} is defined as that part of the operator $\hat{\psi}$ which decreases the number of particles in the condensate by unity, leaving the remaining part of the system unchanged:

$$\psi_{0}(\mathbf{r},\tau)=\langle m,N|\hat{\psi}|m,N+1\rangle\ \text{ as }\ N\to\infty,\ N/V=\text{const},$$

where N is the total number of particles, V the particle volume, and $\tilde{\psi}(\mathbf{r},\tau)$ the operator describing the part over the condensate.

The system of equations for ψ_0 and $\tilde{\varphi}$ is

$$i\frac{\partial\psi_0}{\partial\tau}=-\left[\mu+\frac{[\nabla-2ie\mathbf{A}]^2}{2m^{**}}\right]\psi_0 + \int d^3r'\, v(\mathbf{r}-\mathbf{r}')\{|\psi_0(\mathbf{r}',\tau)|^2\psi_0(\mathbf{r},\tau) + \langle\tilde{\psi}^+(\mathbf{r}',\tau)\tilde{\psi}(\mathbf{r}',\tau)\rangle\psi_0(\mathbf{r},\tau)+\langle\tilde{\psi}^+(\mathbf{r}',\tau)\tilde{\psi}(\mathbf{r},\tau)\rangle\psi_0(\mathbf{r}',\tau)\}, \tag{20}$$

$$i\frac{\partial\tilde{\psi}}{\partial\tau}=-\left[\mu+\frac{(\nabla-2ie\mathbf{A})^2}{2m^{**}}\right]\tilde{\psi}(\mathbf{r},\tau) + \int d^3r'\{|\psi_0(\mathbf{r}',\tau)|^2\tilde{\psi}(\mathbf{r},\tau) + \tilde{\psi}^+(\mathbf{r}',\tau)\tilde{\psi}(\mathbf{r}',\tau)\tilde{\psi}(\mathbf{r},\tau)+\psi_0^*(\mathbf{r}',\tau)\psi_0(\mathbf{r},\tau)\tilde{\psi}(\mathbf{r}',\tau) + \psi_0(\mathbf{r}',\tau)\psi_0(\mathbf{r},\tau)\tilde{\psi}^+(\mathbf{r}',\tau)\}v(\mathbf{r}-\mathbf{r}'). \tag{21}$$

This system of equations must be supplemented by the Maxwell equation

$$\operatorname{rot}\operatorname{rot}\mathbf{A}(\mathbf{r})=4\pi\mathbf{J}_s, \tag{22}$$

where $\mathbf{J}_s$ is the density of the superconducting undamped current. The latter is determined by the order parameter $\psi_s=n_s^{1/2}e^{i\varphi}$, normalized to the density of the superfluid (superconducting) component $n_s=\rho_s/m^{**}$ (see, e.g., Ref. 14). In the weak non-ideality approximation (17), however, it can be assumed that ψ_s and ψ_0 are equal, and $\mathbf{J}_s$ can be expressed in terms of the macroscopic wave function ψ_0. With allowance for gauge invariance we have then

$$\mathbf{J}_s=-i\frac{e}{m^{**}}[\psi_0^*\nabla\psi_0-\psi_0\nabla\psi_0^*]-\frac{4e^2}{m^{**}}|\psi_0|^2\mathbf{A}. \tag{23}$$

Consider now Eq. (20) for ψ_0. In the stationary case, ψ_0 is independent of time: $\partial\psi_0/\partial\tau=0$. Choosing the short-range potential in the form $v(\mathbf{r}-\mathbf{r}')=v_0\delta(\mathbf{r}-\mathbf{r}')$, we rewrite the equation for ψ_0 as follows:

$$\left\{\frac{1}{2m^{**}}[\nabla-2ie\mathbf{A}(\mathbf{r})]^2+\mu\right\}\psi_0(\mathbf{r})-v_0|\psi_0(\mathbf{r})|^2\psi_0(\mathbf{r})=0. \tag{24}$$

Here, too, we have used the condition (17), so that near $T=0$ we can neglect the above-condensate particle density compared with $|\psi_0|^2$.

At low temperatures, a weakly nonideal Bose gas is superfluid. In the present case of a charged Bose gas we can thus determine the critical magnetic fields that destroy the superconductivity (superfluidity) of this system. We point out first the analogy between the system (22)–(24) and the equations of the Ginzburg–Landau phenomenological theory.

Recognizing that the chemical potential of a homogeneous weakly nonideal Bose gas is $\mu\approx nv_0$ (Ref. 15), the characteristic lengths of the problem (22)–(24), given by

$$\delta=[m^{**}/16\pi e^2n]^{1/2},\quad \xi=[2m^{**}nv_0]^{-1/2} \tag{25}$$

are respectively the field penetration depth and the characteristic scale of variation of ψ_0. The Ginzburg-Landau parameter is $\varkappa=m^{**}(v_0/16\pi e^2)^{1/2}$. The large bipolaron mass gives grounds for assuming that the condition for type-II superconductivity is met:

$$\varkappa\gg 1. \tag{26}$$

Indeed, the condition (26) is met if it is assumed that $m^{**}\approx 10^{2-3}m_e$. (Refs. 4, 6, and 12) and $(v_0/16\pi e^2)^{1/2}\approx R_s$ (R_s is a screening radius of the order of the interatomic distance).

Using Eqs. (22)–(24), we calculate the lower critical filed H_{c1} in which a normal vortex appears in a superconducting Bose gas. At large $\varkappa$, just as in the usual superconductivity theory,[16]

$$H_{c1}(T\approx 0)\approx(\Phi_0/4\pi\delta^2)\ln(\delta/\xi),$$

where $\Phi_0=\pi/|e|$ is the magnetic-field flux quantum flux. Using for δ and ξ expressions (25), we get

$$H_{c1}(0)\approx\frac{4\pi en}{m^{**}}\ln\left[m^{**}\left(\frac{v_0}{8\pi e^2}\right)^{1/2}\right]. \tag{27}$$

We calculate now the thermodynamic critical field H_c, meaning the field in which the homogeneous superconducting normal phases are in thermodynamic equilibrium.

We assume that the ground-state energy $E(0)$ is determined by the particle-interaction energy. Neglect of the kinetic energy of the superconducting phase, which is a homogeneous condensate, means neglect of the above-condensate part whose contribution is small in terms of the parameter (17). The normal phase is a homogeneous state in which the particles are on a lower Landau level that is degenerate in the positions of the particle-orbit centers. Allowance for only the interaction energy in this state is equivalent to neglect of the susceptibility.

We obtain thus for the s phase

$$E_s(0)\approx n^2v_0V/2.$$

For normal state

$$E_N(0)\approx v_0\sum_{i,k}n_in_k\int d^3r|\varphi_i(\mathbf{r})|^2|\varphi_k(\mathbf{r})|^2, \tag{28}$$

where n_i is the particle in the state i. By virtue of the homogeneity we have $\Sigma_i n_i|\varphi_i(\mathbf{r})|^2=n$, and we get $E_N(0)\approx n^2v_0V$. Doubling the energy compared with the superconducting state is connected with an additional exchange interaction of the particles on the lower level.

As a result of

$$H_c(0)^2/8\pi=F_N(0)-F_s(0)\approx n^2v_0V/2,$$

the critical field is

$$H_c(0)\approx(4\pi v_0)^{1/2}n. \tag{29}$$

The thermodynamic critical field of a charged Bose gas was already calculated earlier by Schafroth.[10] He regarded an ideal charged Bose gas as a model of a superconductor and obtained for the critical field $H_0=4\pi n\mu_0$, where $\mu_0=e/2m^{**}$. Comparing (29) with Schafroth's result

$$H_c/H_0\approx 4m^{**}R_s\sim\varkappa,$$

we see that when the condition (26) is met the thermodynamic field is determined exclusively by the interaction [see (29)], and the contribution due to the diamagnetism of the charged Bose gas is negligibly small.

3. SPECTRUM OF NORMAL INTERACTING CHARGED BOSE GAS, FIELD H_{c2}

It is known[10] that an ideal charged Bose gas in a homogeneous magnetic field does not condense. We shall show here that the interaction between the particles leads to a finite critical condensation magnetic field, and determine the critical $H_{c2}(T)$ curve.

Consider a normal Bose gas in a magnetic field at finite temperature. It is convenient to treat the interaction of the particles with one another by using temperature Green's functions.

We choose the unperturbed wave functions to be Landau solutions of the Shrödinger equation in a uniform magnetic field:

$$\varphi(r)=\varphi_\nu(r), \quad \nu=(n, p_x, p_z),$$
$$\varepsilon_\nu^0=\frac{p_z^2}{2m^{**}}+\omega\left(n+\frac{1}{2}\right), \quad \omega=\frac{2eH}{m^{**}}. \tag{30}$$

In a magnetic field, the temperature Green's function not perturbed by interaction is equal to

$$G_\nu^0(\omega_j)=[i\omega_j-(\varepsilon_\nu^0-\mu)]^{-1}, \tag{31}$$

where $\omega_j=2\pi jT$, $j=0, \pm 1, \pm 2, \ldots$. The total number of particles in the system is defined as

$$N=T\lim_{\tau\to 0-}\sum_\nu\sum_{\omega_j} G_\nu(\omega_j)\exp(-i\omega_j\tau). \tag{32}$$

Substituting in (32) the Green's function (31), we see that owing to the divergence of the integral with respect to p_z there is no Bose condensation at any temperature. This is due to the quasi-one-dimensional character of the motion in the magnetic field. We write down the Dyson equations

$$\hat{G}^{-1}=\hat{G}_0^{-1}-\hat{\Sigma}. \tag{33}$$

To calculate the self-energy part we confine ourselves to the loop approximation (Fig. 1). The discarded diagrams are small in the parameter Σ/ω, which in turn is small because of the weakness of the interaction (17). In the presentation of the wave functions (30) and (33), we can write

$$G_{\nu\nu'}^{-1}=(G_\nu^0)^{-1}\delta_{\nu\nu'}-\Sigma_{\nu\nu'}. \tag{34}$$

The smallness of the interaction (17) allows us to confine ourselves to the diagonal approximation. It is convenient to write the Dyson equation (33) in the coordinate representation (Fig. 1).

$$G(x,x')=G^0(x,x')+\int dz\,dz'\,G^0(x,z)D(z-z')G(z,z')G(z',x'), \tag{35}$$

where $x=(\mathbf{x},\tau)$, τ is the "imaginary time,"

$$G(x,x')=T\sum_{\nu,\omega_j} G_\nu(\omega_j)\varphi_\nu^*(\mathbf{x})\varphi_\nu(\mathbf{x}')\exp[i\omega_j(\tau-\tau')], \tag{36}$$

$$D(z-z')=T\sum_{q,\omega_j} D(\mathbf{q},\omega_j)\exp[i\omega_j(\tau-\tau')]\exp[i\mathbf{q}(\mathbf{z}-\mathbf{z}')], \tag{37}$$

$$D(\mathbf{q},\omega_j)=\frac{v_0}{1+v_0\Pi(\mathbf{q},\omega_j)}. \tag{38}$$

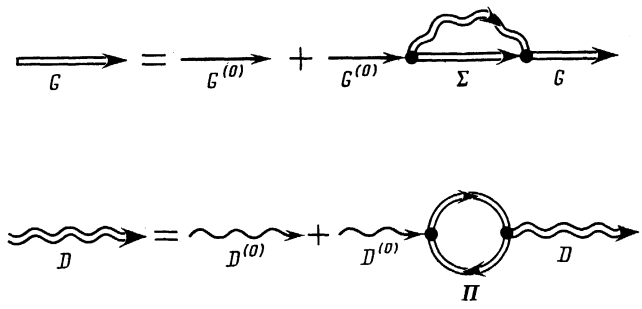

FIG. 1.

Substituting (36) and (37) in (35) we obtain an equation for the self-energy part:

$$\Sigma_\nu(\omega_j)=T\sum_{\nu',\mathbf{k}}\sum_{\omega_j} D(\mathbf{k},\omega_j)G_{\nu'}(\omega_j-\omega_j')|I_\nu^{\nu'}(\mathbf{k})|^2. \tag{39}$$

It suffices to consider the interaction renormalization of the spectrum of the lower Landau level, which makes a singular contribution to (32) at $\omega_j=0$. The Bose-condensation point is determined by the condition

$$\mu=\Sigma_0(0)+\omega/2, \tag{40}$$

where $\Sigma_0(0)$ is the value of Σ_ν at $n=p_z=0$.

The asymptotic behavior of the Green's function at the Bose condensation point is determined at small $|p_z|$ by the critical exponent σ:

$$G_{p_z}(0)\approx|p_z|^{-\sigma}, \quad |p_z|\to 0. \tag{41}$$

If $\sigma<1$, the integral in (32) converges and condensation is possible.

The static component of the self-energy part $\Sigma_\nu(0)$ depends on the renormalized interaction $D(\mathbf{k},0)$, which is determined by the polarization operator

$$\Pi(\mathbf{k})=T\sum_{\nu\nu'}\sum_{\omega_j'} G_\nu(\omega_j')G(-\omega_j')|I_\nu^{\nu'}(\mathbf{k})|^2. \tag{42}$$

Summation over ω_j can be carried out by using

$$T\sum_{\omega_j} G_\nu(\omega_j')\exp(-i\omega_j\tau)|_{\tau\to 0-}=f(\varepsilon_\nu),$$

where $f(\varepsilon_\nu)$ is the Bose distribution function in the interaction-renormalized spectrum $\varepsilon_\nu=\varepsilon_\nu^0+\Sigma_\nu$:

$$\Pi(\mathbf{k})=\sum_{\nu\nu'}\frac{f(\varepsilon_\nu)-f(\varepsilon_{\nu'})}{\varepsilon_{\nu'}-\varepsilon_\nu}|I_\nu^{\nu'}(\mathbf{k})|^2. \tag{43}$$

The overlap functions used in (43) are

$$I_\nu^{\nu'}(\mathbf{k})=\int d\mathbf{r}\exp(i\mathbf{k}\mathbf{r})\varphi_\nu(\mathbf{r})\varphi_{\nu'}(\mathbf{r}). \tag{44}$$

In the Landau gauge, $I_\nu^{\nu'}(\mathbf{k})$ can be rewritten in the form

$$I_\nu^{\nu'}(\mathbf{k})=\delta_{q_x,q_x'+k_x}\delta_{q_z,q_z'+k_z}F_n^{n'}\times\exp\left[-\frac{k_\perp}{4m^{**}\omega}-\frac{i}{m^{**}\omega}\left(k_yq_x-\frac{1}{2}k_xk_y\right)\right],$$

where

$$F_n^{n'}=\left(\frac{n'!}{n!}\right)^{1/2}\left[\frac{k_x+ik_y}{(2m^{**}\omega)^{1/2}}\right]^{n-n'}L_{n'}^{n-n'}\left(\frac{k_\perp^2}{2m^{**}\omega}\right) \tag{45}$$

at $n' \leqslant n$, where $L_{n'}^{n-n'}$ is a Laguerre polynomial.

We seek the lower-level energy in the form

$$\varepsilon(p_z)=\varepsilon_0(p_z)-\mu=p_z^2/2m^{**}+\mathscr{J}|p_z|^\sigma \tag{46}$$

and confine ourselves to the temperature region $T \gg \omega$. We break up the polarization operator into two parts—singular and regular:

$$\Pi(\mathbf{k})=T\frac{m^{**}\omega}{2\pi^2}\int_{-\infty}^{\infty}dq\,\frac{\exp(-k_\perp^2/2m^{**}\omega)}{\varepsilon(q)\varepsilon(q-k_z)}|F_0^0|^2+\Pi_{reg}(\mathbf{k}). \tag{47}$$

We replace the dimensional momentum energy p_z and ε by the dimensionless

$$p_z=(2m^{**}\mathscr{J})^{1/(2-\sigma)}p,\quad \varepsilon(p_z)=(2m^{**})^{\sigma/(2-\sigma)}\mathscr{J}^{2/(2-\sigma)}\varepsilon(p). \tag{48}$$

The asymptote of the singular part of (47) at small k is

$$\Pi(\mathbf{k})=T\frac{m^{**}\omega}{\pi^2\mathscr{J}^{3/(2-\sigma)}}\frac{(2^{\sigma-1}m^{**})}{2|k_z|^{2\sigma-1}}\exp\left(-\frac{k_\perp^2}{2m^{**}\omega}\right)$$
$$\times\left\{B\left(1-\sigma,\frac{1}{2}\right)+B\left(\sigma-\frac{1}{2},\frac{1}{2}\right)\right\},\quad \frac{1}{2}<\sigma<1,$$
$$\Pi(\mathbf{k})=T\frac{m^{**}\omega}{\pi^2\mathscr{J}^2}\ln\frac{1}{|k_z|}\exp\left(-\frac{k_\perp^2}{2m^{**}\omega}\right),\quad \sigma=\frac{1}{2}, \tag{49}$$

where $B(x,y)$ is the Euler beta function.

The effective integration region in Eq. (39) for $\Sigma_{0,p_z}(0)$ is of the order of p. We seek for this equation a solution that takes the asymptotic form (46) at small momenta $p \lesssim 1$. It suffices therefore to consider the singular term (47), and to retain in the sum over ν' in (39) only the term with $n'=0$. Equation (39), which determines $\varepsilon(p)$ in the lowest order in η, is the reduced to on singular nonlinear integral equation:

$$\varepsilon(p)=p^2+\frac{v_0(m^{**}\omega)^2T^2}{16\pi^4\mathscr{J}^{6/(2-\sigma)}}(2m^{**})^{(2-4\sigma)/(2-\sigma)}$$
$$\times\int_{-\infty}^{\infty}\frac{dk}{\varepsilon(k)}\int_{-\infty}^{\infty}\frac{dq}{\varepsilon(q)}\left\{\frac{1}{\varepsilon(q+k+p)}-\frac{1}{\varepsilon(q+k)}\right\}. \tag{50}$$

Taking this to be an iterated integral, substituting $\varepsilon(p)$ in (46), and neglecting the term p^2, we get $\sigma=1/2$,

$$\mathscr{J}^4=(v_0/4\pi^3)(m^{**}\omega)^2T^2. \tag{51}$$

Summation in (32) over all levels but the lowest can be carried out without allowance for quantization, since $T \gg \omega$. For the lower level it is necessary to use the spectrum (46). As a result we obtain for the upper critical field H_{c2}

$$n(1-t^{3/2})=\frac{m^{**}\omega}{2\pi^2}T\int_{-\infty}^{\infty}\frac{dp_z}{\varepsilon(p_z)},\quad \omega=\frac{2eH_{c2}}{m^{**}}. \tag{52}$$

Calculation of the integral yields

$$H_{c2}(T)=0{,}18\Phi_0 m^{**}T_k\eta^{1/2}\frac{(1-t^{3/2})^{3/4}}{t}, \tag{53}$$

where

$$\Phi_0=\frac{\pi}{e},\quad t=\frac{T}{T_k},\quad T_k=3{,}31\frac{n^{2/3}}{m^{**}}.$$

Expression (53) differs from H_{c2} of ordinary superconductors in the sign of the second derivative (which is positive)

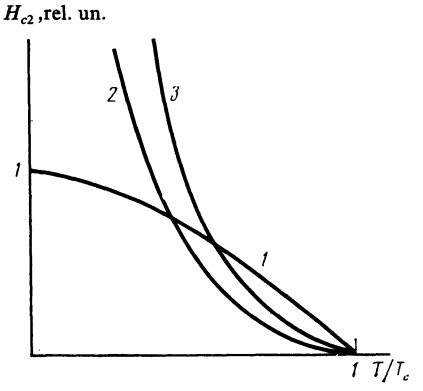

FIG. 2. Temperature dependence of Bose-condensation critical magnetic field: 1—upper critical field of usual superconductor (BCS), 2—$H_{c2}(T)$ for short-range bosons, 3—($H_{c2}(T)$ for bosons interacting with impurities (obtained in Ref. 6).

and in the nonlinear growth near T_k (Fig. 2). The expression for $\mathscr{J}$ is obtained by direct substitution of (53) in (51):

$$\mathscr{J}=1{,}09\eta^{3/4}(1-t^{3/2})^{3/4}T_k^{3/4}/m^{**1/4}. \tag{54}$$

We see hence that the temperature region of validity of (53) ($\Sigma \ll \omega$) is bounded from above: $1-t \gg 0.03\eta$. As $t \to 0$ expression (53) diverges formally. It must be noted, however, that as $t \to 0$ the expansion in powers of η, used to obtain (50), is not valid in view of the possible localization of the bosons in the self-consistent interaction potential, a localization similar to that in the random potential of the impurities.[6] The validity region of (53) is bounded also from below ($T \gtrsim \omega$), viz., $t \gtrsim \eta^{1/4}$.

CONCLUSION

We have determined the critical fields of a weakly interacting charged Bose gas with short-range interaction. It was established that such a gas is a type-II superconductor ($H_{c2} \gg H_c \gg H_{c1}$), the energy spectrum of the normal phase in a magnetic field was found, and the temperature dependence of the upper critical field H_{c2} was shown to differ qualitatively from the $H_{c2}(T)$ of the BCS theory. A similar temperature dependence was obtained earlier for a Bose gas interacting with impurities, i.e., for dirty bipolar superconductors (see Fig. 2).

As shown in the first section of the paper for arbitrary electron densities, and as noted earlier for low atomic densities,[4,6] a charged heavy Bose gas is obtained from a Fermi gas with strong electron—phonon interaction as a result of local lattice instability that leads to a substantial exponential suppression of the kinetic energy (to the polaron effect). We note in this connection that this effect does not occur in the usual theory of electron-phonon interaction in metals, in which one uses a continual approximation, i.e., infinitely broad electron bands with constant density of electronic states $N(0)$.

The existence of heavy charged bosons is possible not only in such d- and f-metal compounds as A-15, Chevrel phases, carbides, oxides, nitrides, or systems with heavy fermions,[6] but also in other substances in which the electron-photon interaction is large ($\lambda > 1$) and there are narrow enough electron bands as a result of large interatomic dis-

tances. These substances include apparently the superconducting ceramic Ba–Pb–Bi–O, a number of anomalous properties of which, including high T_c and the mixed valence of the bismuth, can be explained within the framework of the theory of small-radius polarons.[6] We note in this connection that the high-temperature superconductivity ($T_c > 40$ K) observed recently in the ceramics Ba–La–Cu–O (Ref. 17) and Sr–La–Cu–O (Ref. 18) can also be of the bipolaron type. Small bipolarons can be formed in these compounds on the copper ions as a result of displacement of the surrounding oxygen ions. The decisive role is played in this case by the hard mode corresponding to vibrations of the oxygen ions relative to the copper ion. The role of the barium (strontium) reduces then to a change of the bipolaron density, as a result of which optimal conditions[4,6] are created for bipolaron superconductivity. An estimate of the effective mass of bipolaron from the ciritical temperature (53) of an ideal Bose gas yields $m^{**} \sim 100 m_e$ for $T_c = 100$ K and $n = 10^{22}$ cm^{-3}.

In this case the high-temperature superconducting ceramics[17,18] should feature a large magnetic-field penetration depth, an anomalous temperature dependence of H_{c2} (Fig. 2), an anomalous electronic heat capacity,[6] and a mixed valence of the copper.

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