Nonlinear current-voltage characteristics of metals under size-effect conditions

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Information on the heat flux densities under the conditions of the helium boiling crisis was obtained for zinc, cadmium, tin, and copper whisker crystals at temperatures of 4.2 and 2 K. When the whisker thickness was several microns and the temperature was 2 K, the heat flow fluxes were of the order of 10–20 W/cm^2. Under steady-state conditions it was then possible to pass currents of densities $j > 10^6$ A/cm^2 through these whiskers. The question of whether Ohm's law was satisfied at these current densities was investigated. The resistance decreased to 20% of the initial value. This effect was influenced by the degree of specularity of electron reflection from the surfaces. The results were in qualitative agreement with a model of a magnetodynamic deviation from Ohm's law in the case of "good" metals.

The problem of the limits of validity of Ohm's law in the case of "good" metals is quite old.[1] Certain mechanisms have been suggested to account for a deviation mechanisms that increase the resistance. A characteristic deviation from Ohm's law in the form of reduction of the resistance on increase in the current J is possible when the influence of the scattering of carriers by the surfaces of a sample can no longer be ignored. In the absence of a magnetic field this scattering provides an additional mechanism for the increase in the resistivity of a thin sample (of thickness $d < l$, where l is the mean free path). The transverse magnetic field of the current deflects electrons from the surface toward the axis and thus reduces the influence of the surface scattering. A reduction in the resistance on increase of the current in thin zinc and gallium wires was reported in Refs. 2 and 3 and it was found that the effect became greater as the thickness of the sample decreased. Cylindrical zinc wires with diameters 0.42 and 0.23 mm were characterized by the l/d ratios 5 and 9, respectively, at 4.2 K. An increase of the current to 6 A reduced the resistance of these samples $R = U/J$ by 20–25%. Gallium wires of thickness 0.1, 0.2, and 0.35 cm had the square cross section and the ratio $l/d = 2$–6 at 4.2 K. The fall of the resistance at 4.2 K was very small (3%) and was observed only for the thinnest sample. At 1.2 K an increase in the current to 2.5 A reduced the resistance of all the samples by an amount which reached $\sim 25\%$ for the thinnest.

This reduction in the resistance of gallium samples is attributed in Ref. 4 to a magnetodynamic mechanism involving the trapping of electrons in a potential well formed by the magnetic self-field of the current (inhomogeneous across the thickness and of alternating sign). Consequently, electrons begin to move along the core of the sample along wavelike paths without colliding with the surface. The mean free path of such "trapped" electrons is comparable with l. An increase in the current increases the number of such electrons and this makes a decisive contribution to the conductance of a sample. Therefore, the conductance on the whole rises on increase in J.

It was shown in Refs. 4 and 5 that the nonlinearity should be manifested most clearly under the conditions of a strong size effect ($l \gg d$) in the case of diffuse scattering of electrons by the surface: in the case of samples in the form of plates the current-voltage characteristic should be sublinear $U \propto (JJ_0)^{1/2}$ (J_0 is the current at which the characteristic begins to deviate from linearity) and in the case of a wire with a circular cross section the voltage drop reaches saturation: $U \propto (J/i)(1 + J/i)^{-1}$ (i is the current at which the resistance of the sample decreases by a factor of two). Calculations give

$$i \approx 0.8 \frac{c^2 p_F d}{el}, \quad J_0 \approx 0.5 \frac{c^2 p_F D d \ln^2(d/l)}{el^2}$$

(D is the thickness of the plate). The following conditions should then be satisfied: $d \ll (R_{\mathrm{Larm}} d)^{1/2} \ll l$ for plates and $d \ll R_{\mathrm{Larm}} \ll l$ for wires (R_{Larm} is the Larmor radius of an electron orbit in the magnetic self-field of the current).

A qualitative agreement with the predictions of the theory of Ref. 5, i.e., a fall of the resistance of thin tungsten and cadmium plates on increase in the current, was reported in Ref. 6. However, the observed fall of the resistance changed to an increase at currents lower than those at which the condition $R_{\mathrm{Larm}} > d$ was no longer satisfied. It was pointed out in Ref. 6 that this could be due to the fact that the formulas in Ref. 4 were derived for an infinite plate, whereas the experiments were carried out on plates of width $D < l$.

It should also be mentioned that the deviations of the current-voltage characteristic from linearity (by up to 20–30%) reported in Refs. 2, 3, and 6 were observed when the strong inequalities relating l, d, and R_{Larm} were not satisfied. Therefore, as stressed in Ref. 4, it would be of interest to apply the theoretical predictions to whisker crystals where the ratios are in fact $l/d \approx 10^2$–10^3 and one could expect the conditions $d \ll R_{\mathrm{Larm}} \ll l$ satisfied.

SAMPLES AND MEASUREMENT METHOD

We investigated cadmium, zinc, and tin whiskers which, according to our earlier measurements,[7,8] had values of l at least 200, 300, and 1000 μ, respectively, at 4.2 K. Moreover, we investigated also copper whiskers. Preliminary measurements of the dependences of their resistance on the thickness showed that they were characterized by $l \approx 1$–3 μ. The width and thickness of these samples were approximately the same. The range of the whisker thicknesses was 2–10 μ for zinc and cadmium 0.7–4 μ for tin, and 3.5–21 μ for copper. The thickness of the sample was defined as $d = S^{1/2}$, where S is the whisker cross-section area deduced from the room-temperature resistance. The error in the determination of d, ranging up to 10%, was due to an indeter-

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minacy associated with the width of the potential contacts ($\Delta \approx 100\ \mu$). Therefore, the measurements on copper whiskers were made under the conditions corresponding to $l \lesssim d$, whereas in the case of the other whiskers the conditions corresponded to a strong size effect characterized by $l \gg d$. The investigated zinc, cadmium, and tin whiskers were characterized by a coefficient of specularity of reflection of electrons from the surface amounting to $P \approx 0.6$, 0.4, and 0.5, respectively, whereas in the case of copper and some tin whiskers this coefficient was $P = 0$.

The whiskers were mounted on copper-laminated synthetic-resin bonded plates from which the copper was partly removed in such a way that the remaining strips acted as current and potential contacts of transversely mounted whiskers. Indium was deposited on a copper strip and its surface was carefully leveled. Electrical contacts were made by indenting the whiskers in the indium. Experience showed that this method provided reliable electrical contacts and made it possible to pass steady currents of up to 5×10^5–3×10^6 A/cm^2 density through the whiskers. For contrast, the use of conducting paste contacts (which we employed earlier) resulted in overheating of the samples already at currents corresponding to densities in the range $j \lesssim 10^5$ A/cm^2.

The distances between the current contacts were ~ 2 mm and the potential contacts were ~ 1 mm apart. The temperature during these measurements was 2 K because a special investigation of the liquid helium boiling crisis (see below) indicated that at this temperature it was possible to ensure dissipation of power at densities of ~ 10–20 W/cm^2 in micron-size samples without significant overheating. In the case of tin whiskers the measurements were made only at 4.2 K.

The voltage drop across a sample on increase of the current passing through it was measured with an R-348 potentiometer to within 10^{-7} V, either point by point or by automatic recording.

REMOVAL OF HEAT FROM SAMPLES IN LIQUID HELIUM

Calculations indicated that the reduction in the resistance of samples of thickness of a few microns could be observed only by passing currents of density up to 10^6–10^7 A/cm^2. This would require removal of a dissipated power $q = J^2R/S_s$ (S_s is the area of the lateral surface of a whisker) corresponding to a density of several watts per square centimeter. It was therefore necessary to determine the maximum current density at which there would be no significant heating of the samples. This should be possible under conditions of nucleate boiling of He-I and nonfilm boiling of He-II, when the temperature rises on the surface would be small.[9] Moreover, determination of the maximum value of the heat flux density (q_{cr} in the case of He-I and q^*_{cr} in the case of He-II) removed from samples of micron thickness before the helium boiling crisis (i.e., before the transition to film boiling) was of interest in itself from the point of view of the influence of dimensions on the heat removal rate. The boiling crisis was known to be accompanied by a strong rise of the temperature of the surface of a sample. Our experiments showed that this resulted in thermal fracture of samples of micron thickness. It was also known that in the case of bulk samples the maximum values of the heat flux from a unit surface corresponding to the boiling crisis at 4.2 K was ~ 1 W/cm^2. It was established in Refs. 10 and 11 that when thin wires were used as the samples, a reduction in d increased the values of q_{cr} and q^*_{cr}. For example, in the case of a Manganin wire of thickness $d = 50\ \mu$ the maximum heat flux density was $q_{cr} = 1.6$ W/cm^2 at 4.2 K, whereas in the case of a constantan wire of thickness $d = 17\ \mu$ the corresponding value was $q^*_{cr} = 12$ W/cm^2 at 2 K.

We determined the values of the critical heat fluxes for samples of even smaller thicknesses. In the investigation of the boiling crisis it is usual to determine the dependence of the heat flux density q on the temperature difference ΔT between the liquid and surface. In the case of thin wires it is difficult to determine the temperature of the surface of the sample. Therefore, the sample itself is used as a thermometer. However, in the case of micron thickness whiskers at helium temperatures, when the temperature dependence $R(T)$ is weak, the error in the determination of ΔT by this method is very large. Nevertheless, according to the calculations and experiments reported in Ref. 9, in the case of liquid He-I the value of ΔT at the boiling crisis does not exceed 1 K for thin samples. Therefore, we simply determined the dependences of q_{cr} and q^*_{cr} on the thickness d. The values of q_{cr} and q^*_{cr} were deduced from the current which fractured a sample if up to that moment there was no increase in the resistance ($<1\%$). In other cases the boiling crisis corresponded to the current at which the resistance rose sharply.

Figure 1 shows the results of measurements at 4.2 and 2 K. The scatter of the points was attributed to defects caused by the mounting of whiskers and was sometimes associated with poor electrical contacts between the whiskers and the substrates, resulting in an additional evolution of heat (in both cases the whisker was fractured not in the middle between the potential contacts). Thus, sometimes even before the fracture of a whisker, a considerable rise of its resistance began from $q \approx 0.1$ W/cm^2. This usually happened when the contacts with a whisker were made by means of silver paste (results of this kind were not included in Fig. 1). In spite of the large scatter (which is generally typical of the investigations of properties of whiskers), we found that a reduction in d increased the values of q_{cr} and q^*_{cr}. At 4.2 K we found that the maximum value was $q_{cr} \approx 7$ W/cm^2 and in superfluid helium at 2 K the corresponding value was $q^*_{cr} \approx 20$ W/cm^2. These values corresponded to current densities in the range $j > 10^6$ A/cm^2. It should be stressed that these values could only be underestimated because any mounting defect could only reduce the critical current density j_{cr} and, consequently, the value of q_{cr} or q^*_{cr}. The best results are given in Table I.

RESULTS OF MEASUREMENTS AND DISCUSSION

In the first series of measurements we used whiskers not subjected to chemical etching. It was found that at 2 K the majority of copper whiskers and some of the cadmium whiskers were characterized by a weak nonlinearity of the current-voltage characteristics, corresponding to a fall of the whisker resistance. The maximum deviations of the characteristics from linearity amounted to ~ 2–3% for a cadmium whisker and $\sim 6\%$ for a copper whisker. Similar behavior of the current-voltage characteristics with a maximun deviation of 2% from linearity was recorded only for two out of

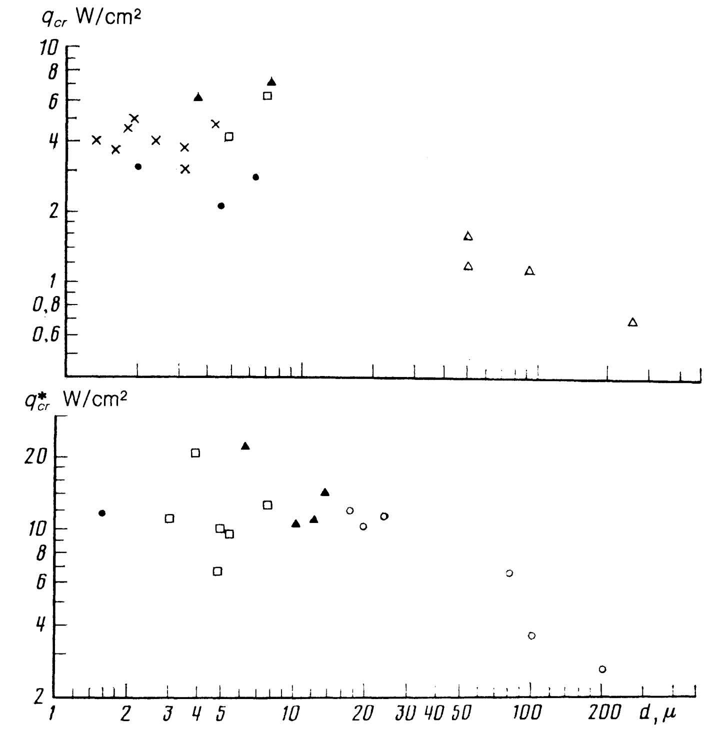

FIG. 1. Dependences of the critical power of the heat flux in liquid helium at 4.2 K (q_{cr}) and at 2 K (q^*_{cr}) on the thickness of a sample d: ×) tin whiskers; □) zinc whiskers; ●) cadmium whiskers; ▲) copper whiskers; △) Manganin wire (Ref. 10); ○) constantan wire (Ref. 11).

30 zinc whiskers. Tin whiskers showed no reduction in the resistance on increase in the current J.

The measurements were then repeated on zinc and cadmium samples subjected to etching in hydrochloric acid vapor. Such etching reduced the specularity coefficient P approximately twofold.[12] In some cases these were the samples belonging to the first batch, which were tested in such a way as to avoid burning through. The results of the measurements are presented in Fig. 2. Clearly, the etched whiskers acquired a considerable nonlinearity of the current-voltage characteristics. In the case of zinc whiskers ($d = 6.6\,\mu$) the deviation from Ohm's law reached $\sim 10\%$, whereas in the case of cadmium whiskers ($d = 6.2\,\mu$) it amounted to $\sim 20\%$. The reduction in the resistance $R = U/J$ in the case of zinc whiskers was observed until the sample burnt through, whereas in the case of cadmium whiskers the evolution of a power of density ~ 5 W/cm² induced a considerable rise of the resistance, which could be attributed to overheating of the sample or to the influence of the magnetoresistance.[13]

The observed reduction in the whisker resistance on increase in the current was in qualitative agreement with the predictions of the magnetodynamic nonlinearity theory. According to the theory, the effect occurs only in the case of diffuse reflection of conduction electrons from the surface of a sample and the results obtained for zinc, cadmium, and copper whiskers indicated a strong influence of the nature of the interaction of conduction electrons with the surface on the current-voltage characteristics of the metals. For example, the high initial specularity of reflection prevented observations of a significant reduction in the resistance of zinc and cadmium whiskers under conditions of a strong size effect, and only a reduction of this specularity by etching revealed the expected reduction. On the other hand, the diffuse nature of electron reflection from the surface of copper whiskers reduced their resistance on increase in the current even in the case when the inequality $l \lesssim d$ was obeyed.

A reduction in the thickness should also enhance the investigated resistance effect. We plotted in Fig. 3 the results obtained in the present study and also those reported in Refs. 2 and 6. Clearly, a reduction in the thickness of the samples enhanced the effect for the same value of the current. However, since $q \propto d^{-4}$, this imposed restrictions on a further reduction in the thickness. This could account for the absence of the resistance effect in the case of tin whiskers which, although they satisfy the conditions of a strong size

TABLE I.

Sample	T, K	d, μ	ρ, Ω·cm	j_{cr}, A/cm²	q_{cr} (q^*_{cr}), W/cm²
Cu-13 *	4,2	7.2	$4.0 \cdot 10^{-8}$	$1.0 \cdot 10^{6}$	7.2
Cu-20	2	13	$2.8 \cdot 10^{-8}$	$1.3 \cdot 10^{6}$	15.4
Cd-9 **	2	3.8	$1.3 \cdot 10^{-8}$	$2.9 \cdot 10^{6}$	10.4
Sn-25	4,2	1.8	$2.0 \cdot 10^{-8}$	$2.3 \cdot 10^{6}$	4.8
Zn-15	2	3.6	$2.6 \cdot 10^{-8}$	$3.0 \cdot 10^{6}$	21

*At high current densities the sample overheated.
**The sample burnt through between the current and potential contacts.

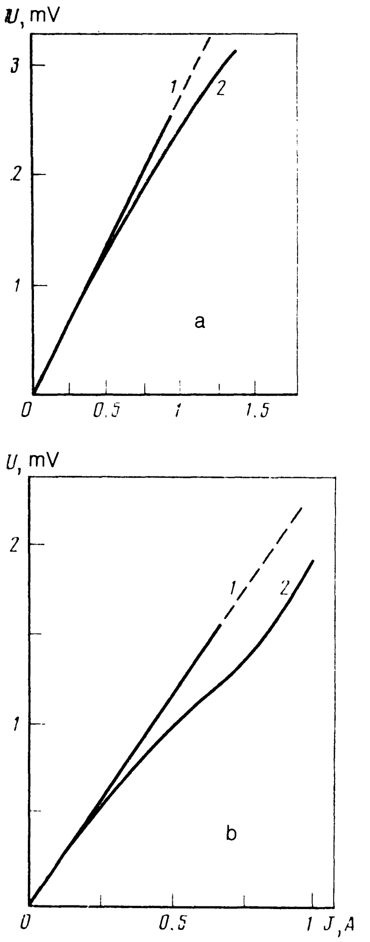

FIG. 2. a) Current-voltage characteristics of zinc whiskers ($d = 6.6\mu$) at $T = 2$ K. Curve 1 was obtained before etching, whereas curve 2 was obtained after etching whiskers in HCl vapor. b) Current-voltage characteristics of cadmium whiskers ($d = 6.2\mu$) at $T = 2$ K after etching in HCl vapor (curve 2). Curve 1 represents linear extrapolation of the initial part of the characteristic. The voltages U are given for etched samples.

effect ($l \gg d$) and of diffuse interaction of electrons with the surface, exhibit the helium boiling crisis for samples of thickness $d < 4\,\mu$ at currents much smaller than i.

A calculation of the characteristic value of the current i, at which the resistance decreased by a factor of two, carried out using the formulas derived in Ref. 4, gave 0.5 and 1 A for zinc ($d = 6.6\,\mu$) and cadmium ($d = 6.2\,\mu$), respectively. However, the experimental results indicated that at these values of the current the effect was much weaker (Fig. 2). One of the possible reasons could be some residual specularity of electron reflection in whisker crystals. Moreover, the theory required that the strong inequalities $d \ll R_{\rm Larm} \ll l$ be satisfied simultaneously, which might not be true in practice. For example, in the case of zinc whiskers of thickness $6.6\,\mu$ for a current of 0.5 A the field on the surface of a sample was $H = 300$ Oe, corresponding to $R_{\rm Larm} \approx 150\,\mu$, i.e., the condition $R_{\rm Larm} \ll l$ was not obeyed; moreover, this condition was disobeyed also in the case of cadmium whiskers of thickness $6.2\,\mu$ for which the value of $R_{\rm Larm}$ corresponding to 1 A was $\approx 100\,\mu$, i.e., $R_{\rm Larm} \lesssim l$. It should be pointed out that the condition $R_{\rm Larm} \ll l$ could not be satisfied at all when the current was $J = i$, because, as demonstrated by the magnetodynamic theory formulas,[4] at the current i we should have $R_{\rm Larm} \approx l$ irrespective of the ratio l/d. Therefore, a correct quantitative comparison with the theory would require the passage of currents $J \gg i$ through a sample.

The results of this investigation demonstrated that a strong deviation from Ohm's law because of the magnetohydrodynamic effect could hardly be detected, in experiments. In fact, an enhancement of the effect would require an increase of the ratio J/i. It should be remembered that $i \propto d/l$. It would be unrealistic to reduce i by increasing l to values much greater than 0.1 cm. Samples of micron thickness with large values of l can only be whiskers. However, in such samples well before the condition $J \gtrsim i$ is attained the current density j would be sufficient to cause the helium boiling crisis. Finally, reduction in the thickness increases the specularity of the surface of a sample because of the increasing contribution of the fraction of "glancing" electrons to the conduction process. It is interesting to note that an increase in l has the same result. One possibility remains: it is necessary to ensure that the reflection of electrons is 100% diffuse without change in l. At present it is not clear how this could be achieved.

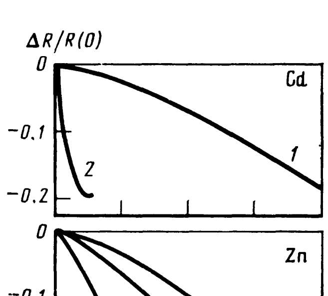

FIG. 3. Dependence of the resistance $\Delta R/R(0) = [R(0) - R(J)]/R(0)$ of thin samples on the current J. The upper part of the figure represents cadmium samples: 1) sample of 0.25×0.12 mm cross section (Ref. 6); 2) cadmium whisker of thickness $d = 6.2\,\mu$. The lower part of the figure represents zinc samples: 1), 2) wires with diameters $d = 0.42$ mm and 0.23 mm, respectively (Ref. 2); 3) whisker of diameter $d = 6.6\,\mu$.

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Dependence of the critical current on the magnetic field applied to disordered Josephson junctions

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A calculation is reported of the dependence of the critical current on the external magnetic field applied to a disordered Josephson contact. It is shown that in strong magnetic fields the field dependence of the critical current may differ greatly from the usual "Fraunhofer" relationship even in the case of a large-area contact. The dependence $I_c(H)$ should exhibit characteristic random fluctuations with an amplitude of the order of the current itself. This "mesoscopic" behavior of a disordered Josephson contact is explained by compensation of the currents of optimal fluctuations over a large part of a sample. The range of existence of the effect is found.

1. INTRODUCTION

Disordered Josephson junctions are currently attracting major interest.[1-6] The disorder of a contact may be associated with a local change in the thickness of the barrier layer, a local reduction in the barrier height,[4] resonant passage of coherent electrons along paths consisting of periodically distributed impurity centers,[5,6] etc. In all these cases the critical current density $j_c(x, y)$ varies along the plane of the junction.

The dependence of the critical current I_c on the external magnetic field H applied to a Josephson junction with structural fluctuations was calculated in Refs. 1–3. The average critical current was shown to fall in strong magnetic fields not to zero but to a constant value I_0 (i.e., a pedestal independent of the magnetic field was observed):

$$\overline{I_c^2}=\frac{\sin^2(\pi\Phi/\Phi_0)}{(\pi\Phi/\Phi_0)^2}I_1^2+I_0^2\left[1-\frac{\sin^2(\pi\Phi/\Phi_0)}{(\pi\Phi/\Phi_0)^2}\right],$$

$$I_0^2=Sr_0^2\overline{(\Delta j_c)^2},\qquad I_1=S\bar{j}_c, \tag{1}$$

where S is the junction area; r^2_0 is the characteristic size of a fluctuation region; $\bar{j}_c$ and $\overline{(\Delta j_c)^2}$ are the average current density and the average square of the fluctuation amplitude; Φ is the external magnetic flux; Φ_0 is a magnetic flux quantum.

It follows from Eq. (1) that if $I_0 \ll I_1$, then the dependence of the average critical current on the magnetic field is close to the usual "Fraunhofer" dependence.[1]

Equation (1) is derived on the assumption of Gaussian fluctuations of the critical current density $j_c(x,y)$. However, as shown in Refs. 4–6, in many cases the probability $F(j)$ of fluctuations of the critical current density differs greatly from the Gaussian value. In the absence of a magnetic field H it is found that the current through a typical Josephson contact found experimentally is equal to the average current [Eq. (1)] only in the case of large-area samples:

$$S\gg r_0^2 j_0\Big/\int F(j)j\,dj, \tag{2}$$

where the optimal fluctuation j_0 is found from the condition for the maximum of the function $W(j) = jF(j)$ (Refs. 4 and 5). In the opposite case the critical current fluctuates from sample to sample by an amount equal to the current itself, and the critical current through a typical junction is then found from the condition

$$I_c=r_0^2 j_1,\qquad (S/r_0^2)\int_{j_1}^{\infty}F(j)\,dj\sim 1.$$

We shall show in Secs. 2 and 3 that in the presence of a magnetic field, even if the junction area satisfies the condition (2), the critical current may vary greatly from sample to sample and may differ from the average current given by Eq. (1). Therefore, in Sec. 2 we shall find the probability of fluctuations of the critical current through a disordered Josephson junction, whereas in Sec. 3 we shall determine the current through a typical Josephson junction.

2. PROBABILITY $P(I_0^2)$ OF FLUCTUATIONS OF THE CRITICAL CURRENT THROUGH A DISORDERED JOSEPHSON JUNCTION

The experimentally observed dependence $I_c(H)$ can be found by calculating the probability $P[I_c^2(H)]$ that a disordered Josephson junction has a critical current I_c.

We shall consider a small Josephson junction of length $L<\lambda_j$ (equal to the Josephson penetration depth) and of width W, subjected to a magnetic field H. The critical current density $j_c(x,y)$ is a random function and its fluctuations are described by the probability density $F(j)$. The radius of action of a fluctuation is r_0. Then, the square of the critical current is given by[1]

$$I_c^2(H)=\left|\int_0^L\int_0^W j_c(x,y)\exp(2\pi i\Phi x/\Phi_0 L)\,dx\,dy\right|^2. \tag{3}$$

(The magnetic field is directed along the y axis.)

The density of the probability $P(I^2_c)$ that a Josephson junction has a critical current I_c is described by the functional integral

$$P(I_c^2)=\int Dj_c\rho(j_c)\delta\Big(I_c^2-\Big|\int_0^L\int_0^W j_c(x,y)\times\exp(2\pi i\Phi x/\Phi_0 L)\,dx\,dy\Big|^2\Big), \tag{4}$$

where $\rho(j_c)$ is the probability of the appearance of a distribution of the critical current density $j_c(x,y)$ in the junction. Knowing the radius of action of the fluctuations and the probability of formation of different fluctuations of j_c, we

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can readily reduce the functional integral to an Nth integral:

$$P(I_c^2)=\int dj_1\ldots dj_N F(j_1)\ldots F(j_N)$$
$$\times\delta\left\{I_c^2-r_0^4\left[\sum_{i=1}^N j_i\exp(2\pi i\Phi x_i/\Phi_0 L)\right]\right.$$
$$\left.\times\left[\sum_{i=1}^N j_i\exp(-2\pi i\Phi x_i/\Phi_0 L)\right]\right\}. \quad (5)$$

In the integral of Eq. (5) we have $N=S/r_0^2$, where $S=WL$ is the Josephson junction area.

We shall transform the integral (5) to

$$P(I_c^2)=(2\pi)^{-1}\int dz\,dt\int dj_1\ldots dj_N F(j_1)\ldots F(j_N)$$
$$\times\delta\left(t-r_0^2\sum_{i=1}^N j_i\exp(-2\pi i\Phi x_i/\Phi_0 L)\right)$$
$$\times\exp\left[izI_c^2-ir_0^2zt\sum_{i=1}^N j_i\exp(2\pi i\Phi x_i/\Phi_0 L)\right]$$
$$=(2\pi)^{-2}\int dz\,dt\,du\int dj_1\ldots dj_N F(j_1)\ldots F(j_N)\exp(izI_c^2+itu)$$
$$\times\exp\left[-ir_0^2zt\sum_{i=1}^N j_i\exp(2\pi i\Phi x_i/\Phi_0 L)\right.$$
$$\left.-iur_0^2\sum_{i=1}^N j_i\exp(-2\pi i\Phi x_i/\Phi_0 L)\right]. \quad (6)$$

If the Josephson phase $\Phi x/\Phi_0 L$ changes little in the fluctuation region ($\Phi r_0/\Phi_0 L\ll 1$), the $(N+3)$rd integral in Eq. (6) can be reduced to the triple integral:

$$P(I_c^2)=(2\pi)^{-2}\int dz\,dt\,du\exp(izI_c^2+itu)$$
$$\times\exp\left[\sum_{i=1}^N \ln\int F(j)\exp\{-ijr_0^2[zt\exp(2\pi i\Phi x_i/\Phi_0 L)\right.$$
$$\left.+u\exp(-2\pi i\Phi x_i/\Phi_0 L)]\}dj\right]$$
$$=(2\pi)^{-2}\int dz\,dt\,du\exp(izI_c^2+itu)$$
$$\times\exp\left\{\frac{W}{r_0^2}\int_0^L dx\ln\int dj\,F(j)\exp\{-ijr_0^2[zt\exp(2\pi i\Phi x/\Phi_0 L)\right.$$
$$\left.+u\exp(-2\pi i\Phi x/\Phi_0 L)]\}\right\}. \quad (7)$$

Using Eq. (7), we can obtain first of all a formula similar to Eq. (1) but for an arbitrary distribution function $F(j)$]:

$$\overline{I_c^2}=\int I_c^2P(I_c^2)\,dI_c^2=I_0^2+\frac{\sin^2(\pi\Phi/\Phi_0)}{(\pi\Phi/\Phi_0)^2}I_1^2,$$
$$I_1=S\bar{j}_c=S\int F(j)j\,dj, \quad (8)$$
$$I_0^2=Sr_0^2(\overline{j_c^2}-\bar{j}_c^2)=Sr_0^2\left[\int F(j)j^2\,dj-\left(\int F(j)j\,dj\right)^2\right].$$

The difference between Eqs. (1) and (8) is the assumption, in the former case, of constancy of the critical current in zero magnetic fields for all the samples.[1-3]

The most interesting results are obtained when the function $F(j)$ differs strongly from the Gaussian form. In the majority of cases, $F(j)$ is given by (Fig. 1)[4-6]

$$F(j)=(1-\gamma)\delta(j-\tilde{j})+F_1(j),\qquad \gamma=\int F_1(j)\,dj\ll 1, \quad (9)$$

where $\tilde{j}$ is the current density in the homogeneous part of the Josephson junction and $F_1(j)$ is the distribution function describing low-probability fluctuations of the critical current density. The average density of the current in the absence of a magnetic field may be determined by rare fluctuations characterized by $\int F_1(j)\,jdj\gg\tilde{j}$.

Substituting Eq. (9) into Eq. (7), we obtain

$$P(I_c^2)=(2\pi)^{-2}\int dz\,dt\,du\exp[izI_c^2+itu-i\tilde{S}\tilde{j}(zt+u)]$$
$$\times\exp\left\{(W/r_0^2)\int_0^L dx\int dj\,F_1(j)\{\exp[-ijr_0^2[zt\exp(2\pi i\Phi x/\Phi_0 L)\right.$$
$$\left.+u\exp(-2\pi i\Phi x/\Phi_0 L)]]\}\right\}, \quad (10)$$
$$\tilde{S}=S\left(\frac{\sin\pi\Phi/\Phi_0}{\pi\Phi/\Phi_0}\right).$$

When the external magnetic flux Φ is small or close to an integral number of flux quanta $((|\Phi/\Phi_0-k|\ll 1$, where k is an integer), the integral with respect to the coordinate x in Eq. (10) can be calculated and we then obtain

$$P(I_c^2)=\int\frac{dz\,dt\,du}{(2\pi)^2}\exp[izI_c^2+itu-i\tilde{S}\tilde{j}(zt+u)]$$
$$\times\exp\left\{\frac{\tilde{S}}{r_0^2}\int dj\,F_1(j)\{\exp[-ijr_0^2(zt+u)]-1\}\right.$$
$$\left.+\frac{S-\tilde{S}}{r_0^2}\int F_1(j)\{J_0(2(ztu)^{1/2}jr_0^2)-1\}dj\right\}, \quad (11)$$

where J_0 is a Bessel function.

Using the smallness of the argument of the function J_0 Eq. (11) can be simplified so that we can obtain the final expression for the probability of fluctuations of the critical junction in a disordered Josephson junction:

$$P(I_c^2)=(2\pi)^{-2}\int dz\,dt\,du\exp[izI_c^2+itu-i\tilde{S}\tilde{j}(zt+u)]$$
$$\times\exp\left\{\frac{\tilde{S}}{r_0^2}\int dj\,F_1(j)\{\exp[-ijr_0^2(zt+u)]-1\}\right.$$
$$\left.-\frac{S-\tilde{S}}{r_0^2}\int F_1(j)j^2\,dj\,ztu\right\}. \quad (12)$$

3. CRITICAL CURRENT IN A TYPICAL JOSEPHSON JUNCTION

Equation (8) describes the average critical current for a large number of samples. However, if the function $F_1(j)$ in Eq. (12) differs greatly from the Gaussian form, the critical current for a typical Josephson junction may differ greatly from the value obtained from Eq. (8). The critical current in a typical disordered Josephson junction can be found from

$$I_T=\exp\langle\ln I_c\rangle,\qquad \langle\ln I_c\rangle={}^1/_2\int_0^\infty dI_c^2(\ln I_c^2)P(I_c^2). \quad (13)$$

We shall calculate the average logarithm in Eq. (13) by transforming Eq. (12) to a more convenient form. We shall do this by substitution of the variables $zt+u=x$ and we shall then calculate the integral with respect to t. We then obtain the following expression for $P(I_c^2)$:

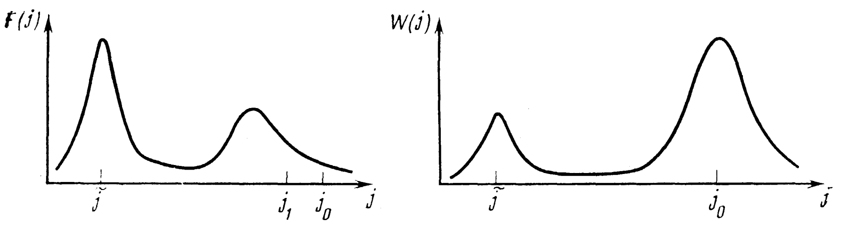

FIG. 1. Probability $F(j)$ of fluctuations of the critical current density: here, j_0 is the optimal fluctuation in the range of magnetic fields $|\sin(\pi\Phi/\Phi_0)(\pi\Phi/\Phi_0)^{-1}| \gg \alpha_1$; j_1 is the optimal fluctuation in the range of magnetic fields $\alpha_2 \ll |\sin(\pi\Phi/\Phi_0)/(\pi\Phi/\Phi_0)^{-1}| \ll \alpha_1$.

$$P(I_c^2) = \frac{1}{4\pi^{3/2}} \int dx\, dz\, z^{-1/2} \left[i - I_0^2 z \left(1 - \frac{\tilde{S}}{S} \right) \right]^{-1/2}$$
$$\times \exp\left\{ i I_c^2 z + \left[i - I_0^2 z \left(1 - \frac{\tilde{S}}{S} \right) \right] \frac{x^2}{4z} \right\}$$
$$\times \exp\left\{ -i\tilde{S}\tilde{j}x + \frac{\tilde{S}}{r_0^2} \int F_1(j) \{\exp(-ijr_0^2 x) - 1\} dj \right\}. \quad (14)$$

Substituting Eq. (14) into Eq. (13) and calculating the integrals for I_c^2 and z, we find with a logarithmic precision that

$$\langle \ln I_c \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dx}{x} \ln\left[\alpha I_0^2 \left(1 - \frac{\tilde{S}}{S} \right) + \frac{1}{x^2} \right]$$
$$\times \exp\left\{ -i\tilde{S}\tilde{j}x + \frac{\tilde{S}}{r_0^2} \int F_1(j) [(\exp(-ijr_0^2 x) - 1) dj] \right\},$$
$$\alpha = e^{-C}, \quad (15)$$

where $C = 0.577...$ is the Euler constant.

When the junction area S is sufficiently large, so that

$$|\tilde{S}| \gg r_0^2 j_0 \left(\int F(j) j\, dj \right)^{-1} \quad (16)$$

[j_0 is found from the condition for the maximum of the function $W = F(j)j$ (Fig. 1)], the integral in Eq. (15) is dominated by small values of x ($j_0 r_0^2 x \ll 1$) and the current for a typical Josephson junction is described by the expression

$$I_t^2 = \alpha I_0^2 \left(1 - \frac{\tilde{S}}{S} \right) + \frac{\tilde{S}^2}{S^2} I_1^2. \quad (17)$$

In the opposite case the integral in Eq. (15) is dominated by those values of x which satisfy the condition $jr_0^2 x \sim 1$ and we then obtain with logarithmic precision

$$I_t^2 = \alpha I_0^2 (1 - \tilde{S}/S) + (j_1 r_0^2)^2, \quad (18)$$

where the current density j_1 is found from the condition

$$\frac{|\tilde{S}|}{r_0^2} \int_{j_1}^{\infty} F_1(j)\, dj \sim 1. \quad (19)$$

The current through a typical Josephson junction can then differ greatly from the average current calculated from Eq. (8). This result has a simple physical meaning. In a disordered Josephson junction of area S satisfying the condition (2) there are many optimal fluctuations of the current density j_0 which determine the Josephson current in zero external magnetic field. However, when the magnetic flux Φ is close to an integral number of magnetic field quanta Φ_0 the optimal fluctuations of the current j_0 distributed over a large part of the area S cease to contribute to the total current, since the currents in these fluctuations flow in opposite directions and the fluctuations compensate one another. Therefore, a part of the sample makes a field-independent contribution to the total current $\sim I_0$ and the dependence of the critical current on the magnetic field is governed solely by fluctuations formed on the "effective" area

$$|\tilde{S}| = S |\sin(\pi\Phi/\Phi_0)(\pi\Phi/\Phi_0)^{-1}|.$$

Then, if the effective area is sufficiently small, there are no fluctuations of the current density j_0 in a typical sample and the critical current is governed by fluctuations which may occur in that small effective area and which also carry the maximum current, i.e., fluctuations of the quantity j_1 found from the condition of Eq. (19) (Fig. 1).

Equations (8) and (18) are valid if the critical current is governed by low-probability fluctuations. However, in the range of small effective areas when

$$|\tilde{S}| \ll r_0^2/\gamma, \quad (20)$$

the dependence of the critical current on the magnetic field is governed by the homogeneous part of the Josephson junction and it is described by

$$I_t^2 = \alpha I_0^2 (1 - \tilde{S}/S) + \tilde{j}^2 \tilde{S}^2. \quad (21)$$

4. DISCUSSION OF RESULTS

The results obtained show that the dependence of the critical current I_c of a disordered Josephson junction on an external magnetic field H (or on the magnetic flux $\Phi = HLd$, where d is the depth of penetration of the magnetic field into this superconductor) may differ greatly from the usual Fraunhofer dependence and from the dependences obtained on the assumption of Gaussian fluctuations of the density of the critical current j_c considered in Refs. 1–3 [see Eq. (1)].

In the range of low magnetic fields the change in the critical current is described by Eq. (17) and the quantity $(I^2 - \alpha I_0^2)^{1/2}$ depends on the magnetic field in the usual Fraunhofer manner (Fig. 2). Then, the probability $P(I_c^2)$ of different values of the critical current is a Gaussian function and the fluctuations of the total critical current from sample to sample are small.

When the external magnetic flux Φ approaches an integral number of flux quanta the condition

$$\left| \frac{\sin \pi\Phi/\Phi_0}{\pi\Phi/\Phi_0} \right| \ll \alpha_1 = \frac{r_0^2}{SF(j_0)} \left(\frac{d^2 \ln F(j_0)}{dj^2} \right)^{1/2} \quad (22)$$

is satisfied, the total critical current flowing in a disordered Josephson junction deviates from the usual Fraunhofer dependence and is described by Eqs. (18) and (19). The dependence of the critical current on the magnetic field is determined by fluctuations of the quantity j_1 over an effective area $|\tilde{S}|$.

It should be pointed out that in this range of fields the probability $P(I_c^2)$ differs greatly from the Gaussian form and the critical current fluctuates from sample to sample by an amount of the order of the current itself, whereas $\ln(I_c - I_0)$ fluctuates by an

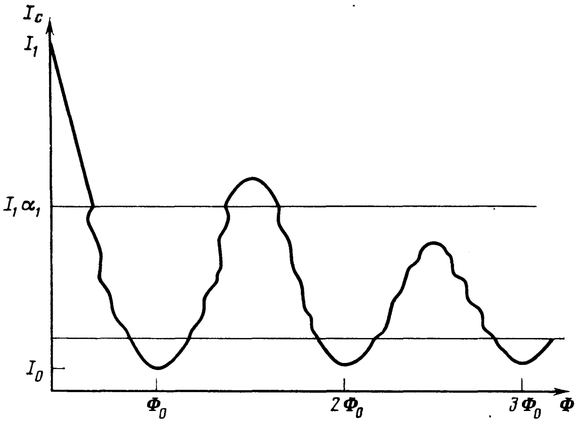

FIG. 2. Dependence of the critical current on the magnetic field in disordered Josephson junctions.

amount of the order of unity. Consequently, the dependence of the critical current on the magnetic field should also exhibit random fluctuations. The amplitude of these fluctuations should be $\ln(I_c - I_0) \sim 1$ and the period of the fluctuations $\Delta\Phi$ is readily found from the condition (19): $\Delta\Phi \propto (|\Phi - k\Phi_0|)$, where k is an integer (Fig. 2).

This behavior of a Josephson junction in an external magnetic field (i.e., the difference between the critical current of a typical Josephson junction and the current averaged for a large number of samples; random fluctuations of the current of magnitude of the order of the current itself when an external parameter is varied) is in many ways similar to other "mesoscopic" phenomena observed in disordered structures.[7,8] Such mesoscopic behavior of a Josephson current is retained as long as the external magnetic flux is sufficiently close to the integral number of flux quanta so that there are no fluctuations over the effective area $|\tilde{S}|$:

$$\left|\frac{\sin \pi\Phi/\Phi_0}{\pi\Phi/\Phi_0}\right| \ll \alpha_2 = \frac{r_0^2}{S\gamma}, \tag{23}$$

whereas the critical current is governed by the homogeneous part of the Josephson junction [Eq. (21)]. In this case the dependence of the critical current on the magnetic field is again of the Fraunhofer type. Therefore, when the magnetic field is varied, the critical current either repeats the Fraunhofer pattern or deviates strongly from it (Fig. 2).

It should be pointed out that in the case of high magnetic fields the critical current through a disordered junction does not approach zero but a constant value $\sim I_0$ independent of the magnetic field. However, in contrast to the current I_1 observed in the absence of a magnetic field, this constant current varies greatly (by an amount equal to its full value) from sample to sample. In fact, in high magnetic fields we can ignore the last exponential function in the integrand in Eq. (14). Calculating the integrals with respect to x and z, we find that

$$P(I_c^2) = I_0^{-2} \exp(-I_c^2/I_0^2).$$

Calculating the average current and its variance with the aid of this function, we can readily show that $\langle(\Delta I)^2\rangle \propto (\langle I\rangle)^2 \propto I_0^2$. Therefore, if the quantity I_0 depends on the external parameter (for example, on temperature T), we may observe mesoscopic fluctuations of the dependence $I_0(T)$. The amplitude of these fluctuations $\ln I_0$ is unity and the fluctuation period is found from the condition $\Delta T \propto (\partial \ln I_0^2/\partial T)^{-1}$. We can therefore use the characteristic dependence of the critical current on the external magnetic field to determine the disorder of a Josephson contact [and the nature of the distribution function $F(j)$ of the critical current density].

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