Anderson transition on a Bethe lattice (the symplectic and orthogonal ensembles)

K. B. Efetov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR
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A model of a granulated disordered metal on a Bethe lattice is studied. The model is invariant under time reversal (the symplectic ensemble and orthogonal ensemble). The existence of a metal-dielectric transition is proved and the critical points are calculated. The density-density correlation function is calculated both in the metallic region and in the dielectric region. The localization length in the dielectric region near the transition point increases as the reciprocal of the distance from the transition point. The diffusion coefficient in the metallic region falls exponentially as the transition point is approached. The form of the density-density correlators and the critical behavior of the orthogonal ensemble, the unitary ensemble, and the symplectic ensemble are the same.

1. INTRODUCTION

In recent papers[1–4] the Anderson metal-dielectric transition in the model of a disordered granulated metal on a Bethe lattice has been studied. Application of the method of supersymmetry[5] has made it possible to reduce the calculation of various correlation functions to the calculation of integrals of the solutions of certain integral equations. In Ref. 1 the author proved the existence of the transition and calculated the correlator of the densities at coinciding points. In Ref. 3 the behavior of a certain two-point correlation function at noncoinciding points was investigated. Finally, in Ref. 4 the density-density correlator at noncoinciding points was calculated directly. The asymptotic form of this correlation function in the limit of large distances and low frequencies has made it possible to calculate the diffusion coefficient in the metallic region and the localization length and permittivity in the dielectric region. The principal assertion of Ref. 4 was the conclusion that near the transition point the diffusion coefficient falls off exponentially. (The assertion made in Ref. 1 that there exists a minimum metallic conductivity was the result of an insufficiently accurate investigation of the integral equation.) In the dielectric region the localization length increases as the reciprocal of the distance from the transition point.[3,4]

However, in Refs. 1–4, only a model with broken time-reversal symmetry was investigated. Physically, this case corresponds to the presence of magnetic fields or magnetic impurities in the granules. Disordered systems that are not invariant under time reversal correspond to the case of an ensemble of unitary matrices,[6] which turns out to be mathematically the simplest case. This was the reason why systems with broken time-reversal symmetry were investigated in Ref. 4. Other possibilities are realized in systems that are invariant under time reversal and possess central symmetry (the orthogonal ensemble), and in systems that are invariant under time reversal but do not possess central symmetry (the symplectic ensemble). The symplectic ensemble corresponds to the situation when there is spin-orbit scattering in the system but magnetic fields and magnetic impurities are absent.

In the present paper we investigate the Anderson metal-dielectric transition on a Bethe lattice for systems that are invariant under time reversal (the symplectic and orthogonal ensembles). The asymptotic form of the density-density correlator in the limit of low frequencies and large distances is calculated. It is shown that in this limit the form of the density-density correlator, both in the metallic and in the dielectric region, is the same for all three types of symmetry. Although the critical points of the transition are different, the critical behavior of the diffusion coefficients in the metallic region and of the localization lengths in the dielectric region is the same. In all three cases the diffusion coefficient decreases exponentially as the transition point is approached, and the localization length is inversely proportional to the distance from the transition point.

2. THE BASIC EQUATIONS

The kinetics of a system of disordered granules is described by the supersymmetric σ-model on a lattice. The effective Hamiltonian in this model is written in the form [5,1]

$$F=-\gamma_0\Big[\sum_{i,j} J_{ij}\,\mathrm{STr}\,Q_iQ_j-\frac{i(\omega+i\delta)}{4}\pi\nu\sum_i V_i\,\mathrm{STr}\,\Lambda Q_i\Big], \tag{1}$$

where $J_{ij}=T_{ij}^2\nu^2V_iV_j$, and T_{ij} is the granule-to-granule hopping amplitude. The first term in (1) describes the interaction of the granules, and the second term is the effective Lagrangian of the isolated granules. The letters ω and ν denote the frequency and density of the levels in the granules, V_i is the volume of the granules, and STr is the supertrace. The supermatrices Q and Λ in (1) have dimensions 8×8 and are equal to

$$Q=UQ_0\bar{U},\quad Q_0=\begin{pmatrix}\cos\hat{\theta} & i\sin\hat{\theta}\\ -i\sin\hat{\theta} & -\cos\hat{\theta}\end{pmatrix},$$
$$U=\begin{pmatrix}u & 0\\ 0 & v\end{pmatrix},\quad \Lambda=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2}$$

The matrix $\hat{\theta}$ has the structure

$$\hat{\theta}=\begin{pmatrix}\theta_{11} & 0\\ 0 & \theta_{22}\end{pmatrix}.$$

A bar above a matrix denotes the charge conjugate. The explicit form of the matrices u, v, and $\hat{\theta}$ is written out in Ref. 5. We note only the following important properties of the matrices:

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$$\bar{u}u=1,\quad \bar{v}v=1,\quad \bar{u}=u^{+},$$

$$\bar{v}=kv^{+}k,\quad \hat{\bar{\theta}}=k\hat{\theta}^{+},\quad k=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2a}$$

The form of the matrices u, v, θ_{11}, and θ_{22} and the coefficient γ_0 depend on the symmetry of the physical interactions in the system. If in the system there is no interaction of a spin and particle with external fields (there are no magnetic or spin-orbit impurities), the particles with different spins can be considered independently. In this case, $\gamma_0=1$. If there are magnetic or spin-orbit impurities, then $\gamma_0=2$.

The calculation of the density correlator $K(r_1, r_2)$ in the model described reduces to the calculation of the following correlator:

$$K(r_1,r_2)=-2\pi^2\nu^2\gamma_0\int (Q_{13}^{12})_{r_1}(Q_{31}^{21})_{r_2}\exp(-F[Q])\prod_i dQ_i, \tag{3}$$

where $F[Q]$ is determined by the expression (1). The superscripts in (3) label the blocks identified explicitly in (2), and the subscripts label the elements in these blocks.

Below we shall calculate the density-density correlator (3) on a Bethe lattice (Cayley tree). As in Refs. 1 and 4, we consider the case of arbitrary branching number m. We assume that only nearest neighbors interact, and that for these all the J_{ij} are the same and equal to J.

The structure of the Bethe lattice makes it possible to reduce the calculation of the integral (3) for arbitrary r_1 and r_2 to the calculation of a certain integral of the solution of certain integral equations. These equations were obtained in Ref. 4. We shall write out these equations and the formula for the density--density correlator in the general form applicable for any type of symmetry:

$$K(r)\equiv N(r)K(r_1,r_2)=-2\pi^2\nu^2\gamma_0\int Q_{13}^{12}P_{31}(r,0)Z(Q)\Psi(Q)\,dQ,$$

$$Z(Q)=\Psi^{m-1}(Q)\exp(\tfrac{1}{4}\beta\,\mathrm{STr}\,\Lambda Q). \tag{4}$$

The supermatrix function $P(r, Q)$ satisfies the linear equation

$$P(r,Q)-[m+\delta(r-1)]\int \exp(\tfrac{1}{4}\alpha\,\mathrm{STr}\,QQ')Z(Q')\times P(r-1,Q')\,dQ'=\delta(r)\Psi(Q)Q^{21}. \tag{5}$$

Finally, for the function Ψ we have the nonlinear integral equation

$$\Psi(Q)=\int \exp(\tfrac{1}{4}\alpha\,\mathrm{STr}\,QQ'+\tfrac{1}{4}\beta\,\mathrm{STr}\,\Lambda Q')\Psi^{m}(Q')\,dQ'. \tag{6}$$

For $\beta=0$, Eq. (6) has the solution

$$\Psi(Q)=1. \tag{6a}$$

This statement is proved by a change of variables in the integral

$$\int \exp(\tfrac{1}{4}\alpha\,\mathrm{STr}\,QQ')\,dQ'=\int \exp(\tfrac{1}{4}\alpha\,\mathrm{STr}\,\Lambda\tilde{Q})\,d\tilde{Q},$$

where $\tilde{Q}$ has the same structure as Q. But the latter integral is equal to unity,[5] and this proves (6a). The solution (6a) corresponds to the dielectric region. In the metallic region a nontrivial solution, which will be discussed later, appears.

In the expressions (4) and (5), r is the distance between sites r_1 and r_2, and $N(r)=m^{r-1}(m+1)$ $(r>1)$ is the number of sites at a distance r from a given site. We assume that $N(0)=1$ and that $P(r, Q)=0$ for $r<0$. The parameters α and β in (4)–(6) are equal to

$$\alpha=8\gamma_0 J,\qquad \beta=-i\gamma_0(\omega+i\delta)\pi\nu V. \tag{7}$$

The function $K(r_1, r_2)$ always decreases exponentially with distance, because of the exponential increase of the number of sites situated at a particular distance r. The factor $N(r)$ in (4) has been included precisely to compensate for this decrease.

For the calculation of the correlator $K(r)$ (4), in a previous paper the author[4] proposed the following scheme. We assumed that the solution P of Eq. (5) has the same structure as the matrix Q^{21}:

$$P=-i\nu P_0(Q_0)\bar{u},\quad P_0=\begin{pmatrix}p_{011} & 0\\ 0 & p_{022}\end{pmatrix}. \tag{8}$$

Substituting P (8) into (5) and integrating over u and v, we were able to reduce Eq. (5) to a system of equations for the components of P_0. After this, we expanded in eigenfunctions of the integral operator appearing in the left-hand side of (5).

However, this scheme is rather cumbersome. It is much simpler to perform the expansion of the solution P in eigenfunctions of the integral operator appearing in the left-hand side of (5) before the integration over u and v. Of course, these two procedures are equivalent to each other. Making the replacement

$$P(r,Q)=-iZ^{-1/2}(Q)\tilde{P}(r,Q), \tag{9}$$

we bring Eq. (5) to the form

$$\tilde{P}(r,Q)-[m+\delta(r-1)]\hat{M}\tilde{P}(r-1,Q)=i\delta(r)\Psi(Q)Q^{21}Z^{1/2}(Q), \tag{10}$$

where

$$\hat{M}\varphi(Q)\equiv\int \exp(\tfrac{1}{4}\alpha\,\mathrm{STr}\,QQ')[Z(Q)Z(Q')]^{1/2}\varphi(Q')\,dQ' \tag{11}$$

for an arbitrary function φ.

We shall consider the eigenfunctions $\varphi_E(Q)$ of the operator $\hat{M}$. These functions satisfy the equation

$$\hat{M}\varphi_E(Q)=E\varphi_E(Q). \tag{12}$$

We shall seek the functions φ_E in the class S_4 of (4×4) supermatrices φ satisfying the condition

$$\varphi^{+}(Q)=\bar{\varphi}(Q)k. \tag{13}$$

In particular, the supermatrix iQ^{21} satisfies the conditions (13).

To expand the solution $\tilde{P}$ in the eigenfunctions φ_E it is necessary to introduce the scalar product in the space of the functions φ. We assume, by definition, that the scalar product of two matrices $\varphi_1(Q)$ and $\varphi_2(Q)$ from the class S_4 is equal to

$$(\varphi_1,\varphi_2)\equiv\int \mathrm{STr}[k\varphi_1^{+}(Q)\varphi_2(Q)]\,dQ. \tag{14}$$

Using the definition (14) and the property (13), we can prove that

$$(\varphi_1, \varphi_2)=(\varphi_2, \varphi_1). \tag{14a}$$

In addition, the scalar product (14) is real:

$$(\varphi_1, \varphi_2)=(\varphi_1, \varphi_2)^*. \tag{14b}$$

It is easy to see that the operator $\hat{M}$ (11) is self-adjoint:

$$(\varphi_1, \hat{M}\varphi_2)=(\hat{M}\varphi_1, \varphi_2). \tag{15}$$

For real β, corresponding to imaginary physical frequencies, the operator $\hat{M}$ is real. In the usual way one can prove that eigenfunctions φ_E of the operator $\hat{M}$ that correspond to different eigenvalues E are orthogonal to each other. The eigenvalues E for real β are real. The normalization condition for the eigenfunctions is written in the form

$$(\varphi_{E'}, \varphi_E)=\delta(E-E'). \tag{16}$$

Expanding the solution $\tilde{P}(r, Q)$ in the eigenfunctions $\varphi_E(Q)$,

$$\tilde{P}(r, Q)=\sum_E A_E(r)\varphi_E(Q),$$

substituting this expansion into (9)–(11), and using (4), we obtain

$$K(r)=2\pi^2\nu^2\gamma_0\left[\frac{m+1}{m}-\frac{1}{m}\delta(r)\right]\sum_E B_E(mE)^r, \tag{17}$$

where

$$B_E=-\int \mathrm{STr}[k\varphi_E^+(Q)Q^{21}]\Psi(Q)Z^{n}(Q)\,dQ \times\int Q_{13}^{12}[\varphi_E(Q)]_{31}\Psi(Q)Z^{n}(Q)\,dQ. \tag{17a}$$

To calculate the density-density correlator at coinciding points we can make use of the completeness property of the eigenfunctions $\varphi_E(Q)$. Performing the summation in (17) for $r=0$, we obtain

$$K(0)=-2\pi^2\nu^2\gamma_0\int Q_{13}^{12}Q_{31}^{21}\Psi^{m+1}(Q)\exp(\tfrac{1}{4}\beta\,\mathrm{STr}\,\Lambda Q)\,dQ. \tag{18}$$

This expression was written out in Ref. 1.

Any eigenfunction φ_E of the operator $\hat{M}$ (11) can be represented in the form

$$\varphi_E(Q)=v^nR(\hat{\theta})\bar{u}^m, \quad R(\hat{\theta})=\begin{pmatrix} f(\hat{\theta}) & 0 \\ 0 & if_1(\hat{\theta})\end{pmatrix} \tag{19}$$

(n and m are integers). This follows from the invariance of the kernel of $\hat{M}$ under the simultaneous replacements

$$Q\to U_0Q\bar{U}_0, \quad Q'\to U_0Q'\bar{U}_0,$$

where U_0 has the same structure as U (2). In the integrals for B_E in (17a) a contribution is made only by eigenfunctions φ_E of the form (19) with $n=m=1$.

We note that all the formulas obtained have a general character and are correct for all three types of symmetry. The subsequent calculations must be performed separately for each type of symmetry. The study of Eqs. (6) and (11) for arbitrary frequencies is very difficult. In the following sections we shall study only the most interesting, low-frequency limit.

3. THE DIELECTRIC REGION

The symplectic ensemble

First we shall consider the region of sufficiently small α, in which, as will be seen from the following, the system is a dielectric. For the calculation of the correlator $K(r)$ from formula (17) it is necessary first of all to find the solutions of Eq. (6). As in Refs. 1, 4, 6, and 7, we assume that the solution Ψ depends only on $\hat{\theta}$ (of course, this is true not only for the dielectric region). This form of the solution makes it possible to perform the integration over u' and v' in (6) immediately. The integration over u' and v' is not complicated for the unitary model, and was carried out in Ref. 1. However, in the cases of the symplectic and the orthogonal ensemble this integration leads to immensely cumbersome expressions. Nevertheless, in the region of low frequencies, as in the unitary model, Eq. (6) and all the other formulas needed for the calculation of the density-density correlator become significantly simpler.

First we perform the calculation for the symplectic ensemble. In this case the matrices θ_{11} and θ_{22} appearing in (2) have the form

$$\theta_{11}=\begin{pmatrix}\theta_1 & \theta_2\\ \theta_2 & \theta_1\end{pmatrix}, \quad \theta_{22}=i\begin{pmatrix}\theta & 0\\ 0 & \theta\end{pmatrix},$$

$$\theta>0, \quad 0<\theta_1<\pi, \quad 0<\theta_2<\frac{\pi}{2}. \tag{20}$$

The integration over u' and v' in (6) is implemented as follows. By the change of variables $U'\to UU'$ we can get rid of the matrices U. After this it is necessary to write in explicit form the expression for S:

$$S=\tfrac{1}{4}\alpha\,\mathrm{STr}(Q_0U'Q_0'\bar{U}'). \tag{21}$$

This expression is a polynomial in Grassmann variables. Next we expand $\exp S$ in the Grassmann variables (this again gives a polynomial) and integrate over these variables. After this we can integrate over F and Φ, which, together with the Grassmann elements, specify the matrices u and v (Ref. 5). All the expressions are very cumbersome. However, in the limit of low frequencies the large $\theta\sim\theta'\sim\ln(1/\beta)$ become important. The same situation also obtained in the study of the unitary ensemble. This considerably simplifies the resulting expressions. Assuming that the solution Ψ depends only on θ and integrating over all the remaining variables, we bring Eq. (6) to the form

$$\Psi(\theta)=\int_0^\infty L(\theta-\theta')\exp\left[\frac{1}{2}(\theta-\theta'-\beta e^{\theta'})\right]\Psi^m(\theta')\,d\theta', \tag{22}$$

where

$$L(\theta)=\left(\frac{\alpha}{2\pi}\right)^{1/2}\left[\frac{\alpha S(\alpha)}{2}\mathrm{sh}^2\theta+\left(\mathrm{sh}\,\alpha-\frac{S(\alpha)}{\alpha}\right)\mathrm{ch}\,\theta + \mathrm{ch}\,\alpha-\frac{\mathrm{sh}\,\alpha}{2\alpha}-\frac{S(\alpha)}{8\alpha}\right],$$

$$S(\alpha)=\int_0^\alpha\frac{\mathrm{sh}x}{x}\,dx.$$

Equation (22) is valid for large θ, θ'. In the derivation of this equation we used the formulas obtained in Ref. 5 for the Jacobian that arises in the transformation to the variables u, v, θ.

By direct integration we can verify that

$$\int_{-\infty}^{\infty} L(\theta) e^{-\theta/2}\, d\theta = 1. \tag{23}$$

It follows from the equality (23) that for $\theta \gg 1$ and $\beta = 0$ Eq. (22) has the solution

$$\Psi(\theta) = 1. \tag{24}$$

This solution agrees with the general form of the solution (6a), which is applicable not only for large θ. However, this solution is realized only in the dielectric region $\alpha < \alpha_c$. Only in this region does the solution of Eq. (22) [which for $\beta = 0$ coincides with (24)] ensure the analyticity of the physical quantities in the upper half-plane of the frequencies.

For the calculation of the physical quantities for small but nonzero β we make the change of variables $t = \theta + \ln \beta$. The important contribution to the physical quantities is made by the region $t \sim 1$. Correspondingly, Eq. (24) can be rewritten in the form

$$\Psi(t) = \int_{-\infty}^{\infty} L(t-t') \exp\left(\frac{t-t'}{2}\right) \Psi^m(t') \exp\left(-\frac{e^{t'}}{2}\right) dt'. \tag{25}$$

Using the equality (23), we can determine the asymptotic form as $t \to -\infty$:

$$\Psi(t) \to 1, \quad t \to -\infty. \tag{25a}$$

As $t \to \infty$ the function $\Psi(t)$ tends to zero. The function Ψ has exactly the same asymptotic forms in the unitary-ensemble model.[1] In the interval between these asymptotes $\Psi(t)$ decreases monotonically. This solution is possible for $\alpha < \alpha_c$.

The calculation of the critical point α_c is performed in exactly the same way as in the unitary-ensemble model.[1] The procedure for finding this point consists in the linearization of Eq. (25) about the value $\Psi = 1$ and the investigation of the Green function of the linearized equation. Performing calculations analogous to those in Ref. 1, we obtain the equation for the critical point: $1 = m\Gamma_0(\alpha_c)$, with

$$\Gamma_\varepsilon(\alpha) = \left(\frac{\alpha}{2\pi}\right)^{1/2} \left\{ \frac{i\varepsilon S(\alpha)}{2} [K_{1+i\varepsilon}(\alpha) - K_{1-i\varepsilon}(\alpha)] + \operatorname{sh}\alpha [K_{1+i\varepsilon}(\alpha) + K_{1-i\varepsilon}(\alpha)] + \left[2\operatorname{ch}\alpha - \frac{\operatorname{sh}\alpha}{\alpha} - \frac{S(\alpha)}{4\alpha}\right] K_{i\varepsilon}(\alpha) \right\}, \tag{26}$$

where $K_\nu(\alpha)$ is the Macdonald function.

For $\alpha < \alpha_c$ the Green function has no singularities in the region $\operatorname{Re}\beta > 0$. This corresponds to the dielectric region. For $\alpha > \alpha_c$ the Green function has singularities for $\operatorname{Re}\beta > 0$. In this region of values of α Eq. (24) also has a nontrivial solution for $\beta = 0$, and this leads to the appearance of diffusion.

In the limiting cases of large m and values of m close to unity, the solution of Eq. (26) can be obtained analytically. For large m the critical α_c is small, while for $m \to 1$ it is large. Using the asymptotic forms of the function $K_\nu(\alpha)$, we obtain

$$\alpha_c = \frac{1}{8(m-1)}, \quad m \to 1; \qquad \frac{3}{4} m \left(\frac{\alpha_c}{2\pi}\right)^{1/2} \ln\frac{2}{\alpha_c} = 1, \quad m \gg 1. \tag{27}$$

We note that in the formal limit $m \to 1$ the critical values α_c for the unitary model and the symplectic model are large and coincide with each other. In the limit $\alpha \gg 1$ Eqs. (22) and (25) become differential equations and coincide with the corresponding equations of the unitary-ensemble model, which are written out in Ref. 1. This property was noted in the problem of wires in Refs. 5 and 7, which corresponds to the case $m = 1$. A comparison of the formulas (27) with the corresponding formulas for the unitary model shows that the inequality $\alpha_{cu} < \alpha_{cs}$ is fulfilled. [The equation for the critical point α_{cu} in Ref. 1 differs from Eq. (27) in that it does not contain the numerical factor 3/4 in the left-hand side.] Comparison of the functions $\Gamma_0(\alpha)$ for the symplectic ensemble and the unitary ensemble makes it possible to conclude that for all m the inequality

$$^9/_{16} < \alpha_{cu}/\alpha_{cs} < 1 \tag{28}$$

is valid. We recall that we must distinguish the unitary models with a magnetic field [the model (11a)] and with magnetic impurities [the model (11b)]. The quantities α in these models, for the same densities of states and the same hopping amplitudes, differ by a factor of 2 (Ref. 5).

We turn now to the calculation of the correlation functions. Having obtained the solution $\Psi(t)$ of Eq. (25), we can immediately use the expression (18) to write the density-density correlator at coinciding points. Integrating over u and v in (18) and changing to the variable t in place of θ we can verify that the correlator $K(0)$ for small β is inversely proportional to β. Using (7) and going over to the time representation $\tilde{K}(0, t)$, we obtain

$$\tilde{K}(0, t \to \infty) \equiv p_\infty(0) = \frac{2\pi\nu}{V} \int_{-\infty}^{\infty} \Psi^{m+1}(t) \exp\left(-\frac{e^t}{2}\right) e^t\, dt. \tag{29}$$

The quantity $p_\infty(0)$ is proportional to the probability that the particle will be found after an infinite time at the point from which it began its motion. This quantity, as in the unitary model, is finite for all $\alpha < \alpha_c$, and tends to a finite, nonzero limit as the transition point α_c is approached. As the critical value is passed, the quantity $p_\infty(0)$ drops discontinuously to zero.

To calculate the correlation function $K(r)$ at large distances we make use of the expansion (17). In the integrals for B_E (17a) there are functions $\varphi_E(Q)$ of the form (19) with $n = m = 1$. In Eq. (12) we go over to the limit of large $\theta \sim \ln(1/\beta)$. By performing the integration in (12) and separating out the terms of leading order in $1/\beta$, we bring the formulas (12), (17a) to the form

$$B_E = \frac{1}{\pi} \left[\int_{-\infty}^{\infty} \Psi(t) Z^{1/2}(t) e^{t/2} \tilde{\varphi}_E(t)\, dt \right]^2. \tag{30}$$

The functions $\tilde{\varphi}_E(t)$ are eigenfunctions of a real symmetric operator $\bar{M}$:

$$\int_{-\infty}^{\infty} \bar{M}(t, t') \tilde{\varphi}_E(t')\, dt' = E\tilde{\varphi}_E(t), \tag{31}$$

where $\bar{M}(t,t') = L(t-t')[Z(t)Z(t')]^{1/2}$, and the function $L(t)$ is defined in (24).

The scheme of the subsequent calculation in (17), (30),

and (31) is completely analogous to that presented in Refs. 3 and 4. First of all it is proved that the spectrum of the eigenfunctions $\tilde{\varphi}_E(t)$ (31) is continuous. Next, there exists a maximum eigenvalue $E_{\max} = \Gamma_0(\alpha)$. At large distances the principal contribution is made by states with eigenvalues close to $E_{\max}$. As a result, as in Refs. 3 and 4, we obtain

$$K(r) = \frac{2\pi^2\nu^2}{\beta}\frac{m+1}{m}a^2\exp\left(-\frac{r}{4l}\right)\int_0^\infty \varepsilon^2\exp(-rb\varepsilon^2)\,d\varepsilon, \quad (32)$$

where

$$a = \int_{-\infty}^{\infty} V(t)\Psi(t)Z^{1/2}(t)e^{t/2}\,dt, \quad b = -\frac{1}{2}\frac{\partial^2\Gamma_\varepsilon(\alpha)}{\partial\varepsilon^2}\Big|_{\varepsilon=0}$$
$$l^{-1} = -4\ln(mE_{max}) = -4\ln[m\Gamma_0(\alpha)],$$

in which $V(t) = \lim_{\varepsilon\to 0}[\tilde{\varphi}_E(t)/\varepsilon]$, and $V(t)$ tends to a finite limit as $\alpha \to \alpha_c$.

Going over to the time representation $\tilde{K}$ by means of (7) and calculating the integral in (32), we obtain

$$\tilde{K}(r, t\to\infty) \equiv p_\infty(r) = \frac{\pi^{1/2}\nu}{2V}\frac{m+1}{m}a^2(br)^{-1/2}e^{-r/4l}. \quad (33)$$

The formula (33) is applicable for $r \gg 1$, and the coefficients a and b remain finite as the transition point α_c is approached. The only quantity with singular behavior is the localization length l. Comparing (32) with (26), we obtain the asymptotic behavior of l for $\alpha \to \alpha_c$:

$$l = \text{const}/(\alpha_c - \alpha). \quad (34)$$

The formulas (32)–(34) coincide fully with the corresponding formulas for the unitary ensemble. Of course, the dependence of the coefficients a and b and of the localization length on α is different for these two models (only the dependence of l on $\alpha_c - \alpha$ in the critical region $\alpha_c - \alpha \ll \alpha_c$ is the same). Despite the power dependence (34) of l on $\alpha_c - \alpha$, the formula (33) does not agree with scaling theory, since the pre-exponential factor in (33) tends to a constant as $\alpha \to \alpha_c$. For agreement with ordinary scaling we need an extra factor $l^{1/2}$ in the pre-exponential factor in (33).

The orthogonal ensemble

The investigation of localization in the orthogonal-ensemble model is the most difficult in the technical aspect. The complexity of the calculations greatly exceeds the complexity arising in the investigation of the symplectic ensemble, although, at first sight, these models are very similar. In the orthogonal model the matrices θ_{11} and θ_{22} appearing in (2a) have the form[5]

$$\theta_{11} = \begin{pmatrix}\theta & 0\\ 0 & \theta\end{pmatrix}, \quad \theta_{22} = i\begin{pmatrix}\theta_1 & \theta_2\\ \theta_2 & \theta_1\end{pmatrix},$$

$$0<\theta<\pi, \quad 0<\theta_1<\infty, \quad 0<\theta_2<\infty. \quad (35)$$

As in the preceding section, we reduce the integration over U' in (6) to the integration of $\exp S$, where S is written out in (21). In the dielectric region in the limit of small β, as in the preceding models, large values of θ_1 and θ_2 are important. In this case $\tau_1 = \theta_1 + \theta_2 \sim \ln(1/\beta)$. At the same time, the important values of $\tau_2 = \theta_1 - \theta_2$ are of order unity. Despite the great simplifications that arise for small β, the necessary calculations are extremely lengthy. Assuming that Ψ does not depend on the elements of the matrices u and v, we bring Eq. (6) to the form

$$\Psi(\tau_1,\tau_2,\theta) = \frac{1}{16\pi^3}\left(\frac{\pi}{\alpha}\right)^{1/2}\int\frac{\exp[(\tau_1-\tau_1')/2]\sin^3\theta'}{(a'-c')^2(1+x^2)^{1/2}}$$
$$\times\exp\{\alpha\mathbf{n}\mathbf{n}' - {}^1/_2\alpha(1+x^2)[\operatorname{ch}(\tau_1-\tau_1') + \operatorname{ch}(\tau_2-\tau_2')]\}$$
$$\times\{\alpha^4(a-c)(a'-c')[\operatorname{ch}(\tau_1-\tau_1')(\operatorname{ch}(\tau_2-\tau_2') + \mathbf{n}\mathbf{n}')(1+x^2)$$
$$+\mathbf{n}\mathbf{n}'(\operatorname{ch}(\tau_2-\tau_2') + \mathbf{n}\mathbf{n}') + x^2(\operatorname{ch}(\tau_2-\tau_2')\mathbf{n}\mathbf{n}'+1)]$$
$$+2\alpha^3[(a-c)(a'-c')(\operatorname{ch}(\tau_2-\tau_2')(1+x^2/2)$$
$$+\operatorname{ch}(\tau_1-\tau_1')(1+x^2)+2\mathbf{n}\mathbf{n}')$$
$$+{}^1/_2(ss'\cos\rho - bb')(\mathbf{n}\mathbf{n}'+\operatorname{ch}(\tau_1-\tau_1')(1+x^2))$$
$$+{}^1/_2x^2(\mathbf{n}\mathbf{n}'\operatorname{ch}(\tau_2-\tau_2')$$
$$+1-ca-c'a'+2(a-c)b'\operatorname{sh}(\tau_2-\tau_2'))]$$
$$+2\alpha^2[\operatorname{ch}(\tau_1-\tau_1')(1+x^2)$$
$$+\mathbf{n}\mathbf{n}'+{}^1/_2(ss'\cos\rho - bb') + (a-c)(a'-c') + x^2\operatorname{ch}(\tau_2-\tau_2')]\}$$
$$\times\Psi^m(\tau_1',\tau_2',\theta')\exp\left(\frac{\beta}{4}e^{\tau_1'}\right)d\theta'\sin^2\rho\,d\rho\,dx\,d\tau_1'\,d\tau_2', \quad (36)$$

where

$$a=\operatorname{ch}\tau_2,\ a'=\operatorname{ch}\tau_2',\ b=\operatorname{sh}\tau_2,\ b'=\operatorname{sh}\tau_2',\ c=\cos\theta,\ c'=\cos\theta',$$
$$s=\sin\theta,\ s'=\sin\theta'.$$

The vectors $\mathbf{n}$ and $\mathbf{n}'$ have components $\mathbf{n} = (\sin\theta, \cos\theta)$, $\mathbf{n} = (\sin\theta'\cos\rho, \cos\theta')$. The integration over τ_1' extends from 0 to ∞, while that over τ_2' extends from $-\infty$ to ∞.

Equation (36) is very complicated, and we have not succeeded in investigating it for arbitrary α. However, this equation becomes considerably simpler in the limiting cases $\alpha \gg 1$ and $\alpha \ll 1$. It makes sense to describe the dielectric region in the limit $\alpha \gg 1$ if we consider the formal limit $m \to 1$. For $\alpha \gg 1$ all the integrals in (36) become Gaussian, and Eq. (36) itself becomes a differential equation. As in Refs. 5 and 7, we assume that Ψ depends on τ_1. Changing to the variable $z = 1/2\exp\tau_1$ we obtain for $\alpha \gg 1$ and $m - 1 \ll 1$ the equation

$$\frac{z^2}{\alpha}\frac{d^2\Psi}{dz^2} - \frac{\beta z}{4}\Psi + (m-1)\Psi\ln\Psi = 0, \quad (37)$$

which coincides in form with the corresponding equation for the unitary and symplectic ensembles. The only difference is that the coefficient of the first term in (37) is twice as large, and the coefficient of the second term is half as large, as the corresponding coefficients in the equation for the unitary and symplectic ensembles. By means of Eq. (37) we immediately determine the transition point α_c:

$$\alpha_c = 1/4(m-1). \quad (38)$$

The fact that Eq. (37) coincides in form with the corresponding equation for the unitary and symplectic ensembles makes it possible to apply all the formulas obtained for these models to the orthogonal ensemble as well.

In the limit $\alpha \ll 1$ the solution Ψ also depends only on τ_1. This follows from the fact that in the presently considered

region $\tau_2, \tau_2' \sim 1$, $\tau_1, \tau_1' \gg 1$ all the variables τ_2 drop out. Performing the integration in (36) over τ_2', x, ρ, and θ', we obtain in the leading approximation in α

$$\Psi(\tau_1)=\int_0^\infty L(\tau_1-\tau_1')\exp\left(\frac{\tau_1-\tau_1'}{2}\right)\Psi^m(\tau_1') \times \exp\left(-\frac{\beta}{4}e^{\tau_1'}\right)d\tau_1', \tag{39}$$

where

$$L(\tau)=\frac{\alpha^2}{4\pi^2}\left(\frac{\pi}{\alpha}\right)^{1/2}\operatorname{ch}\tau\left[K_0\left(\frac{\alpha}{4}\operatorname{ch}\tau\right)+K_1\left(\frac{\alpha}{4}\operatorname{ch}\tau\right)\right] \times \exp\left(-\frac{\alpha}{4}\operatorname{ch}\tau\right).$$

For the function $L(\tau)$ in the leading approximation in α the equality

$$\int_{-\infty}^{\infty} L(\tau)\exp(-\tau/2)\,d\tau=1 \tag{39a}$$

is fulfilled. From (39a), as usual, it follows that for $\tau_1 \gg 1$, for which Eq. (39) is valid, we have for $\beta = 0$ the solution $\Psi = 1$. This is in agreement with the general form (6a).

Making in (39) the change of variables $t = \tau_1 + \ln(\beta/2)$, we arrive at Eq. (25), in which, as $L(t)$, we must take the expression (39). The subsequent scheme of the calculations is entirely analogous to that followed in the preceding section. Introducing the function Γ_ε:

$$\Gamma_\varepsilon=\int_{-\infty}^{\infty} L(\tau)\cos(\varepsilon\tau)\,d\tau, \tag{40}$$

where L is defined in (39), we obtain Eq. (26) for the transition point. Expanding (26), (40), and (39) in explicit form, we obtain

$$1=2m\left(\frac{\alpha_c}{\pi}\right)^{1/2}\int_0^\infty K_1\left(\frac{\alpha_c}{2}(1+u^2)\right)(1+u^2)^{1/2}\,du$$

$$=\frac{2^{3/2}m}{\pi}\left(\frac{\alpha_c}{2\pi}\right)^{1/2}\ln\frac{\gamma}{\alpha_c}, \tag{41}$$

where γ is a number of order unity.

Comparing (38) and (41) with the corresponding formulas for the unitary ensemble,[1] we can see that in the presently considered limits $\alpha \gg 1$ and $\alpha \ll 1$ the inequality

$$\alpha_{co} > \alpha_{cu} \tag{42}$$

is fulfilled. It is entirely natural to assume that the inequality (42) is also valid for $\alpha \sim 1$. The case $\alpha \sim 1$ has not yet been subjected to analytical investigation. However, it is hard to see why we should expect a qualitative difference between the results for $\alpha \sim 1$ and those for $\alpha \ll 1$ and $\alpha \gg 1$.

Using the formulas (17) or (17a) and the lack of dependence of Ψ on τ_2, we again arrive at the expressions (29)–(31). Correspondingly, the final formulas (32)–(34) are valid for the orthogonal ensemble. Only the dependence of l on α far from the transition, the critical value α_c, and the numerical coefficients a and b turn out to be different in (33). The quantity p_∞ (0) (29) depends on the model, but remains nonzero as the transition point is approached.

4. THE METALLIC REGION

We turn to the calculation of the density-density correlation function $K(r)$ in the metallic region. In Ref. 4 it was shown that this correlator in the region of low frequencies and large distances in the unitary model always has the form

$$K(r)=\frac{(2\pi\nu)^2}{cm}\exp\left(-\frac{\beta r}{cm}\right), \tag{43}$$

where c is a coefficient that depends on α, and β is related to the physical frequency by the formula (7). The coefficient

$$D=\frac{cm}{m-1}\frac{1}{\pi\nu V} \tag{43a}$$

was identified with the diffusion coefficient. The expression (43) corresponds to classical diffusion on a Bethe lattice, the coefficient D being proportional to the frequency of hopping to any of the neighboring sites per unit time.[4] In Ref. 1 the form of the correlation function $K(r)$ (43) was obtained for all three types of symmetry in the limit $\alpha \gg 1$. The derivation given in Ref. 4 of formula (43) for the unitary ensemble was based on the use of the equation obtained from Eq. (10) by integration over u and v. However, even for the unitary ensemble this derivation required very cumbersome calculations. For the orthogonal ensemble and symplectic ensemble this procedure does not seem possible at all. Below we propose a simpler scheme of calculations.

It turns out that the formula (43) is a consequence of the invariance of the Lagrangian (1) for $\omega = 0$ under the replacement

$$Q_i \to VQ_i\bar{V}, \tag{44}$$

where V is an arbitrary supermatrix satisfying the condition

$$\bar{V}V=1. \tag{44a}$$

Naturally, with regard to the structure of the matrix V we should also impose certain conditions, such that the supermatrix $VQ\bar{V}$ has the same structure as Q. These conditions can be written in the form[5]

$$\bar{V}=KV^+K, \quad K=\begin{pmatrix}1 & 0\\ 0 & K\end{pmatrix}. \tag{44b}$$

The invariance (44) makes it possible to obtain an explicit expression for the diffusion coefficient D in terms of the solution (taken for $\beta = 0$) of Eq. (6). Formally, the metallic region differs from the dielectric region in that, even for $\beta \to 0$, the solution Ψ_0 of Eq. (6) differs from unity and varies from 1 to 0 upon change of the parameters of the matrix θ_{22} in (2) and (2a) from 0 to ∞. The solution that goes over to $\Psi_0 = 1$ at $\beta = 0$ must be discarded, since it does not give analyticity of the physical quantities in the upper half-plane of the frequencies.

To derive (43) we note that if $\Psi_0(Q)$ is a certain solution of Eq. (6) for $\beta = 0$, then $\Psi_0(VQ\bar{V})$ is also a solution of this equation for any V satisfying the conditions (44a) and (44b). This fact is a consequence of (44).

The solution $\Psi_0(Q)$ in fact depends only on Q_0 (2). This implies that $\Psi_0(Q)$ can be represented in the form

$$\Psi_0(Q)=F(\operatorname{STr}f_1(\Lambda Q),\ \operatorname{STr}f_2(\Lambda Q),\ldots,\operatorname{STr}f_N(\Lambda Q)), \tag{45}$$

where F is a function of N variables, and the f_i are functions of matrices. The number N in (45) depends on the dimen-

sions of the matrix Q. We represent V (44), (44a), (44b) in the form

$$V=(1+iH)(1-iH)^{-1}, \quad H=\begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}. \tag{46}$$

We shall make use of the fact that $\Psi_0(VQ\bar{V})$ is a solution of Eq. (6) with $\beta=0$ for any H, including small H. Expanding $\Psi(VQ\bar{V})$ in (6) in H, equating the terms linear in H, and using the representations (2), (2a), we obtain

$$\sum_i v\left[\exp i\begin{pmatrix} 0 & \hat{\theta} \\ \hat{\theta} & 0 \end{pmatrix}\cdot f_i'\left(\exp i\begin{pmatrix} 0 & \hat{\theta} \\ \hat{\theta} & 0 \end{pmatrix}\right)\right]^{21} \bar{u}F_i'$$

$$= m\int \exp\left(\frac{\alpha}{4}\operatorname{STr} QQ'\right)\Psi_0^{m-1}(Q')\sum_i v'\left[\exp i\begin{pmatrix} 0 & \hat{\theta}' \\ \hat{\theta}' & 0 \end{pmatrix}\right.$$

$$\left.\times f_i'\left(\exp i\begin{pmatrix} 0 & \hat{\theta}' \\ \hat{\theta}' & 0 \end{pmatrix}\right)\right]^{21}\bar{u}'F_i'\,dQ', \tag{47}$$

where F_i' denotes the derivative with respect to the ith argument. In the left-hand side F_i' depends on the elements of the matrix θ, and in the right-hand side F_i' depends on the elements of the matrix $\hat{\theta}'$.

We introduce the function $\Phi_\beta(Q)$:

$$\Phi_\beta(Q)=v\frac{\partial\Psi}{\partial\hat{\theta}}\bar{u}Z^{1/2}(Q)\equiv v\begin{pmatrix} \partial\Psi/\partial\theta_{11} & 0 \\ 0 & \partial\Psi/\partial\theta_{22} \end{pmatrix}\bar{u}Z^{1/2}(Q). \tag{48}$$

The functions $\Psi(Q)$ and $Z(Q)$ (4) in (48) are taken for arbitrary β. In the unitary model,

$$\partial\Psi/\partial\theta_{11}\equiv\partial\Psi/\partial\theta, \quad \partial\Psi/\partial\theta_{22}\equiv\partial\Psi/\partial\theta_1, \tag{48a}$$

in the orthogonal model,

$$\partial\Psi/\partial\theta_{11}\equiv\partial\Psi/\partial\theta, \quad \partial\Psi/\partial\theta_{22}\equiv\partial\Psi/\partial\theta_1+\tau_1\partial\Psi/\partial\theta_2, \tag{48b}$$

and in the symplectic model,

$$\frac{\partial\Psi}{\partial\theta_{11}}\equiv\frac{\partial\Psi}{\partial\theta_1}+\frac{\partial\Psi}{\partial\theta_2}\tau_1, \quad \frac{\partial\Psi}{\partial\theta_{22}}\equiv\frac{\partial\Psi}{\partial\theta}, \quad \tau_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculating the derivatives with respect to the elements of the matrix $\hat{\theta}$ in (48) and comparing with (47), we can rewrite Eq. (47) in the form

$$\Phi_0(Q)=m\int\exp\left(\frac{\alpha}{4}\operatorname{STr} QQ'\right)[Z_0(Q)Z_0(Q')]^{1/2}\Phi_0(Q')\,dQ', \tag{49}$$

where

$$\Phi_0(Q)=[\Phi_\beta(Q)]_{\beta=0}.$$

We turn now to the calculation of the correlator $K(r)$, using the expression (17). In the metallic region the eigenvalue spectrum of the operator $\hat{M}$ (11) is discrete, and, at least, the difference between the zeroth and the first value does not tend to zero at $\beta=0$. This is a consequence of the fact that the solution Ψ of Eq. (6) decreases to zero as $\theta\to\infty$ in the symplectic model and as $\tau_1\to\infty$ in the orthogonal model. In the unitary model, Ψ decreases to zero as $\theta_1\to\infty$. This case was discussed in more detail in Refs. 3 and 4. The decrease of the solution Ψ leads to the result that the kernel of $\hat{M}$ that arises after the integration of $\hat{M}$ over U differs substantially from zero in a finite range of variation of the elements of the matrix $\hat{\theta}$. From this follows the discreteness of the eigenvalues of the operator $\hat{M}$ (11).

At large distances the main contribution in the sum in (17) is made by just the state with the largest eigenvalue $E_0(\beta)$. For $\beta=0$ the largest eigenvalue and the eigenfunction $\varphi_0(Q)$ corresponding to it can be found exactly. Comparing Eqs. (12) and (11) with the identity (49), we obtain

$$\varphi_0(Q)=c_1\Phi_0(Q), \quad E_0(0)=1/m, \tag{50}$$

where c_1 is a normalization factor, equal to

$$c_1=\left[\int\operatorname{STr} k\Phi_0^+(Q)\Phi_0(Q)\,dQ\right]^{-1/2}. \tag{50a}$$

To calculate the largest eigenvalue $E_0(\beta)$ for finite values of β we take the scalar product of both sides of Eq. (12) with the function $\Phi_\beta(Q)$ (48). We have in mind the scalar product defined by the formula (14). The result of the multiplication is written in the form

$$(\Phi_\beta,\ \hat{M}\varphi_E)=E(\beta)(\Phi_\beta,\ \varphi_E). \tag{51}$$

The function Φ_β in (51) contains derivatives with respect to $\hat{\theta}$ (48). Integrating over $\hat{\theta}$ by parts in the left-hand side of (51), we bring (51) to the form

$$E(\beta)(\Phi_\beta,\varphi_E)=-\frac{i}{m}\beta\int\operatorname{STr}[kQ^{12}k\varphi_E(Q')]Z^{1/2}(Q')\Psi^m(Q)$$

$$\times\exp\left[\frac{\beta}{4}\operatorname{STr}\Lambda Q+\frac{\alpha}{4}\operatorname{STr} QQ'\right]dQ'\,dQ$$

$$-\frac{1}{m}\int\exp\left(\frac{\beta}{4}\operatorname{STr}\Lambda Q\right)\Psi^m(Q)$$

$$\times Z^{1/2}(Q')\operatorname{STr}\left[ku\frac{\partial}{\partial\hat{\theta}}\bar{v}k\varphi_E(Q')\right]\exp\left(\frac{\alpha}{4}\operatorname{STr} QQ'\right)dQ'\,dQ. \tag{52}$$

The second term in the right-hand side of (52) contains the matrix $\partial/\partial\hat{\theta}$, which is in fact defined in (48a). It is assumed that the operator of derivatives with respect to the elements of the matrix $\hat{\theta}$ acts on all the factors standing to the right of $\partial/\partial\hat{\theta}$ (including the Jacobian that arises in the integration over $\hat{\theta}$, u, and v). Comparing Eq. (49) with the derivative with respect to $\hat{\theta}$ of both sides of Eq. (6) taken for $\beta=0$, we obtain

$$\int v\frac{\partial}{\partial\hat{\theta}}\bar{u}\exp\left(\frac{\alpha}{4}\operatorname{STr} QQ'\right)\Psi_0^m(Q')\,dQ'$$

$$=-\int\Psi_0^m(Q')v'\frac{\partial}{\partial\hat{\theta}'}\bar{u}'\exp\left(\frac{\alpha}{4}\operatorname{STr} QQ'\right)dQ'. \tag{53}$$

However, the equality (53) is valid not only for the solution Ψ_0 of Eq. (6). It is fulfilled for all functions Ψ_s that depend only on Q_0 (2). This statement can be verified by substituting an arbitrary function Ψ_s in place of Ψ_0 in the left-hand side of Eq. (6) taken for $\beta=0$. Repeating the transformations (45)–(49), in the left-hand side of (49) we obtain the matrix $\partial\Psi_s/\partial\hat{\theta}$ in place of $\partial\Psi_0/\partial\hat{\theta}$. After this we can derive (53), but now Ψ_0 in (53) is not necessarily a solution of Eq. (6). Varying with respect to Ψ_0 in (53), substituting the result into (52), and using (6), (48) for $\beta=0$, we obtain

$$\left[E(\beta)-\frac{1}{m}\right](\Phi_\beta,\varphi_E)=-i\frac{\beta}{m}\,(Z^{1/2}(Q)\Psi(Q)Q^{12},\hat{M}\varphi_E(Q))$$
$$=\frac{i\beta E(\beta)}{m}\,(Z(Q)\Psi(Q)Q^{12},\varphi_E(Q)). \qquad (54)$$

The equation (54), which is valid for all β, is simplified greatly for values of β that are small in comparison with the difference between the eigenvalues. Retaining only terms linear in β, and using (50) and (50a), we find

$$mE(\beta)=1-\frac{ic_1^2\beta}{m}\int \mathrm{STr}\left(kQ^{12}kv\frac{\partial\Psi_0}{\partial\hat{\theta}}\bar{u}\right)\Psi_0{}^m(Q)\,dQ. \qquad (55)$$

The subsequent calculations are rather simple. First it is necessary to integrate over u and v in (55). The resulting expression must be integrated over the elements of the matrix $\hat{\theta}$. The integrand then turns out to be an exact divergence in the space of the elements of the matrix $\hat{\theta}$ and the integral is transformed to a surface integral over an infinitesimal surface about the coordinate origin $\theta=\theta_1=\theta_2=0$. At this point $\Psi_0=1$, and this makes it possible to calculate the integral in explicit form. For the unitary model more-detailed calculations are given in Ref. 4.

Calculating next the integrals in (17a) and (50a) over u and v, retaining in (17), at large distances, only the largest eigenvalue, and going over to the physical frequencies (7), we obtain

$$K(r)=\frac{4\pi\nu}{V(m-1)D}\exp\left[\frac{i(\omega+i\delta)r}{(m-1)D}\right]. \qquad (56)$$

The diffusion coefficient D in (56) is equal to

$$D=\frac{m}{2\pi\nu\gamma_0 V}\frac{m+1}{m-1}\int\left[\left(\frac{\partial\Psi_0}{\partial\theta}\right)^2+\left(\frac{\partial\Psi_0}{\partial\theta_1}\right)^2+\left(\frac{\partial\Psi_0}{\partial\theta_2}\right)^2\right]$$
$$\times\Psi_0^{M-1}\bar{J}\,d\theta\,d\theta_1\,d\theta_2. \qquad (56a)$$

The formulas (56) and (56a) are applicable both for the orthogonal and for the symplectic ensemble. The limits of integration over θ, θ_1, and θ_2 are indicated in (20) and (35). The quantity $\bar{J}$ is proportional to the Jacobians J (Ref. 5):

$$\bar{J}=2^9J.$$

The corresponding expression for the diffusion coefficient of the unitary model is written out in Ref. 4. It differs from (56a) by the absence of the third term in the integrand. Naturally, we have taken the Jacobian of the unitary model here.

In the derivation of (56) and (56a) we used only the assumptions that β is small and that r is large. For the calculation of the coefficient D for arbitrary α it is necessary to use numerical methods. An analytical investigation can be carried out only in the limits $\alpha\gg1$ and $\alpha-\alpha_c\ll1$. In the limit $\alpha\gg1$ the coefficient c in (43) and the diffusion coefficient D (43a) are the same for all three types of symmetry and are equal to[1]

$$c=\alpha(m-1)/m,\quad D=\alpha/\pi\nu V. \qquad (57)$$

It is possible to arrive at the expressions (57) both by means of the direct calculation proposed in Ref. 1 and by using (56).

As in the localized region, large values of the elements of the matrix θ_{22} in Q (2), (2a) become important near the transition point. This makes it possible to make use of the equations (24), (36), and (39) obtained in this limit, in which we must set $\beta=0$. A scheme for finding the solution Ψ for the unitary ensemble was developed in Refs. 1, 3, and 4. Similar arguments can be carried through for the symplectic and orthogonal ensembles as well. As a result, for the symplectic and orthogonal ensembles we obtain

$$D=\frac{p(m+1)m}{12\pi\nu V\gamma_0(m-1)}\frac{\exp[-q(\alpha-\alpha_c)^{-1/2}]}{(\alpha-\alpha_c)^{3/2}},$$
$$q=\pi[b^{-1}(\alpha_c)\partial\Gamma(\alpha_c)/\partial\alpha]^{1/2}, \qquad (58)$$

where p is a numerical factor. The quantity $b(\alpha)$ is defined in (32). The formula (58) coincides with the corresponding formula of Ref. 4, obtained for the unitary ensemble. Only the coefficients p and q and the critical value α_c itself depend on the model under consideration.

5. CONCLUSION

The investigation carried out above, and also the results of Refs. 1, 3, and 4, show that the kinetics of disordered metals on a Bethe lattice is the same for all physical types of symmetry. Neither a magnetic field nor magnetic or spin-orbit impurities alter the critical behavior near the metal-dielectric transition point. In all cases the density-density correlator has the form (33) in the localized region and the form (56) in the conducting region. The dependence of the localization length l on α in the critical region $\alpha_c-\alpha\ll\alpha_c$ is described by the formula (34), and the dependence of the diffusion coefficient for $\alpha-\alpha_c\ll\alpha_c$ is described by the formula (58). The form of the density-density correlator (56) is a consequence of the invariance of the Lagrangian (1) for $\omega=0$ under the rotation (44).

The dependence of the localization length l (34) on the distance from the transition point coincides with the corresponding dependence obtained in the Anderson model on a Bethe lattice in Ref. 8. The sharp dependence (58) of the diffusion coefficient on $\alpha-\alpha_c$ agrees with the sharp dependence obtained in Ref. 9, although it is difficult to carry out a detailed comparison because of the small number of points calculated near the transition in Ref. 9. The formulas obtained for the localized region for $m=1$ agree completely with the corresponding formulas obtained in one-dimensional models differing from those considered above.[10] The only discrepancy is a discrepancy with the result of Ref. 11, in which a linear decrease of the diffusion coefficient near the transition point was obtained. In Ref. 4 a possible reason for this discrepancy was discussed in detail.

The results obtained above have made it possible to determine the direction of the shift of the transition point α_c under different physical perturbations. For example, an applied magnetic field, according to (42), expands the metallic region. In the absence of a magnetic field the critical value α_c for a system with spin-orbit impurities is higher than α_c in a system with magnetic impurities [see (28)]. It is necessary to recall that a magnetic field and magnetic impurities induce different shifts of the transition point because of the difference of the coefficient γ_0 in (1). Combining the formulas (28) and (42) with the formula (7), we can consider the remaining cases.

[1]K. B. Efetov, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 17 (1984) [JETP Lett. **40**, 738 (1984)]; Zh. Eksp. Teor. Fiz. **88**, 1032 (1984) [Sov. Phys. JETP **61**, 606 (1984)].
[2]M. R. Zirnbauer, Nucl. Phys. B **265**, 375 (1986).
[3]M. R. Zirnbauer, Phys. Rev. **34**, 6394 (1986).
[4]K. B. Efetov, Zh. Eksp. Teor. Fiz. **92**, 638 (1987) [Sov. Phys. JETP **65**, 360 (1987)].
[5]K. B. Efetov, Adv. Phys. **32**, 53 (1983).
[6]F. J. Dyson, J. Math. Phys. **3**, 140, 157, 166 (1962).
[7]K. B. Efetov and A. I. Larkin, Zh. Eksp. Teor. Fiz. **85**, 764 (1983) [Sov. Phys. JETP **58**, 444 (1983)].
[8]H. Kunz and B. Souillard, Commun. Math. Phys. **78**, 201 (1980); J. Phys. (Paris) Lett. **44**, L503 (1983).
[9]S. M. Girvin and M. Jonson, Phys. Rev. B **22**, 3583 (1980).
[10]V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. **65**, 1251 (1973) [Sov. Phys. JETP **38**, 620 (1974)]; A. A. Gogolin, V. I. Mel'nikov, and É. I. Rashba, Zh. Eksp. Teor. Fiz. **69**, 327 (1975) [Sov. Phys. JETP **42**, 168 (1975)].
[11]B. Shapiro, Phys. Rev. Lett. **50**, 747 (1983).

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