The Arnowitt-Deser-Misner principle and geodesic flows in Wheeler–De Witt superspace

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An approach is developed to the investigation of the stability of Hamiltonian systems in a Wheeler–De Witt superspace (a manifold with pseudo-Riemannian structure) on the basis of the Arnowitt-Deser-Misner variational principle. For inflationary solutions, describing the dynamics of a universe with a massive scalar field, stability criteria (decay laws for disturbances) are derived for each form of the scalar potential.

1. INTRODUCTION

The problem of stability of cosmological solutions, being of fundamental importance in investigations of the evolution of the universe, is a problem of stability in Wheeler-De Witt superspace with a pseudo-Riemannian (Lorentz) metric.[1] In this connection there arises the problem of investigating the stability of Hamiltonian systems by studying the global geometric properties of these spaces (another example is real space-time, which also exhibits a Lorentz signature of the metric). Our approach is based on the consideration of nearby geodesics on some metric manifold which is the configuration space of the Hamiltonian system.[2] The appeal of the geodesic method is important in principle for this problem, since it differs from a simple analysis of the perturbed solution of the differential equation by allowing one to follow the genuine divergence of nearby trajectories and not only of separate points on them. Thus, one can see that in our problem two points on nearby geodesics may become separated, whereas the geodesics themselves may not (cf. Ref. 3).

A first step on this path is a representation of a model of the universe filled with a scalar field as a Hamiltonian system with a prescribed Hamiltonian. This can be achieved by making use of the Hilbert variational principle in the Arnowitt-Deser-Misner (ADM) modification (Ref. 4, cf. also Refs. 5, 6).

As an important application we have considered in detail the behavior of solutions of inflationary type, to which present-day hopes for an understanding of a number of key cosmological problems are tied (Refs. 7–10). Our analysis, which in a certain sense is a continuation of the papers in Ref. 11, has made it possible to derive the stability laws (decay of disturbances) for each type of scalar field.

In considering the dynamics of geometry in the ADM formulation[4] the 3-geometries of the initial and final Cauchy hypersurfaces are assumed given. The action integral is extremal with respect to the choice of a space-time between these two hypersurfaces.

In what follows we shall consider locally isotropic and homogeneous cosmological models with a scalar field, i.e., when the metric on the compact 3-manifold S depends on a single parameter a:

$$h_{ij}=\sigma^2a^2\tilde{h}_{ij},$$

where

$$\sigma^2=\frac{4\pi G}{3}\Big[\int_S\tilde{h}^{1/2}\,d^3x\Big]^{-1},\qquad \tilde{h}\equiv\det\tilde{h}_{ij}.$$

The curvature of the metric $\tilde{h}_{ij}$ equals (cf. also Ref. 1):

$${}^3\tilde{R}_{ijkl}=k(\tilde{h}_{ik}\tilde{h}_{jl}-\tilde{h}_{il}h_{jk})$$

with $k=+1$ (when S is a 3-sphere or a 3-sphere factored by a discrete group), $k=0$ (when S is a 3-torus or any other flat space), or $k=-1$ (when S is a 3-hyperbolic space, factored with respect to a discrete group). For $k=0$ we require that the condition $\int_S d^3x\tilde{h}^{1/2}=1$ should be satisfied.

Near the manifold S the 4-metric has the form

$$ds^2=-N^2dt^2+h_{ij}dx^idx^j.$$

Keeping in mind that the Lagrangian of the scalar field is

$$L_\phi=-g^{-1/2}[\phi_{,a}\phi_{,b}g^{ab}+V(\phi)],$$

the action $I=I_g+I_\phi$ can be written in the form (cf. Ref. 6)

$$I=\int p_\alpha\,d\alpha+p_\chi\,d\chi-N\mathscr{H}_{\rm ADM}\,dt,$$

where

$$\mathscr{H}_{\rm ADM}=\frac{e^{-3\alpha}}{2}[-p_\alpha{}^2+p_\chi{}^2]+e^{3\alpha}\Big[U(x)-\frac{k}{2}e^{-2\alpha}\Big],$$

$$\alpha=\ln a,\qquad \chi=\sigma\phi,\qquad U(x)=\frac{4\pi G\sigma^2}{3}V(\phi),$$

and the variation is carried out with respect to $\alpha, \chi, p_\alpha, p_\chi$, and N. Variation with respect to N leads to the equation $\mathscr{H}_{\rm ADM}=0$.

2. REDUCTION OF THE HAMILTONIAN SYSTEM TO A GEODESIC FLOW ON A PSEUDO-RIEMANNIAN SPACE

We have thus derived a Hamiltonian system with the Hamiltonian

$$\mathscr{H}={}^1/_2g^{ab}p_ap^b+V(x)$$

and the constraint equation $\mathscr{H}=0$.

The motion of this system is along the extremals of the action functional

$$I=\int p_a\,dx^a-\mathscr{H}(p,x)\,d\tau$$

with the subsidiary condition $\mathscr{H}=0$.

We introduce some notations to be used below

$$v^a\equiv\dot{x}^a=\frac{dx^a}{d\tau}=p_ag^{ab},\qquad u^a=\frac{dx^a}{ds},\qquad p_a=g_{ab}v^b,\qquad g_{ab}g^{bc}=\delta_a{}^c$$

(for the moment s is an arbitrary parameter). We also denote the following regions on the hypersurface:

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$$W^+\{x|V(x)>0\},\quad W^-\{x|V(x)<0\},\quad \text{ext}\, I=\gamma_{ext},$$

where $\delta I(\gamma_{\text{ext}})=0$. We then have

$$\text{ext}\, I|_{\mathscr{H}=0}=\text{ext}\int p_a\,dx^a|_{\mathscr{H}=0,p_a=g_{ab}v^b}$$

$$=\text{ext}\int g_{ab}v^av^b\,d\tau|_{\mathscr{H}(g_{ab}v^b,x)=0},\qquad (1)$$

where

$$\mathscr{H}(g_{ab}v^av^b,\ x)={}^1/_2g_{ab}v^av^b+V(x)=0.\qquad (2)$$

Assuming that g is a Riemannian metric one can write in the region W^-

$$g_{ab}v^av^b=-2V>0,\quad d\tau=(g_{ab}u^au^b/-2V)^{1/2}ds$$

and, making use of Eq. (1), we have

$$\text{ext}\, I|_{\mathscr{H}=0}=\text{ext}\int\left[-2V\left(\frac{g_{ab}u^au^b}{-2V}\right)^{1/2}\right]ds$$

$$=\text{ext}\,2^{1/2}\int(-Vg_{ab}u^au^b)^{1/2}\,ds=\text{ext}\int(G_{ab}u^au^b)^{1/2}\,ds,$$

where $G_{ab}=-Vg_{ab}$ is also a Riemannian metric.

We choose the parameter s so as to satisfy the conditions[2] $\|u\|^2=G_{ab}u^au^b=1$,

$$G_{ab}u^au^b=-Vg_{ab}v^av^b(d\tau/ds)^2=2V^2(d\tau/ds)^2,$$

whence

$$ds=2^{1/2}(-V)d\tau;$$

the extremals of the action I

$$\text{ext}\int(G_{ab}u^au^b)^{1/2}\,ds=\text{ext}\frac{1}{2}\int G_{ab}u^au^b\,ds$$

reduce to the geodesics in the region W^{-1}:

$$\mathscr{H}={}^1/_2g^{ab}p_ap_b+V\leftrightarrow\{G_{ab}=-Vg_{ab},\ ds=2^{1/2}(-V)d\tau,\ \|u\|^2=1\}.\qquad (3)$$

As regards the region W^+, it is obvious that a classical system cannot end up inside it.

Now let g be a pseudo-Riemannian metric with signature $(-,-,..+,+,...,+)$. Then for x in the region W^- (cf. Ref. 2) we have

$$g_{ab}v^av^b=-2V>0,$$

and we are again led to the representation (3). For $V=U-E$ this representation is a consequence of the Maupertuis action principle.[2] Proceeding similarly for x in W^+ we are led to the representation

$$\mathscr{H}\leftrightarrow\{|V|g_{ab},\ {}^1/_2|V|d\tau,\ -\text{sign}\ V\}.$$

Thus, the Hamiltonian system is represented as a geodesic flow on a pseudo-Riemannian manifold. In order to study the stability of this flow it is necessary to start from the Jacobi equation (Ref. 2; for details for the case under consideration see Refs. 12, 13):

$$\ddot{Y}+[\omega-({}^1/_2\dot{\gamma}+\gamma^2/4)]Y=0,$$

where

$$\gamma=-\frac{d}{d\tau}\ln|V|,\quad \omega=2V^2K\|u\|^2,\quad Y=Z|V|^{-1/2},$$

$$K=\frac{\langle E,R(u,E)u\rangle}{\|u\|^2}$$

(E is a Fermi basis, $Z^-=ZE$ is a vector separating the geodesics).

In accord with the expressions for the action I given above, we have

$$\mathscr{H}_{\text{ADM}}\leftrightarrow\{G\eta_{ab},\ 2^{1/2}|V|d\tau,\ -\text{sign}\ V\},$$

where

$$Ndt=d\tau,\quad G=e^{6\alpha}|u-ke^{-2\alpha/2}|,\quad |V|=e^{3\alpha}|u-ke^{-2\alpha}/2|.$$

In the case $k=0$ these equations simplify even further:

$$G=e^{6\alpha}|U|,\quad |V|=e^{3\alpha}|U|,$$

$$\gamma(\tau)=-3\frac{d\alpha}{d\tau}-\frac{1}{U}\frac{dU}{d\tau}=-3H-\frac{1}{U}\frac{dU}{d\tau},$$

$$K(\tau)=-\frac{1}{2}\frac{\Box\ln G}{G}=-\frac{1}{2}\frac{\Box\ln(e^{6\alpha}|U|)}{e^{6\alpha}|U|}=-\frac{1}{2}\frac{\Box\ln|V|}{e^{6\alpha}|U|},$$

$$\omega(\tau)=2e^{6\alpha}U^2\left[-\frac{1}{2}\frac{\Box\ln|U|}{e^{6\alpha}|U|}\right](-\text{sign}\ U)$$

$$=U\Box\ln|U|=U''-\frac{U'^2}{U},$$

$$\Box\equiv-\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\chi^2},\quad U'\equiv\frac{dU}{dx}.\qquad (4)$$

3. THE INFLATIONARY PHASE

Now we can start investigating the inflationary solutions. Setting $k=0$ below, we start from a known property of this phase, namely its rapid transition to flat space even for an initial value $k\neq0$. As was shown in Ref. 11, the inflationary stage can be realized for sufficiently general initial conditions. The following inequalities hold during this stage:

$$\dot{H}\ll H^2,\quad \dot{\chi}^2\ll U,\quad |\dot{\chi}/\chi|\ll H,$$
$$\chi\gg1\quad\text{for}\quad\alpha\to-\infty.$$

We assume the potential of the scalar field in the form of a power function

$$U=\lambda\chi^n/n.$$

Sometimes, making use of Eqs. (4), we have

$$\gamma=-3H-n\dot{\chi}/\chi\sim-3H,\quad \omega=-nU/\chi^2,$$

whence

$$\dot{\gamma}=-3\dot{H},\quad {}^1/_2\dot{\gamma}+\gamma^2/4\approx{}^9/_4H^2.$$

In this case the Jacobi equation takes on the form

$$\ddot{Y}-[-nU/\chi^2+({}^3/_2H)^2]Y=0.\qquad (5)$$

From the Einstein equation

$$\dot{H}+3H^2=6U$$

and the condition $\dot{H}\ll H^2$, to which the Hubble constant is subject, we get

$$H^2\sim2U.$$

If $\chi \gg (2n)^{1/2}/3$, then

$$nU/\chi^2 \sim nH^2/2\chi^2 \ll {}^9/_4 H^2.$$

Consequently, Eq. (5) has the simple form

$$\ddot{Y} - ({}^3/_2 \dot{\alpha})^2 Y = 0,$$

and for $\alpha \to -\infty$, as is easily verfied by direct substitution (neglecting the term $\ddot{\alpha} = \dot{H}$ is related to the condition $\dot{H} \ll H^2$), it has the solution $Y \approx \text{const} \exp(\pm 3/2\alpha)$ or

$$z_\pm \approx e^{{}^3/_2\alpha} |U|^{1/2} \text{const}\, e^{\pm {}^3/_2 \alpha},$$
$$z_+ \approx \text{const}\, |U|^{1/2}, \quad z_- \approx \text{const}\, |U|^{1/2} e^{3\alpha}.$$

We finally have for z (in view of the fact that $z_- \ll z_+$)

$$z = c_+ z_+ + c_- z_- \approx \text{const}\, |U|^{1/2},$$

hence

$$\dot{z} \approx \frac{1}{2} \frac{U'}{U^{1/2}} \dot{\chi}. \tag{6}$$

Making use of the relation[15] $\dot{\chi} \approx \text{const} \cdot U/U^{1/2}$, we obtain from Eq. (6) (for more details, see Ref. 12)

$$\delta^2 = z^2 + \dot{z}^2 \approx \text{const}\, U + \text{const}' \frac{U'^4}{U^2} \approx \text{const}\, \chi^n + \text{const}' \chi^{2n-4}. \tag{7}$$

During the expansion ($H > 0$) the potential of the scalar field χ decreases, i.e., for any $n \geqslant 2$, δ decreases, and consequently one may say that the inflationary solutions under discussion are Lyapunov-stable. The equation (7) allows one to obtain also the law of decay of disturbances for each concrete form of the potential. Thus for the often discussed case $n = 2$, i.e., for $U = \lambda\chi^2/2$ we have $z \approx \chi \approx -t$,

$$\dot{z} \approx -1, \quad \delta = |z^2 + \dot{z}^2|^{1/2} \approx -t.$$

In the case under consideration the inflationary solutions are linearly Lyapunov-stable. For $n = 4$ the disturbances decay exponentially; the larger n, the more stable the solution is.

Thus, we have proposed an approach to the investigation of Hamiltonian system on pseudo-Riemannian manifolds. Making use of the ADM principle we have in this way studied the Wheeler-de Witt superspace. We have studied separately the solutions of equations of the exponentially inflating type, describing the dynamics of a universe filled with a massive scalar field. We have studied the decay of disturbances for fields with a power-law potential. Together with previous results (Refs. 8, 11) this shows the convincing character of the concept of an inflationary stage in the early universe.

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[1]V. G. Gurzadyan and A. A.Kocharyan, Preprint, EPI-921(72), Yerevan, 1986.
[2]V. I. Arnol'd, *Mathematical Methods of Classical Mechanics*, Springer Verlag, New York, 1978.
[3]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, London–New York, 1973.
[4]R. Arnowitt, S. Deser, and C. Misner, "The dynamics of general relativity" in *Gravitation: An Introduction to Current Research*, Ed. by L. Witten, J. Wiley, New York, 1962, pp. 227-265.
[5]K. Kuchař, in *Relativity, Astrophysics, and Cosmology*, Ed. W. Israel, Reidel, Dordrecht, 1973.
[6]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman, S. Francisco, 1973.
[7]A. D. Linde, Zh. Eksp. Teor. Fiz. **87**, 369 (1984) [Sov. Phys. JETP **60**, 211 (1984)].
[8]L. A. Kofmann, A. D. Linde, and A. A. Starobinsky, Phys. Lett. **157B**, 361 (1985).
[9]T. Piran, IAS Preprint, Princeton, 1986.
[10]A. D. Linde, IITP Preprint IC/86/75, Trieste, 1986.
[11]V. A. Belinskiĭ, L. P. Grishchuk, I. M. Khalatnikov, and Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **89**, 346 (1985) [Sov. Phys. JETP **62**, 195 (1985)]; Phys. Lett. **162B**, 281 (1985).
[12]V. G. Gurzadyan and A. A. Kocharyan, Dokl. Akad. Nauk SSSR **287**, 813 (1986) [Sov. Phys. Doklady **31**, 275 (1986)].
[13]V. G. Gurzadyan and A. A. Kocharyan, Preprint EPI-920(71), Yerevan, 1986.
[14]A. D. Linde, Phys. Lett. **162B**, 281 (1985).
[15]S. W. Hawking, Phys. Lett. **150B**, 339 (1985).

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