Stochastic dynamics of atoms in the field of plane light waves

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The motion of two-level atoms in the field of two traveling light waves is studied for the case of atom-field interaction times much longer than the radiative relaxation times. It is shown that the dynamics of the atoms is described by the Fokker-Planck equation, whose coefficients (the radiative force and the momentum diffusion tensor) are functions of the atomic velocity and position. It is shown that, in a field of arbitrary form, the radiative force and diffusion tensor can be represented as series containing both spatially homogeneous and oscillating terms. The contributions of the nonlinear multiresonance atom-field interaction processes to the radiative force are investigated. The resulting theory is used to find the time required for atoms to become localized at the nodes or antinodes of a low-intensity standing wave. The relations connecting the radiative force for a field of two plane waves to the forces acting on an atom in a running plane wave, a standing plane wave, and a Gaussian beam are determined.

1. INTRODUCTION

The development of methods of controlling the motion of atoms with the aid of the resonant laser radiation pressure has recently stimulated the formulation of the theoretical principles of the stochastic dynamics of atoms in a resonant light field. Here and below we use the term "stochastic dynamics" to describe the motion of atoms for atom-field interaction times much longer that the radiative relaxation times for the atomic levels. For such times the radiative force on the atom is always responsible for changes in its resonant-radiation field, while the fluctuations of the momentum of the atom are responsible for the diffusion of the atom in momentum space.

Thus far, stochastic dynamics has been studied most fully in the light-field configurations of traveling and standing waves. These field configurations have played an important role in the investigation of a number of kinetic effects, on the basis of which there have been developed such new methods of controlling atomic motion as velocity monochromatization of atomic ensembles, the focusing and defocusing of atomic beams, and longitudinal and transverse radiative cooling of atomic beams to temperatures two to three orders of magnitude smaller than the temperature of liquid helium.[1]

But the one-dimensional light-field models thus far investigated (traveling and standing waves) are found to be inadequate for the analysis of such methods, presently under study, of controlling atomic motion as collimation and compression of atomic beams.[1,2] In these methods light fields containing noncollinear waves are used. Accordingly, of importance in investigations of the stochastic dynamics of atoms is the behavior of the atomic motion under conditions when the interaction with the field depends not on one component of the velocity, as in the case of the one-dimensional field, but on the total atomic velocity vector. The solution to the present problem naturally cannot be obtained through a direct generalization of the theory in the case of the one-dimensional field, since because of the saturation effect, the contributions of the individual waves to the change in the momentum of the atom are not additive.

The main aim of the present paper is to investigate the stochastic dynamics of atoms in a light field produced by two plane waves propagating in arbitrary directions. The results of the investigation show that the stochastic dynamics of an atom in the field of the plane waves is determined by the atomic-velocity-vector-dependent multiresonance nonlinear atom-field interaction processes.

Besides this objective, the study of the motion of an atom in the field of two plane waves is of interest for the establishment of the relation between the forces acting on an atom in such physically important light-field configurations as a plane traveling wave, a plane standing wave, and a light ray. As is well known, in a plane traveling wave the radiative force, which has the meaning of a light pressure force, does not depend on the location of the center of mass of the atom.[3–6] On the other hand, in the case of a plane standing wave the expression for the radiative force contains, besides a spatially homogeneous term, terms that oscillate at the wavelength of the field.[7,8] And in the case of a light ray the expression for the radiative force contains a spatially homogeneous longitudinal component (a light pressure force) and an atomic-position-dependent transverse component (the gradient force).[6] Thus, in the field configurations in question the radiative force is determined by qualitatively different relations. At the same time all these types of fields can be considered to be particular cases of the field produced by two plane waves. Accordingly, the study of the force for the field consisting of two plane waves allows us to study how the general relation for the radiative force goes over, as the angle between the two plane waves is varied, into the relations determining the forces for the plane traveling wave, the plane standing wave, and the light ray.

As an example of the application of the equation of motion of an atom in the field of two plane waves, below we present for the first time (as far as we can determine from the literature) an estimate for the time of localization of an atom at a node or antinode of a weak standing wave.

The possibility of localizing an atom in a region with dimensions of the order of the wavelength of the field has been pointed out before by Letokhov.[9] The first indirect experimental proof of the localization of atoms at the nodes or antinodes of a standing wave was recently published by Prentiss and Ezekeil.[10] In the present paper we show that the

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physical cause of the escape of atoms initially localized at the nodes or antinodes of a weak standing wave is the diffusion of the momenta of the atoms in the radiation field. At physically reasonable values of the standing-wave intensity, the momentum diffusion limits the time of localization of the atoms at the nodes or antinodes of the field to a value in the range from 0.01 to 0.1 sec.

2. BASIC RELATIONS

We shall assume that a two-level atom resonantly interacts with the field of two plane running waves of the same frequency ω and amplitude E_0, propagating in the directions of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ ($\mathbf{k}_1=\mathbf{e}_1k$, $\mathbf{k}_2=\mathbf{e}_2k$, $k=\omega/c$):

$$\mathbf{E}=\mathbf{e}E_0\cos(\mathbf{k}_1\mathbf{r}-\omega t)+\mathbf{e}E_0\cos(\mathbf{k}_2\mathbf{r}-\omega t). \tag{1}$$

For the two-level atom model to be applicable, we assume that the waves are linearly polarized in the direction of the unit vector $\mathbf{e}=\mathbf{e}_x$, perpendicularly to the plane, yz, containing the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. As the y axis we choose the line symmetrically located with respect to the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ by setting $\mathbf{k}_1\mathbf{e}_y=\mathbf{k}_2\mathbf{e}_y=k\cos\varphi$ (Fig. 1). Let us assume that the lower level of the atom is the ground level, and that the upper level spontaneously decays to the ground level at a rate $2\gamma=4d^2\omega_0^3/3\hbar c^3$, where d is the matrix element of the x component of the atom's dipole moment and ω_0 is the frequency of the atomic transition.

For the description of the motion of the atom in the field (1) we shall use the atomic density matrix in the Wigner representation $\rho_{\alpha\beta}=\rho_{\alpha\beta}(\mathbf{r},\mathbf{p},t)$, where α, $\beta=1$, 2. The equations describing the evolution of $\rho_{\alpha\beta}(\mathbf{r},\mathbf{p},t)$ in the standard rotating-wave approximation, after the substitution

$$\rho_{21}\to\rho_{21}\exp(-i\Omega t+iky\cos\varphi),\ \Omega=\omega-\omega_0,$$

which allows us to eliminate the explicit time dependence from the equations, has been made, are the following (see, for example, Refs. 4–6 and 8):

$$\begin{aligned}
\frac{d}{dt}\rho_{22}(\mathbf{p})&=ige^{iakz}\rho_{12}\left(\mathbf{p}-\frac{\hbar\mathbf{k}_1}{2}\right)+ige^{-iakz}\rho_{12}\left(\mathbf{p}-\frac{\hbar\mathbf{k}_2}{2}\right)\\
&\quad+\text{c.c.}-2\gamma\rho_{22}(\mathbf{p}),\\
\frac{d}{dt}\rho_{11}(\mathbf{p})&=-ige^{iakz}\rho_{12}\left(\mathbf{p}+\frac{\hbar\mathbf{k}_1}{2}\right)-ige^{-iakz}\rho_{12}\left(\mathbf{p}+\frac{\hbar\mathbf{k}_2}{2}\right)\\
&\quad+\text{c.c.}+2\gamma\int d\mathbf{n}\,\Phi(\mathbf{n})\rho_{22}(\mathbf{p}+\mathbf{n}\hbar k), \qquad (2)\\
\frac{d}{dt}\rho_{21}(\mathbf{p})&=ige^{iakz}\left[\rho_{11}\left(\mathbf{p}-\frac{\hbar\mathbf{k}_1}{2}\right)-\rho_{22}\left(\mathbf{p}+\frac{\hbar\mathbf{k}_1}{2}\right)\right]\\
&\quad+ige^{-iakz}\left[\rho_{11}\left(\mathbf{p}-\frac{\hbar\mathbf{k}_2}{2}\right)-\rho_{22}\left(\mathbf{p}+\frac{\hbar\mathbf{k}_2}{2}\right)\right]\\
&\quad+[i(\Omega-bkv_y)-\gamma]\rho_{21}(\mathbf{p}),
\end{aligned}$$

where

$$g=dE_0/2\hbar,\ a=\sin\varphi,\ b=\cos\varphi,$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial\mathbf{r}},\qquad \Phi(n)=\frac{3}{8\pi}[1-(\mathbf{n}\mathbf{e}_x)^2],$$

and $\mathbf{n}$ is the unit vector for determining the direction of propagation of the spontaneously emitted photon.

At the times $t\gg\gamma^{-1}$ of interest to us, the atomic density matrix, considered on a scale larger than the photon momentum $\hbar k$, is a smooth function of the atomic momentum.[6]

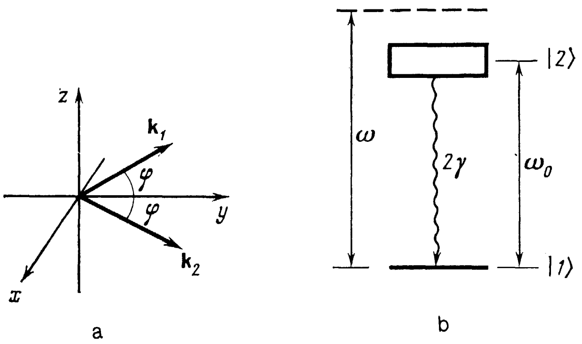

FIG. 1. Propagation geometry for the plane waves (1) and the two-level scheme of interaction of an atom with a light-wave field (b).

Therefore, the equations (2) can be expanded in series in powers of $\hbar k$. Then introducing the real Wigner function

$$w=\rho_{11}+\rho_{22}$$

and Wigner-Bloch functions

$$u=\rho_{11}-\rho_{22},\ c=\rho_{21}+\rho_{12},\ is=\rho_{21}-\rho_{12},$$

which are convenient functions for the investigation, we obtain from (2) the equations

$$\frac{d}{dt}w=-2\hbar kgb\cos(akz)\frac{\partial}{\partial p_y}s+2\hbar kga\sin(akz)\frac{\partial}{\partial p_z}c$$
$$+\frac{1}{2}\hbar^2k^2\gamma\sum_i\alpha_{ii}\frac{\partial^2}{\partial p_i^2}(w-u)+\dots, \tag{3a}$$

$$\frac{d}{dt}u=2\gamma(w-u)-4g\cos(akz)s+\dots, \tag{3b}$$

$$\frac{d}{dt}c=-(\Omega-bkv_y)s-\gamma c+2\hbar kga\sin(akz)\frac{\partial}{\partial p_z}w+\dots, \tag{3c}$$

$$\frac{d}{dt}s=(\Omega-bkv_y)c+4gu\cos(akz)-\gamma s$$
$$-2\hbar kgb\cos(akz)\frac{\partial}{\partial p_y}w+\dots, \tag{3d}$$

where

$$i=x,\ y,\ z;\ \alpha_{xx}=1/5,\ \alpha_{yy}=\alpha_{zz}=2/5.$$

3. KINETIC EQUATION

In the equations (3) the terms containing the momentum derivatives are, as compared to the other terms, of the same order as $\varepsilon=\hbar k/\Delta p$, where $\Delta p\approx M\gamma/k$ is the characteristic momentum range within the limits of which the atom interacts with the resonance radiation. Using the value of the spontaneous emission rate $\gamma\approx d^2\omega_0^3/\hbar c^3$, the value of the matrix element of the dipole moment $d=er_0$, and the relation between the optical transition frequency ω_0 and the Bohr radius r_0 ($\omega_0\approx\hbar(m_er_0)^{-1}$), we find that the small parameter is of the order of

$$\varepsilon\approx\frac{\hbar k^2}{M\gamma}=\frac{m_e}{\alpha M}\ll1,$$

where α is the fine structure constant, m_e is the electron mass, and M is the mass of the atom.

In conformity with the smallness of the terms containing the momentum derivatives, the equations (3) go over in the kinetic stage of the evolution (i.e., for $t \gg \gamma^{-1}$) into the kinetic equation describing the classical motion of the atom. To establish the form of the classical kinetic equation, let us consider the equations (3) in increasing orders in the small parameter ε. In the zeroth approximation in ε, it follows from (3a) that the function w satisfies the phase-density conservation equation:

$$\frac{\partial}{\partial t} w + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} w = 0. \tag{4}$$

For the subsequent analysis of the classical limit of the equations (3), we shall use Bogolyubov's idea, according to which the rapidly relaxing (relaxation time $\sim \gamma^{-1}$) functions $h = u, c$, and s should, in the kinetic stage of the evolution, be regarded as functionals of the distribution function $w = w(\mathbf{r},\mathbf{p},t)$ (see, for example, Ref. 11).

To lowest order in ε, the functional relations are, in accordance with the structure of the right members of (3b)–(3d), linear:

$$h = H^0 w, \tag{5}$$

where the functions $H^0 = U^0, C^0$, and S^0 do not depend explicitly on the time. Accordingly, to lowest order in ε, the time derivatives in the left members of (3b)–(3d) can be computed with the use of (5) and (6):

$$\frac{dh}{dt} = H^0 \frac{dw}{dt} + \mathbf{v} w \frac{\partial H^0}{\partial \mathbf{r}} = \mathbf{v} w \frac{\partial H^0}{\partial \mathbf{r}}. \tag{6}$$

Since the right members of Eqs. (3b)–(3d) contain oscillating terms, the functions H^0 can naturally be represented in the form of series:

$$H^0 = U^0, C^0, S^0 = \sum_{n=-\infty}^{+\infty} h_n^0 e^{inakz}. \tag{7}$$

The new unknown functions $h_n^0 = u_n^0, c_n^0$, and s_n^0 satisfy the following equations, which are obtained from (3b)–(3d):

$$\begin{aligned} &u_n^0 (2\gamma + inakv_z) = -2g(s_{n-1}^0 + s_{n+1}^0) + 2\gamma\delta_{n,0}, \\ &c_n^0 (\gamma + inakv_z) = -(\Omega - bkv_y) s_n^0, \\ &s_n^0 (\gamma + inakv_z) = (\Omega - bkv_y) c_n^0 + 2g(u_{n-1}^0 + u_{n+1}^0). \end{aligned} \tag{8}$$

Using the solutions to the equations (8), we can now find the equation for w to first order in ε. Indeed, retaining in the right member of (3a) the terms linear in ε, and using the representation in the form (5), we obtain from (3a) the Liouville equation

$$\frac{\partial}{\partial t} w + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} w + \frac{\partial}{\partial \mathbf{p}} (\mathbf{F} w) = 0, \tag{9}$$

where the force is $\mathbf{F} = \mathbf{e}_y F_y + \mathbf{e}_z F_z$. The force components, which are given by the solutions to the equations (8), are

$$\begin{aligned} F_y &= F_y^0 + \sum_{n=1}^{\infty} [F_y^{2nc} \cos(2ankz) + F_y^{2ns} \sin(2ankz)], \\ F\ &= F_z^0 + \sum_{n=1}^{\infty} [F_z^{2nc} \cos(2ankz) + F_z^{2ns} \sin(2ankz)], \end{aligned} \tag{10}$$

where

$$\begin{aligned} &F_y^0 = 2\hbar kgb \operatorname{Re} s_1^0, \quad F_z^0 = 2\hbar kga \operatorname{Im} c_1^0, \\ &F_y^{2nc} = 2\hbar kgb \operatorname{Re}(s_{2n+1}^0 + s_{2n-1}^0), \\ &F_z^{2nc} = 2\hbar kga \operatorname{Im}(c_{2n+1}^0 - c_{2n-1}^0), \\ &F_y^{2ns} = -2\hbar kgb \operatorname{Im}(s_{2n+1}^0 + s_{2n-1}^0), \\ &F_z^{2ns} = 2\hbar kga \operatorname{Re}(c_{2n+1}^0 - c_{2n-1}^0). \end{aligned} \tag{11}$$

The solutions to the equations (8) are

$$\begin{aligned} &c_n^0 = -(\Omega - bkv_y) s_n^0 / (\gamma + iankv_z), \\ &c_{-n}^0 = (c_n^0)^*, \ n = +1, \pm 3, \ldots, \\ &s_n^0 = -(Q_{n-1}/D_{n-1}) u_{n-1}^0, \quad s_{-n}^0 = (s_n^0)^*, \qquad n = \pm 1, \pm 3, \ldots, \\ &u_n^0 = -(Q_{n-1}/D_{n-1}) s_{n-1}, \quad u_{-n}^0 = (u_n^0)^*, \qquad n = 0, \pm 2, \pm 4, \ldots, \\ &u_0^0 = (1 + 2 \operatorname{Re} Q_0)^{-1}. \end{aligned} \tag{12}$$

Here the Q_m are convergent nonterminating continued fractions ($m = 0, \pm 1, \pm 2, \ldots$):

$$Q_m = \cfrac{p_m}{1 + \cfrac{p_{m+1}}{1 + \cfrac{p_{m+2}}{1 + \ldots}}}, \tag{13}$$

with numerators

$$p_n = G \frac{\gamma + in_1 akv_z}{2\gamma + in_2 akv_z} \frac{2\gamma^2}{(\Omega - bkv_y)^2 + (\gamma + in_1 akv_z)^2},$$

in which

$n_1 = n + 1$, $n_2 = n$, when n is even,

$n_1 = n$, $n_2 = n + 1$, when n is odd,

and $G = 2g^2/\gamma^2$ is the saturation parameter.

The quantities D_n are given by the relations

$$D_n = \begin{cases} -2g/(2\gamma + inakv_z) & \text{for even } n, \\ 2g(\gamma + iankv_z)/[(\gamma + inakv_z)^2 + (\Omega - bkv_y)^2] & \text{for odd } n. \end{cases}$$

Thus, at $t \gg \gamma^{-1}$, and up to first order in ε, the basic microscopic quantum-mechanical equations for the Wigner density matrix reduce to the Liouville equation for the classical distribution function w and functional relations, (5) and (7), for the Wigner-Bloch functions

$$h = u, c, s = \left[\sum_{n=-\infty}^{\infty} h_n^0 e^{iankz} \right] w.$$

Let us now continue the derivation of the classical kinetic equation. Through first order in ε inclusively, the functional relations have, in accordance with the right members of (3b)–(3d), the form

$$h=H^0w+\hbar kbH^{1,y}\partial w/\partial p_y+\hbar kaH^{1,z}\partial w/\partial p_z, \tag{14}$$

where the functions $H^{1,y}$ and $H^{1,z}$ do not depend on the time and the position y. In first order in ε, the derivatives on the left-hand sides of (3b)–(3d) are, according to (9), equal to

$$\frac{dh}{dt}=-H^0\left(F_y\frac{\partial w}{\partial p_y}+F_z\frac{\partial w}{\partial p_z}\right)+\hbar kbv_z\frac{\partial H^{1,y}}{\partial z}\frac{\partial w}{\partial p_z}$$
$$+\hbar kav_z\frac{\partial H^{1,z}}{\partial z}\frac{\partial w}{\partial p_z}. \tag{15}$$

Let us now represent the functions $H^{1,i}=H^{1,y}$, $H^{1,z}$ in the form of the series (7):

$$H^{1,i}=\sum_{n=-\infty}^{\infty}h_n^{1,i}e^{iankz}. \tag{16}$$

Next, separating out in (3b)–(3d) the terms proportional to the $\partial/\partial p_i$ derivatives, and of first order in ε, we obtain from these equations the system of recurrence equations

$$(2\gamma+iankv_z)u_n^{1,y}+2g(s_{n-1}^{1,y}+s_{n+1}^{1,y})=\sum_l u_l{}^0f_y^{n-l},$$
$$(2\gamma+iankv_z)u_n^{1,z}+2g(s_{n-1}^{1,z}+s_{n+1}^{1,z})=\sum_l u_l{}^0f_z^{n-l},$$
$$(\gamma+iankv_z)c_n^{1,y}+(\Omega-bkv_y)s_n^{1,y}=\sum_l c_l{}^0f_y^{n-l},$$
$$(\gamma+iankv_z)c_n^{1,z}+(\Omega-bkv_y)s_n^{1,z}=\sum_l c_l{}^0f_z^{n-l}+ig(\delta_{n,1}-\delta_{n,-1}), \tag{17}$$
$$(\gamma+iankv_z)s_n^{1,y}-(\Omega-bkv_y)c_n^{1,y}-2g(u_{n-1}^{1,y}+u_{n+1}^{1,y})$$
$$=g(\delta_{n,1}+\delta_{n,-1})+\sum_l s_l{}^0f_y^{n-l},$$
$$(\gamma+iankv_z)s_n^{1,z}-(\Omega-bkv_y)c_n^{1,z}-2g(u_{n-1}^{1,z}+u_{n+1}^{1,z})=\sum_l s_l{}^0f_z^{n-l},$$

where the quantities f_y^n and f_z^n are given by the relations

$$F_y=\hbar kb\sum_{n=-\infty}^{\infty}f_y{}^{2n}e^{2iankz},\qquad F_z=\hbar ka\sum_{n=-\infty}^{+\infty}f_z{}^{2n}e^{2iankz},$$

and the summation is over the index l, which varies from $-\infty$ to $+\infty$.

Finding the solution to the present system, and substituting (14) into (3a), we obtain for w in second order in ε the Fokker-Planck equation

$$\frac{d}{\partial t}w+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}w=-\frac{\partial}{\partial \mathbf{p}}(\mathbf{F}w)+\sum_{ij}\frac{\partial^2}{\partial p_i\,\partial p_j}(\mathscr{D}_{ij}w), \tag{18}$$

where the force is given by the relations (10) and the momentum diffusion tensor is

$$\mathscr{D}_{ij}=\frac{1}{2}\hbar^2k^2\gamma\left\{d_{ij}{}^0+\sum_{n=1}^{\infty}[d_{ij}^{2nc}\cos(2ankz)\right.$$
$$\left.+d_{ij}^{2ns}\sin(2nakz)]\right\},$$

$$d_{xx}{}^0=\alpha_{xx}(1-u_0{}^0),\quad d_{yy}{}^0=\alpha_{yy}(1-u_0{}^0)-(4g/\gamma)b^2\,\mathrm{Re}\,s_1{}^{1,y},$$
$$d_{zz}{}^0=\alpha_{zz}(1-u_0{}^0)-(4g/\gamma)a^2\,\mathrm{Im}\,c_1{}^{1,z},$$
$$d_{yz}{}^0=-(4g/\gamma)ab(\mathrm{Re}\,s_1{}^{1,z}+\mathrm{Im}\,c_1{}^{1,y}),$$
$$d_{xx}^{2nc}=-2\alpha_{xx}\,\mathrm{Re}\,u_{2n}{}^0,\qquad d_{xx}^{2ns}=2\alpha_{xx}\,\mathrm{Im}\,u_{2n}{}^0, \tag{19}$$
$$d_{yy}^{2nc}=-2\alpha_{yy}\,\mathrm{Re}\,u_{2n}{}^0-(4g/\gamma)b^2\,\mathrm{Re}(s_{2n-1}^{1,y}+s_{2n+1}^{1,y}),$$
$$d_{yy}^{2ns}=2\alpha_{yy}\,\mathrm{Im}\,u_{2n}{}^0+(4g/\gamma)b^2\,\mathrm{Im}(s_{2n-1}^{1,y}+s_{2n+1}^{1,y}),$$
$$d_{zz}^{2nc}=-2\alpha_{zz}\,\mathrm{Re}\,u_{2n}{}^0-(4g/\gamma)a^2\,\mathrm{Im}(c_{2n+1}^{1,z}-c_{2n-1}^{1,z}),$$
$$d_{zz}^{2ns}=2\alpha_{zz}\,\mathrm{Im}\,u_{2n}{}^0-(4g/\gamma)a^2\,\mathrm{Re}(c_{2n+1}^{1,z}-c_{2n-1}^{1}),$$
$$d_{yz}^{2nc}=-(4g/\gamma)ab[\mathrm{Re}(s_{2n+1}^{1,z}+s_{2n-1}^{1,z})+\mathrm{Im}(c_{2n+1}^{1,y}-c_{2n-1}^{1,y}),$$
$$d_{yz}^{2ns}=(4g/\gamma)ab[\mathrm{Im}(s_{2n+1}^{1,z}+s_{2n-1}^{1,z})-\mathrm{Re}(c_{2n+1}^{1,y}-c_{2n-1}^{1,y}).$$

The quantities u_m ($m=0,\ \pm2,\ \pm4,\ \ldots$) entering into (19) can be expressed in terms of the convergent infinite continued fractions from (13). After the system (17) has been solved, the quantities $s_m^{1,y}$, $s_m^{1,z}$, $c_m^{1,y}$, $c_m^{1,z}$ ($m=\pm1,\ \pm3\ldots$ can also be expressed in terms of convergent infinite continued fractions.

We can, continuing the process of derivation of the classical kinetic equation, establish the fact that the function w satisfies the infinite (generalized) Fokker-Planck equation

$$\frac{d}{dt}w+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}w=\sum_{lmp}\frac{\partial^{l+m+p}}{\partial p_x{}^l\partial p_y{}^m\partial p_z{}^p}(A_{x,y,z}^{lmp}w), \tag{20}$$

whose coefficients $A_{x,y,z}^{lmp}$ are functions of the atomic center of mass coordinate and the velocity components v_y and v_z. The dominant contribution to the distribution function in Eq. (20) is made by the first two terms. The remaining terms make contributions that are of the order of the parameter $1/\gamma t$ in smallness. Thus, the ratio of the third term to the second is of the order of

$$\hbar k/\Delta p\approx\hbar k/(\mathscr{D}t)^{1/2}\sim1/(t\gamma)^{1/2}\ll1.$$

The same estimate follows from the central-limit theorem.[12]

Let us note here that, since the Fokker-Planck equation does not contain information about the internal state of the atom, it is equally applicable both to a single atom with stochastically prescribed position and momentum distributions and to an ensemble of noninteracting atoms.

4. MULTIRESONANCE PROCESSES

A characteristic of the coefficients of the Fokker-Planck equation (20) is their critical dependence on the velocity components v_y and v_z. As an example, we show in Figs. 2 and 3 the plots of the force components F_y^0 and F_z^0 as functions of the velocity component v_z for different values of the velocity component v_y. The cause of the critical oscillatory dependence of the force components on the velocity vector is the nonlinear multiresonance processes of interaction of the atom with the field of the two running waves. For small values of the saturation parameter the positions of the

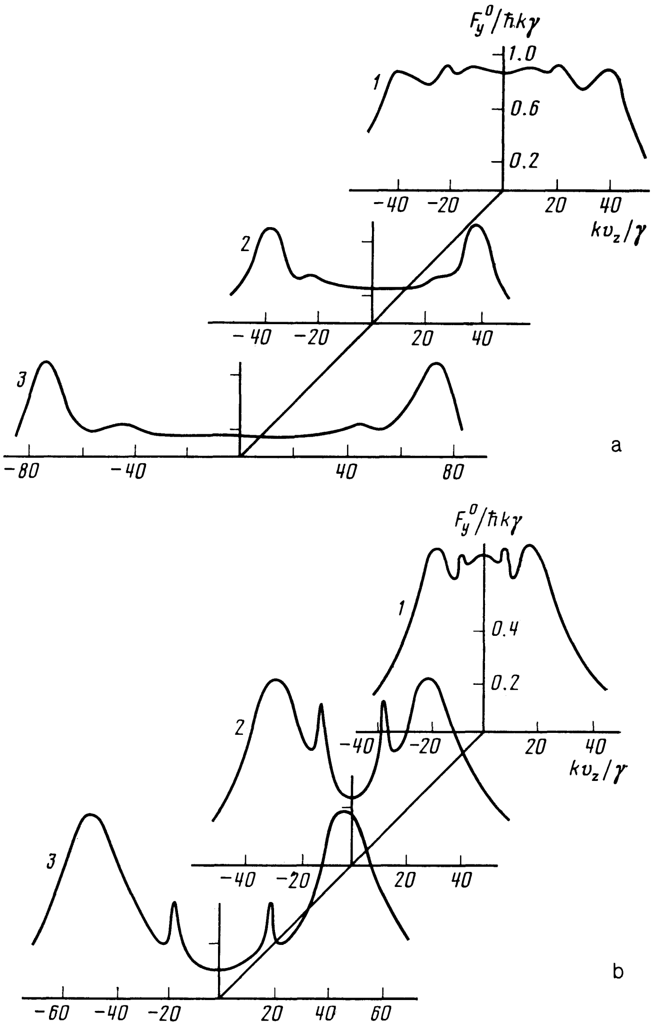

FIG. 2. Dependence of the radiative-force component $F_y^0/\hbar k\gamma$ on the velocity component kv_z/γ, as computed in the case when the angle $\varphi = 15°$ (a) and 45° (b) for $G = 100$, $\Omega = -3\gamma$, and different values of kv_y/γ: 1) 0; 2) −20; and 3) −40.

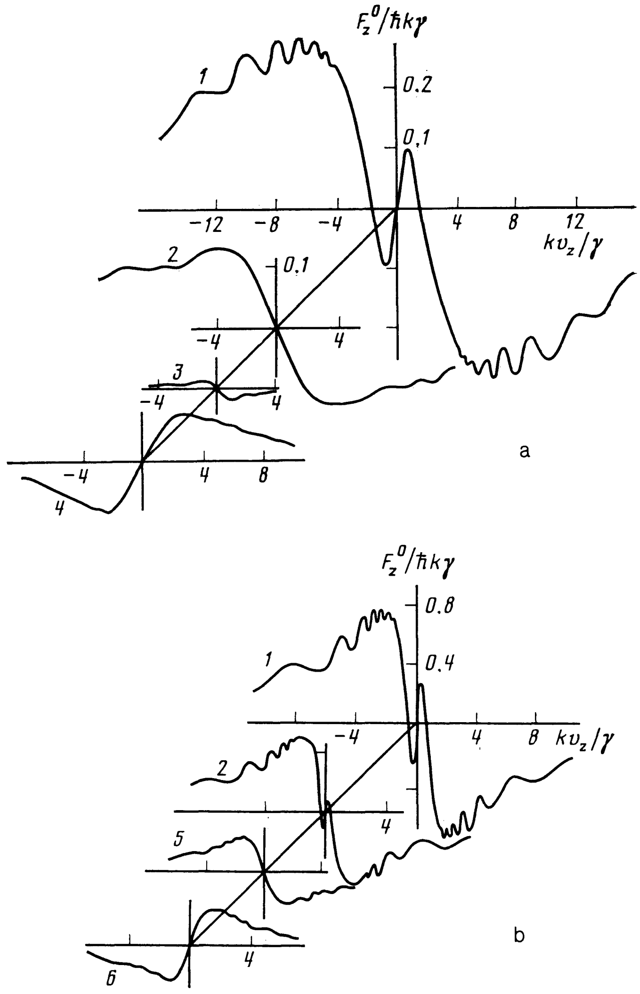

FIG. 3. Dependence of the radiative-force component $F_z^0/\hbar k\gamma$ on the velocity component kv_z/γ, computed in the cases when the angle is $\varphi = 15°$ (a) and 45° (b) for $G = 100$, $\Omega = -3\gamma$ and different values of kv_z/γ: 1) 0; 2) −2; 3) −3; 4) −4; 5) −3.4; and 6) −5.2.

resonances can be found on the basis of the energy conservation law, applied to a definite interaction process. Let us, for definiteness, consider the interaction processes in the rest frame of the atom.

The simplest interaction process is the single-resonance process, in which the atom is excited into the upper state by the field of any one of the waves. This process, according to the energy conservation law, occurs at velocities satisfying one of the conditions:

$$\omega_0 = \omega - \mathbf{k}_1\mathbf{v},\quad \omega_0 = \omega - \mathbf{k}_2\mathbf{v}.$$

A second-order process occurs when the atom absorbs a photon from one wave and returns to the initial state by emitting the photon into the other wave. According to the energy conservation law

$$(\omega - \mathbf{k}_1\mathbf{v}) - (\omega - \mathbf{k}_2\mathbf{v}) = 0,$$

this process is effective when $v_z \approx 0$. A third-order process occurs at velocities satisfying one of the following conditions:

$$\omega_0 = (\omega - \mathbf{k}_1\mathbf{v}) - (\omega - \mathbf{k}_1\mathbf{v}) + (\omega - \mathbf{k}_1\mathbf{v}),$$
$$\omega_0 = (\omega - \mathbf{k}_2\mathbf{v}) - (\omega - \mathbf{k}_1\mathbf{v}) + (\omega - \mathbf{k}_2\mathbf{v}).$$

Similarly, we can write out the conditions under which higher-order multiresonance processes occur. At large values of the saturation parameter the resonances undergo displacements, and their widths increase.

In the case of unidirectional waves ($\varphi = 0$), when the field (1) reduces to a plane running wave, the conditions for the occurrence of the processes of even order are fulfilled for any velocities, and all the odd-order processes occur at one resonance velocity, $v_y = \Omega/k$. In this case the relations (10) determine the light-pressure force in the running wave.[1] In the case of waves propagating in opposite directions ($\varphi = \pi/2$) the odd-order processes occur at the velocities $kv_z = \pm\Omega/(2n+1)$. The relations (11) in this case give the force acting on the atom in the plane standing wave.[7,8]

5. THE LIMITING CASES

Let us consider the dependence of the radiative force (10) on the angle between the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ (Fig. 1a), having in mind the establishment of the connection between the general relation (10) and the relations for the radiative force in the particular cases of a traveling and a standing light wave, and also in the case of a light ray.

a. A plane traveling wave ($\varphi=0$)

In this case $a=0$, $b=1$, and the series (10) reduce to the relations

$$F_y=F_y^0+\sum_{n=1}^{\infty}F_y^{2nc}, \quad F_z=0, \tag{21}$$

where

$$F_y^0=2\hbar kgs_1^0, \quad F_y^{2nc}=2\hbar kg(s_{2n+1}^0+s_{2n-1}^0).$$

Summing the series in (21), we have

$$F_y=4\hbar kg(s_1^0+s_3^0+s_5^0+\ldots)=\hbar k\gamma\frac{G_1}{1+G_1+(\Omega-kv_y)^2/\gamma^2}, \tag{22}$$

where $G_1=2(2g/\gamma)^2$ is the saturation parameter for a plane running wave with amplitude $2E_0$.

The relation (22) coincides with the well-known expression for the light-pressure force acting on an atom in a plane traveling wave.[3–6]

b. A plane standing wave ($\varphi=\pi/2$)

In this case $a=1$, $b=0$, and the series (10) reduce to the relations

$$F_y=0, \quad F_z=F_z^0+\sum_{n=1}^{\infty}[F_z^{2nc}\cos(2nkz)+F_z^{2ns}\sin(2nkz)], \tag{23}$$

where

$$F_z^0=2\hbar kg\,\mathrm{Im}\,c_1^0,$$

$$F_z^{2nc}=2\hbar kg\,\mathrm{Im}(c_{2n+1}^0-c_{2n-1}^0), \quad F_z^{2ns}=2\hbar kg\,\mathrm{Re}(c_{2n+1}^0-c_{2n-1}^0),$$

and the quantities c_m^0 can be expressed in terms of the convergent infinite continued fractions (13). The force given by (23) coincides with the force found earlier in Refs. 7 and 8.

In the case of weak saturation of the atomic transition, when the condition

$$G\ll(1+\Omega^2/\gamma^2)^{1/2} \tag{24}$$

is fulfilled, all the multiresonance processes, except the first-order processes, can be neglected in (23). Then the expression for F_z reduces to[13,14]

$$F_z=2\hbar k\gamma G\frac{L_--L_+}{1+G(L_-+L_+)}\sin^2 kz$$
$$+\hbar k\Omega G\frac{(1-kv_z/\Omega)L_-+(1+kv_z/\Omega)L_+}{1+G(L_-+L_+)}\sin 2kz, \tag{25}$$

where $G=2(g/\gamma)^2$, and we have introduced the functions

$$L_\pm=[1+(\Omega\pm kv_z)^2/\gamma^2]^{-1}. \tag{26}$$

In the limit of a large detuning ($|\Omega|\gg k|v_z|$) the force (25) has, to first order in kv_z/Ω, the form

$$F_z=F_{fr}+F_{osc}=8\hbar k^2G\frac{\Omega/\gamma}{(1+\Omega^2/\gamma^2)^2}v_z\sin^2 kz$$
$$+2\hbar k\Omega\frac{G}{1+\Omega^2/\gamma^2}\sin 2kz. \tag{27a}$$

If, moreover, $|\Omega|\gg\gamma$, then (27a) goes over into the relation

$$F_z=F_{fr}+F_{osc}=16\hbar k^2\frac{g^2\gamma}{\Omega^3}v_z\sin^2 kz+4\hbar k\frac{g^2}{\Omega}\sin 2kz, \tag{27b}$$

the second part of which coincides with the expression obtained in Refs. 15 and 16. The latter, in turn, coincides with the expression found earlier by Letokhov[9] on the basis of the classical formulas.[17,18]

c. Light ray ($\varphi\ll\pi/2$)

If $\varphi\ll\pi/2$, then the field (1) at small values of z ($|z|\ll\pi/ka$) reduces to the field of a Gaussian light beam:

$$\mathbf{E}=2\mathbf{e}E_0\exp(-a^2k^2z^2/2)\cos(ky-\omega t). \tag{28}$$

In this case the longitudinal component F_y of the force coincides with (22). The transverse component F_z of the force is obtained from (11) with allowance for the condition $ak|z|\ll\pi$:

$$F_z=F_z^0+\sum_{n=1}^{\infty}F_z^{2nc}+\sum_{n=1}^{\infty}2ankzF_z^{2ns}. \tag{29}$$

Summing the series (29), we obtain

$$F_z=2\hbar kga\,\mathrm{Im}(c_1^0+c_3^0-c_1^0+c_5^0-c_3^0+\ldots)$$
$$+4\hbar g(ka)^2z\,\mathrm{Re}(c_3^0-c_1^0+2c_5^0$$
$$-2c_3^0+3c_7^0-3c_5^0+\ldots)=\frac{\hbar z}{q^2}\frac{(\Omega-kv_y)G_1(z)}{1+G_1(z)+(\Omega-kv_y)^2/\gamma^2}, \tag{30}$$

where we have introduced the beam radius $q=1/ka$ and the atomic-coordinate dependent saturation parameter

$$G_1(z)=2[2g(z)/\gamma]^2,\ g(z)=dE_0\exp(-a^2k^2z^2/2)/2\hbar.$$

The force (30) coincides with the gradient force obtained in Ref. 6.

6. THE RATE EQUATION APPROXIMATION

In the important case of weak atomic-transition saturation, we can neglect all the multiresonance processes, except the first-order processes, in the coefficients of the Fokker-Planck equation. Formally, the approximation in question is valid when we can limit ourselves to considering only the first numerator in the nonterminating continued fractions determining the coefficients of the kinetic equation. For $\varphi=\pi/2$ (the standing-wave case) the condition for the applicability of the weak-saturation (rate equation) approximation has been written out above in (24). In the general case of an arbitrary angle φ the condition of applicability of the weak saturation approximation is

$$G\ll[1+(\Omega-bkv_y)^2/\gamma^2]^{1/2}. \tag{31}$$

In this approximation the components of the force are given by the relations

$$F_y=2\hbar k\gamma Gb\left[\frac{L_-+L_+}{1+G(L_-+L_+)}\cos^2(akz)\right.$$
$$\left.+\frac{1}{2\gamma}\frac{(\Omega-bkv_y+akv_z)L_+-(\Omega-bkv_y-akv_z)L_-}{1+G(L_-+L_+)}\sin(2akz)\right], \tag{32a}$$

$$F_z=2\hbar k\gamma Ga\left[\frac{L_--L_+}{1+G(L_-+L_+)}\sin^2(akz)\right.$$
$$\left.+\frac{1}{2\gamma}\frac{(\Omega-bkv_y-akv_z)L_-+(\Omega-bkv_y+akv_z)L_+}{1+G(L_-+L_+)}\sin(2akz)\right], \tag{32b}$$

where

$$L_\pm=(1+(\Omega-bkv_y\pm akv_z)^2/\gamma^2)^{-1}.$$

In the general case of arbitrary φ, the momentum diffusion tensor, even in the weak saturation approximation, is given by extremely unwieldy formulas. Therefore, here we shall write it out only for the practically important case of a standing wave, when $\varphi = \pi/2$, $a = 1$, and $b = 0$. In this case

$$\mathscr{D}_{ij}={}^1/_2\hbar^2k^2\gamma d_{ii}\delta_{ij}, \tag{33}$$

where for $i = x, y$

$$d_{ii}=\alpha_{ii}\frac{G}{1+G(L_-+L_+)}(L_-+L_++\mu\cos 2kz+\nu\sin 2kz),$$
$$\mu=\frac{\gamma^2(L_-+L_+)+kv_z(\Omega_-L_--\Omega_+L_+)}{\gamma^2+k^2v_z^2},$$
$$\nu=\gamma\frac{kv_z(L_-+L_+)+(\Omega_+L_+-\Omega_-L_-)}{\gamma^2+k^2v_z^2}, \tag{34a}$$

and for $i = z$

$$d_{zz}=\alpha_{zz}\frac{G}{1+G(L_-+L_+)}(L_-+L_++\mu\cos 2kz+\nu\sin 2kz)$$
$$-2d_{zz}^{2c}\sin^2 kz+d_{zz}^{2s}\sin 2kz,$$
$$-d_{zz}^{2c}=\frac{G(L_-+L_+)+4G^2L_-L_+}{1+G(L_-+L_+)}$$
$$\cdot\left\{1+\frac{G(L_--L_+)^2[1-4(L_-+L_+)-8GL_-L_+]}{[1+G(L_-+L_+)]^2[L_-+L_++4GL_-L_+]}\right\}$$
$$-\frac{2G^2L_-L_+/\gamma^2}{[1+G(L_-+L_+)]^3}\{2\Omega(\Omega_+L_++\Omega_-L_-)$$
$$+G[(L_--L_+)^2(\gamma^2-4\Omega_-\Omega_+)$$
$$+2\Omega(\Omega_+L_++\Omega_-L_-)(L_-+L_+)]\},$$
$$d_{zz}^{2s}=\frac{G}{\gamma}\frac{\Omega_-L_-(1+2GL_+)-\Omega_+L_+(1+2GL_-)}{1+G(L_-+L_+)}$$
$$+\frac{G^2}{\gamma}\frac{(L_+-L_-)}{[1+G(L_-+L_+)]^3}\cdot$$
$$\cdot[(\Omega_-L_-+\Omega_+L_+)(1+4GL_-L_+)$$
$$+2\Omega_+L_+(2L_++GL_-+GL_+-2GL_-^2)$$
$$+2\Omega_-L_-(2L_-+GL_-+GL_+-2GL_+^2)+4\Omega(1+G)L_-L_+], \tag{34b}$$

where $\Omega_\pm = \Omega \pm kv_z$, and the functions $L_\pm$ are given in (26).

7. ESTIMATION OF THE TIME OF LOCALIZATION OF ATOMS IN THE FIELD OF A STANDING LIGHT WAVE

Let us consider as an example of the application of the relations obtained above the question of the time of localization of atoms at the nodes or antinodes of a weak standing light wave. We shall assume that the standing wave is oriented along the $z(\varphi = \pi/2)$ axis, and that the detuning is negative and large ($|\Omega| \gg k|v_z|, g, \gamma$). In this case the atoms in the field of the standing wave are acted upon by the force (27b), the second part, $F_{\rm osc}$, of which produces the periodic potential

$$U(z)=-U_0\cos 2kz,\quad U_0=2\hbar g^2/|\Omega|. \tag{35}$$

If we place a cold atom at one of the minima of the potential (35), e.g., at the point $z = 0$, the atom will be localized in the potential well until its kinetic energy attains, as a result of the diffusional heating, a value equal to the depth U_0 of the potential well (Fig. 4). The first part $F_{\rm fr}$ of the force (27b)

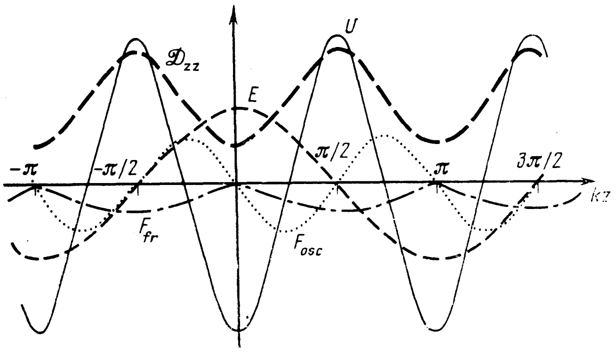

FIG. 4. Dependence of the standing-light-wave-field amplitude E, potential U, potential force $F_{\rm osc}$, frictional force $F_{\rm fr}$, and diffusion coefficient $\mathscr{D}_{zz}$ on the coordinate z of the atom for $\Omega < 0$ and a low Rabi frequency g $(4G \ll 1 + \Omega^2/\gamma^2)$.

does not in this case stabilize the atom in the vicinity of the bottom of the well, since the force $F_{\rm fr}$ is small in the vicinity of the point $z = 0$, and is equal to zero at the $z = 0$ point itself.

Using the diffusional law of increase of the kinetic energy of the atom, we find for the time of localization of the atom in the potential well U the expression

$$\tau=\langle(\Delta p)^2\rangle/\mathscr{D}_{zz}, \tag{36}$$

where $\langle(\Delta p)^2\rangle = 2MU_0$, and the coefficient of diffusion at the point $z = 0$ is, according to (34b), given by the formula

$$\mathscr{D}_{zz}=4\hbar^2k^2\gamma\alpha_{zz}g^2/\Omega^2.$$

From this we finally have

$$\tau=\frac{\hbar}{2R}\frac{|\Omega|}{\gamma\alpha_{zz}},\qquad R=\frac{\hbar^2k^2}{2M}. \tag{37}$$

Thus, the time of localization of atoms in a weak standing wave always increases when the detuning is increased. It should, however, be borne in mind that the depth U_0 of the potential well decreases when the detuning is increased. In view of the limitation $g \ll |\Omega|$, the maximum depth of the potential well is attained at $g \sim |\Omega|$. In this case $U_0 \approx 2\hbar|\Omega| \approx 2\hbar g$, and the localization time is of the order of

$$\tau\approx(\hbar/2R)(G/2)^{1/2}\alpha_{zz}^{-1}.$$

For example, for $G \approx 10^8$, when $g \approx |\Omega| \sim 10^4$, the localization time is $\tau \sim 0.1$ sec.

Let us note that, in the case of a high-intensity standing wave, the radiative force (23) also cannot stabilize the atoms at the minima of the periodic potential. At low velocities the force (23) reduces, after the summation of the series has been carried out, to the relation[19]

$$F_z=f_0+f_1, \tag{38}$$

where

$$f_0=\hbar k\Omega\frac{2G\sin 2kz}{1+\Omega^2/\gamma^2+4G\cos^2 kz},$$
$$f_1=\hbar k\Omega\frac{8G\sin^2 kz(1+\Omega^2/\gamma^2-4G\cos^2 kz-8G^2\cos^4 kz)}{(1+\Omega^2/\gamma^2+4G\cos^2 kz)^3}\frac{kv_z}{\gamma}.$$

The first part of the force (38) produces a periodic potential whose minima in the case when $\Omega > 0$ are located at the

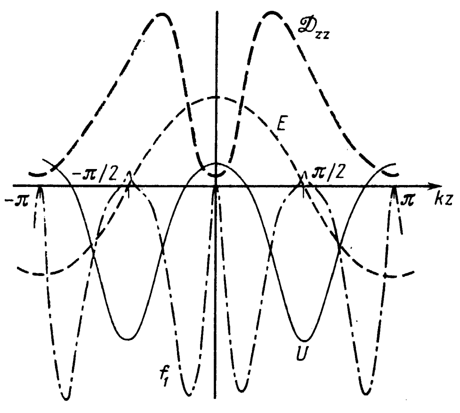

FIG. 5. Dependence of the standing-light-wave-field amplitude E, potential U, force f_1, and diffusion coefficient $\mathscr{D}_{zz}$ on the coordinate z of the atom for $\Omega > 0$ and a high Rabi frequency g $(4G \gg 1 + \Omega^2/\gamma^2)$.

points $kz = n\pi/2$ $(n = \pm 1, \pm 3, \ldots)$ (Fig. 5). But in the vicinity of these points the second part of the force (38) is always directed along the velocity of the atoms, causing, together with diffusion, the escape of the atoms from the minima of the periodic potential.

8. CONCLUSION

We note in conclusion that as the above analysis shows, the stochastic dynamics of atoms in a light field of any type is governed by the radiative force and the effect of the momentum diffusion. In the general case the radiative force and momentum diffusion tensor depend on the position and velocity of the atom, and can be represented by series containing spatially homogeneous terms and terms that oscillate in space with periods $2\pi/\Delta k_{ij}^{\alpha}$, where $\Delta k_{ij}^{\alpha} = k_i^{\alpha} - k_j^{\alpha}$; k_i and k_j being the wave vectors of the field.

In the case of a weak standing light wave the atoms can execute both finite motion at the minima of the periodic potential and free motion with periodic variation of the velocity and coordinate. If the atom was initially localized at one of the minima of the periodic potential (see Fig. 4), then after a characteristic time τ, (37), it will leave the well as a result of the diffusional heating, and go over into the state of free motion. In the case of negative detuning (i.e., for $\Omega < 0$), when the field on the average cools the atomic ensemble, the atoms can be captured again and again at the minima of the potential.

Thus, from the statistical standpoint the initial atomic ensemble is always divided into two in the field of a standing light wave. One of them is made up of cold atoms localized in the potential wells of the spatially periodic potential. The second is made up of atoms executing stochastic motion above the surface of the periodic potential. These ensembles continuously exchange atoms. On the one hand the diffusional heating of the atomic pulses leads to the escape of the atoms from the potential wells, and, on the other, the cooling of the freely moving atoms by the standing-wave field leads to the continuous capture of the atoms in the potential wells.

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