Collective hydrodynamic effects in disperse systems

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The hydrodynamic drag force that acts on a spherical particle surrounded by a large number of similar particles in a fluid flow unrestricted at infinity is calculated by the renormalization group method. Stokes and inertial (in the case of moderate Reynolds numbers) flows-around are considered for particles located at one-, two-, and three-dimensional lattice sites. It is shown that the drag force is much weaker in systems with a large number of particles because of collective screening.

The question of the hydrodynamic resisting force exerted on a particle surrounded by a large number of other particles in a flow unrestricted at infinity arises in connection with the motion of particle clouds, in the determination of the shape of macromolecular structures from viscosimetric data, etc.[1] In such situations the resisting force is weaker because of the fact that, in a collective system, the particles screen each other off, reducing the effective difference between the velocity of an individual particle and the velocity of the fluid flowing past it. The resisting force was first computed by Smoluchowskii[1] for many-particle systems in unrestricted Stokes flow. His result corresponds to the weak-interaction limit.

Let us consider a system of spheres of the same radius a. In the case of a one-dimensional disposition ($d = 1$) the centers of the spheres are located on a straight line Γ at distances l from each other. In the cases of two- and three-dimensional ($d = 2$ and 3) dispositions of the spheres the centers of the spheres occupy the sites of a square ($L \times L$, where L is the length of a side; $L \gg l$) and a simple cubic ($L \times L \times L$) lattice. We shall, for definiteness, assume that the system of spheres is immersed in a fluid flowing with velocity U in the direction perpendicular to the straight line Γ ($d = 1$); the plane in which the centers of the spheres are located ($d = 2$); and one of the faces of the cubic lattice ($d = 3$). Let us first consider the case of slow Stokes flow at a Reynolds number defined in terms of the diameter of a sphere: $\mathrm{Re} \to 0$. Let us take a group of m spheres. Let us note that in the $d = 2$ and 3 cases, as such a group, we shall take spheres whose centers occupy the sites of a $(m^{1/2}l \times m^{1/2}l)$ square and a $(m^{1/3}l \times m^{1/3}l \times m^{1/3}l)$ cubic lattice. On the group of m spheres a total force $f_m = F(m,y,f_0)$, where F is an unknown function, $y = l/a$, and $f_0 = 6\pi\mu a U$ is the force acting on an isolated moving sphere (in the present case the Stokes force; μ is the viscosity of the fluid). From dimensional considerations in the situation in question, we obtain

$$F(m, y, f_0) = f_0\varphi(m, y), \tag{1}$$

where φ is some other unknown function.

Let us associate with this group a sphere of radius a_m, such that the Stokes force $6\pi\mu a_m U$ that acts on it during its motion in the absence of other spheres is equal to f_m. Then from (1) we obtain

$$a_m = a\varphi(m, y). \tag{2}$$

We take N/m groups, similar to the first group, of spheres of radius a. Let us form from them a new group A geometrically similar to the first group, and with the same relative separation y of the spheres. In this unified group A a sphere is, on the average, acted upon by the force $F(N,y,f_0)/N$, while each of the groups (of m spheres) forming the group A is acted upon by the force

$$\psi_A = mF(N, y, f_0)/N = mf_0\varphi(N, y)/N. \tag{3}$$

We now combine the N/m spheres of radius a_m into a group B geometrically similar to the original group of m spheres of radius a. The centers of the spheres in the group B will be located at distances l_m from each other. Then a sphere (of radius a_m) in the group B will, on the average, be acted upon by the force

$$\psi_B = F(N/m, y_m, f_m)(N/m)^{-1} = mf_m\varphi(N/m, y_m)/N,$$
$$y_m = l_m'/a_m. \tag{4}$$

On the other hand, the total force acting on the group A is determined by the interaction of the N/m original groups of m spheres, and, consequently, is equal to

$$F_1\left(\frac{N}{m}, \frac{l_m}{ma}, f_m, \varepsilon_i\right) = f_m\varphi_1\left(\frac{N}{m}, \frac{l_m}{ma}, \varepsilon_i\right), \tag{5}$$
$$l_m = m^{1/d}l, \quad i = 1, 2, \ldots$$

Here F_1 and φ_1 are some unknown functions; the extent to which the flux of each of the original groups is impeded is characterized by the quantity ma and some set of dimensionless parameters ε_i of its shape, that take account of, in particular, the separation of the spheres in the group.

Consequently,

$$\psi_A = f_m\varphi_1(N/m, y/m^{1-1/d}, \varepsilon_i)(N/m)^{-1}. \tag{6}$$

Similarly, for the group B we have

$$\psi_B = f_m\varphi_1(N/m, y_m, \varepsilon_i')(N/m)^{-1}, \tag{7}$$

with ε_i' corresponding to the interaction of the spheres, so that $\varphi_1(N/m, y_m, \varepsilon_i') = \varphi(N/m, y_m)$.

Let us choose l_m' such that

$$y_m = y/m^{1-1/d}, \tag{8}$$

i.e., allowing for (2) and the expression for y_m in (4), we

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take $l_m' = l\varphi(m,y)/m^{1-1/d}$.

The basic assumption used below consists in the fact that the correction factor to the force acting on an isolated moving body (in the present case f_m), a factor which is due to the collective hydrodynamic interaction, practically does not depend on the shape of the body, and is determined only by the distances between, and the effective dimensions of, the bodies. In other words, it is assumed that the values of the function φ_1 do not depend on the magnitudes of the arguments ε_i.

That the dependence of the correction factor η on the shape parameters ε_i should indeed be quite weak is borne out by the following examples. For $y \gg 1$ and $N \gg 1$, we can, using Smoluchowskii's well-known result,[1] obtained from the correction factor to the Stokes force in the group A in the form ($d = 1$)

$$\eta_S = \frac{\varphi_S(N,y)}{N} = 1 - \frac{3}{4y}\frac{1}{N}\sum_{k=1}^{N}\sum_{i=1,i\neq k}^{N}\frac{1}{|i-k|}$$

$$= 1 - \frac{3}{2}\frac{\ln N}{y}. \qquad (9)$$

As the number N of spheres increases, η_S deviates more and more from unity, which indicates the dominant influence of the long-range interaction and, consequently, in the general case, the loss of information about the shape of outlying bodies, which impede the flux in the vicinity of the body under consideration.

The second example is connected with flows round two parallel prolate spheroids against which a Stokes flow runs along the normal to the semimajor axis. Calculations[1] show that the dependence of η on the ratio of the semiaxes of the spheroids is much weaker than the dependence on the relative distance between the two bodies.

Thus, on account of the assumption made above, we have, in accordance with (6)–(8), $\psi_A = \psi_B$, and, consequently, using (2)–(4), we arrive at the functional renormalization-group (RG) equation[2]

$$\varphi(N,y) = \varphi(m,y)\varphi\left(\frac{N}{m}, \frac{y}{m^{1-1/d}}\right), \quad \varphi(1,y) = 1. \qquad (10)$$

The functional equation (10), under the assumption that φ is a smooth function of its arguments, reduces to the differential equation

$$N\frac{\partial\varphi(N,y)}{\partial N} + \left(1 - \frac{1}{d}\right)y\frac{\partial\varphi(N,y)}{\partial y} = \varphi(N,y)\left.\frac{\partial\varphi(m,y)}{\partial m}\right|_{m=1} \qquad (11)$$

In the $d = 1$ case, using the expression for $\varphi(2,y)$ from Ref. 1, we obtain for the RG function the equation

$$\left.\frac{\partial\varphi(m,y)}{\partial m}\right|_{m=1} = \varphi(2,y) - \varphi(1,y) = 1 - \frac{3}{2y} + \frac{9}{8y^2} - \frac{59}{32y^3} + \frac{465}{128y^4} - \frac{15813}{3584y^5} = \nu(y). \qquad (12)$$

Solving (11) for the $d = 1$ case with allowance for (12), we find

$$\varphi(N, y) = N^{\nu(y)}, \quad \eta_1 = \varphi(N, y)/N = N^{\nu(y)-1}. \qquad (13)$$

In the weak-interaction limit (i.e., for $\ln N/y \ll 1$) we obtain (9) from (13). On the other hand, Eqs. (13) can be obtained directly with the aid of (10) and (9), but then the expression for $\nu(y)$ will have the same degree of accuracy as (9), i.e., will be accurate to within quantities $O(y^{-2})$.

In the two- and three-dimensional cases it is possible to determine the RG function only to within $O(y^{-2})$. Using Smoluchowskii's formula,[1] we obtain

$$\varphi(2^d, y) = 2^d(1 - k/y),$$
$$k = \frac{3}{4}\left(2 + \frac{1}{\sqrt{2}}\right) = 2.03, \quad d=2$$
$$k = \frac{3}{4}\left(4 + \frac{4}{\sqrt{2}} + \frac{4}{3\sqrt{3}}\right) = 5.7, \quad d=3 \qquad (14)$$

Hence

$$\left.\frac{\partial\varphi(m,y)}{\partial m}\right|_{m=1} = 1 - \frac{k_1}{y}, \quad k_1 = \frac{2^d k}{2^d - 1}, \qquad (15)$$

and the solution to Eq. (11) in the $d = 2$ and 3 cases will be

$$\varphi(N,y) = N\exp\left[\frac{1}{1-1/d}\frac{k_1}{y}(1 - N^{1-1/d})\right],$$
$$\eta_{2,3} = \exp\left[-\frac{1}{1-1/d}\frac{k_1}{y}(N^{1-1/d} - 1)\right]. \qquad (16)$$

As the dimensionality of the space increases, the collective effect is enhanced and the quantity η progressively decreases at fixed values of N and y.

The drag force is much weaker in systems with a large number of particles because of the collective screening.

It is of interest to obtain estimates for the case of inertial flow past a system of spheres. In the region $2\times10^4 \lesssim \mathrm{Re} \lesssim 2\times10^5$ the drag coefficient C_D for a sphere practically does not depend on the value of the Reynolds number.[3] It is reasonable to assume that the correction to the drag coefficient for a sphere in this region of Re values also does not depend on the Reynolds number. Consequently,

$$f_0 = {}^1\!/_2 C_D\rho U^2\pi a^2, \quad f_m = {}^1\!/_2 C_D\rho U^2\pi a_m^2 = f_0\varphi(m, y), \qquad (17)$$

whence

$$a_m = a[\varphi(m, y)]^{1/2} \qquad (18)$$

(ρ is the fluid density in the free stream).

The subsequent discussion is similar to the analysis for the case of slow Stokes flow, so that

$$l_m' = l[\varphi(m, y)]^{1/2}/m^{1-1/d}, \qquad (19)$$

and we again arrive at Eqs. (10) and (11).

In the case of inertial flow the expression for the RG function in the $d = 1$,2, and 3 cases apparently has the form (15) at sufficiently high values of y, but the quantity k_1 should naturally differ from the quantities given in (12) and (15). In principle, in the inertial regime the coefficients k_1 for $d = 1$, 2, and 3 can be determined experimentally, after which the results will be given, as before, by the relations (13) and (16) with the new k_1 values. In the $d = 3$ case the quantity k_1 can be estimated in the following manner. Let us consider a group of eight spheres located at the vertices of a cube in the $y = 2$ case (i.e., the spheres touch). Let us replace this group by an effective sphere with radius $a_* = 2a$ and, hence, the same volume. The drag force acting on this group will be

$$\tfrac{1}{2} C_D \rho U^2 \pi a^2 \varphi(8, a) \approx \tfrac{1}{2} C_D \rho U^2 \pi a_*^2. \qquad (20)$$

Approximating the function $\varphi(8,y)$ in the entire range of values of $y \geqslant 2$ by the expression $\varphi(8,y) = 8(1 - k/y)$, and using (20), we find that $k = 1$ and $k_1 = 8/7$. In the $d = 1$ and 2 cases similar estimates should, apparently, lead to greater errors.

By comparing the value $k_1 = 8/7$ obtained here with the value $k_1 = 6.51$ for the case of Stokes flow, we can verify that the screening of the particles in the inertial regime is weaker (and that η_3 is greater) than in creeping flow.

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[2] N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Wiley-Interscience, New York, 1980.
[3] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 1959.

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