The structure of the near zone and resonance cones of oscillating dipoles in a magnetized plasma

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We study the field of electric and magnetic dipoles of infinitesimal dimensions in the post-quasi-static approximation. We find expressions which completely describe the structure of the singularities of the field in the near zone of the source and the region of the resonance cones. We obtain the conditions for the applicability of the quasi-static approximation. In the case of a weakly gyrotropic medium we find the field at any distance from the dipoles and the resonance cones.

1. INTRODUCTION

The field of antennae in anisotropic media and particularly in a magnetized plasma has continued to be of interest already during several decades and has thus been the subject of many papers (see, e.g., Ref. 1, and also the review articles of Refs. 2, 3, and the literature cited therein). The timeliness of this problem has recently increased considerably in connection with active experimentation in the ionospheric and magnetized plasmas. It then turned out that not only the field in the wave zone of the antenna but also the field in the near zone and the resonance cones, which reach appreciable values, are important. The basic equations for the near zone and the resonance cones in an anisotropic medium were obtained in the "quasi-static" approximation in Ref. 4. Their applications to dipole and ring antennae were considered in Refs. 4–6. However, the equations of the quasi-static approximation contain only the main order singularities.

In the present paper we consider the next approximation for electric and magnetic dipoles. We consider also the important limiting case of radiation in weakly dispersive media. We start from the general equations for the electromagnetic field of the source in an anisotropic medium,[7] which we rewrite in a form convenient for our study. We then assume everywhere the plasma to be collisionless, cold, and fixed relative to the source, thus neglecting dissipation and spatial dispersion. The expressions obtained are applied further to study the fields of infinitesimal electric and magnetic dipoles which are oriented along the external magnetic field. Using an approach based upon the analysis of the behavior of the Fourier components of the Green function for large wave numbers, we find explicit formulae for the fields near the source and the resonance zones. We show that they describe completely the structure of the field singularities which occur here. From them it follows that there are added to the expression with the main singularities, which follow from the quasi-static approximation, a series of terms containing weaker singularities.[1] With increasing distance from the source and the resonance cones, the relative contribution from these terms increases and they become important on going to the wave zone. In particular, the results lead to the the necessary conditions for the applicability of the quasi-static approximation.

In the last section we consider the case of radiation in a weakly gyrotropic medium. In first approximation in the gyrotropy parameter we find expressions, which are valid in the whole of space, for the fields of electric and magnetic dipoles oriented along the external magnetic field. Comparing these expressions with the formulae for the field near the source and the resonance cones in the case of arbitrary gyrotropy parameters, we can trace the connection between the quasi-static and the wave zones and thus to obtain a qualitative idea about the structure of the transition region for dipole antennae in an anisotropic medium and, in particular, in a cold magnetoactive plasma.

From what we have said above it is clear that we ignore in the present paper a number of factors which in some specific problem or other may turn out to be important. For instance, together with the spatial dispersion we neglect all effects caused by the interaction of charged particles with the surface of the source, such as double layers and the absorption of particles by the surface. Double layers can sometimes be taken into account phenomenologically by introducing an appropriate form factor. The role of absorption often turns out to be small thanks to the special coatings applied to the antennae or to their small dimensions. In any case, this large class of problems falls outside the scope of the present paper. Finally, we neglect non-linearities, which have partly already been considered in several of the papers cited above, and to which we intend to return separately. We note that the criteria obtained here for the quasi-static approximation turn out to be useful also for the study of non-linear effects.

An extended variant of the present paper was published in the form of a preprint.[8]

2. BASIC EQUATIONS

Assuming that

$$\mathbf{j}(\mathbf{R}, t) = \text{Re}\{\mathbf{j}(\mathbf{R})e^{-i\omega t}\}, \quad \mathbf{E}(\mathbf{R}, t) = \text{Re}\{\mathbf{E}(\mathbf{R})e^{-i\omega t}\},$$

$$\mathbf{H}(\mathbf{R}, t) = \text{Re}\{\mathbf{H}(\mathbf{R})e^{-i\omega t}\},$$

we shall start from the Maxwell equations for the complex amplitudes: (2.1)

$$\hat{D}_{ik}E_k(\mathbf{R}) = \frac{4\pi i\omega}{c^2}j_i(\mathbf{R}), \quad \mathbf{H}(\mathbf{R}) = -\frac{ic}{\omega}\text{rot}\,\mathbf{E}(\mathbf{R}), \quad (2.1)$$

where

$$\hat{D}_{ik} = \frac{\partial^2}{\partial x_i\,\partial x_k} - \delta_{ik}\Delta - \frac{\omega^2}{c^2}\varepsilon_{ik}, \quad (2.2)$$

the indices i and k take on the values x, y, and z, Δ is the Laplace operator, and ε_{ik} is the dielectric permittivity tensor

whose non-vanishing components have in the "cold" plasma approximation the form (the z axis is directed along the external magnetic field)

$$\varepsilon_{xx}=\varepsilon_{yy}=\varepsilon, \quad \varepsilon_{xy}=-\varepsilon_{yx}=-ig, \quad \varepsilon_{zz}=\eta. \tag{2.3}$$

One can find, e.g., in Ref. 9, how the quantities ε, η, and g depend on the frequency and the plasma parameters.

The solution of the first Eq. (2.1) using a Fourier transformation gives[7]

$$E_i(\mathbf{R})=\omega^{-1}\hat{A}_{ik}\int G(\mathbf{R}-\mathbf{R}')j_k(\mathbf{R}')\,d\mathbf{R}', \tag{2.4}$$

where

$$\hat{A}_{ik}=\frac{1}{2}e_{imn}e_{kls}\hat{D}_{lm}\hat{D}_{sn}, \tag{2.5}$$

$$G(\mathbf{R})=-\frac{4\pi ic}{\omega}\int\frac{d\mathbf{n}}{(2\pi)^3}\frac{\exp\{i(\omega/c)\mathbf{nR}\}}{P(n,\theta)}, \tag{2.6}$$

$$P(n,\theta)=n^4(\varepsilon\sin^2\theta+\eta\cos^2\theta)-n^2[\varepsilon\eta(1+\cos^2\theta)$$
$$+(\varepsilon^2-g^2)\sin^2\theta]+\eta(\varepsilon^2-g^2). \tag{2.7}$$

Here e_{ijk} is a completely antisymmetric third rank unit pseudo-tensor, and θ is the angle between the z axis and the dimensionless "wave vector" $\mathbf{n}$. The components of the differential operator $\hat{A}_{ik}$ are explicitly written out in Ref. 8.

We give also for the function $G(\mathbf{R})$ another representation which is obtained from (2.6) after changing to cylindrical coordinates $x=r\cos\varphi$, $y=r\sin\varphi$, z and, correspondingly, $n_x=n_\perp\cos\psi$, $n_y=n_\perp\sin\psi$, $n_z=n_\parallel$. It is convenient in that case to write the function (2.7) in the form

$$P(n_\parallel,n_\perp)=\eta(n_\parallel^2-n_1^2)(n_\parallel^2-n_2^2), \tag{2.8}$$

where

$$n_{1,2}^2(n_\perp)=\left(\varepsilon-\frac{\varepsilon+\eta}{2\eta}n_\perp^2\right)\pm\left[\frac{(\varepsilon-\eta)^2}{4\eta^2}n_\perp^4-\frac{g^2}{\eta}n_\perp^2+g^2\right]^{1/2}, \tag{2.9}$$

and to integrate in (2.6) over ψ and $n_\parallel$. In the case when $n_{1,2}^2 > 0$, the function $P^{-1}(n_\parallel-n_\perp)$ has poles on the real axis. One must go around them assuming that the frequency ω has a small positive imaginary part, i.e., one must substitute $\omega\to\omega+i\delta$, $\delta\to+0$. After this the poles $\pm n_1$ and $\pm n_2$ acquire non-vanishing imaginary parts. To be specific, we assume in what follows that $n_1(n_\perp)$ and $n_2(n_\perp)$ lie in the upper half-plane. As a result we get the following expression for $G(\mathbf{R})$:[7]

$$G(\mathbf{R})=G(r,z)$$

$$=\frac{c}{\omega\eta}\int_0^\infty n_\perp\frac{J_0(\omega n_\perp r/c)}{n_1^2-n_2^2}\left(\frac{e^{i\omega n_1|z|/c}}{n_1}-\frac{e^{i\omega n_2|z|/c}}{n_2}\right)dn_\perp. \tag{2.10}$$

As the simplest examples we consider an electrical and magnetic dipole of infinitesimal size with moments which oscillate at a fixed frequency ω.

Let $\mathbf{p}(t)=e(t)\mathbf{l}=\mathrm{Re}\{\mathbf{p}e^{-i\omega t}\}(l\to0,\ p\neq0)$ be the electrical dipole moment. The current amplitude is then

$$\mathbf{j}(\mathbf{R})=-i\omega\mathbf{p}\delta(\mathbf{R}). \tag{2.11}$$

We assume that the dipole is oriented along the z axis, i.e., $p_x=p_y=0$, $p_z=p$. Equations (2.4), (2.5) then give

$$E_x=-\frac{\partial\Psi^{(p)}}{\partial x}+pg\frac{\omega^2}{c^2}\frac{\partial^2G}{\partial y\,\partial z}, \tag{2.12}$$

$$E_y=-\frac{\partial\Psi^{(p)}}{\partial y}-pg\frac{\omega^2}{c^2}\frac{\partial^2G}{\partial x\,\partial z}, \tag{2.13}$$

$$E_z=-\frac{\partial\Psi^{(p)}}{\partial z}-ip\varepsilon\frac{\omega^2}{c^2}\Delta G-ip(\varepsilon^2-g^2)\frac{\omega^4}{c^4}G, \tag{2.14}$$

where

$$\Psi^{(p)}=ip\frac{\partial}{\partial z}\left(\Delta+\frac{\omega^2}{c^2}\varepsilon\right)G. \tag{2.15}$$

Changing to the cylindrical coordinates r, φ, z we get instead of (2.12) and (2.13)

$$E_r=-\frac{\partial\Psi^{(p)}(r,z)}{\partial r}, \tag{2.16}$$

$$E_\varphi=-pg\frac{\omega^2}{c^2}\frac{\partial^2G(r,z)}{\partial r\,\partial z}. \tag{2.17}$$

These formulae together with E_z from (2.14) and $\mathbf{H}$ from (2.1) give the general expressions for the field of an electrical dipole oriented along the z-axis.

For a "point" magnetic dipole with moment $\mathbf{M}(t)=\mathrm{Re}\{\mathbf{M}e^{-i\omega t}\}$ ($M=\pi a^2I/c$, $a\to0$, $M\neq0$) we have

$$\mathbf{j}(\mathbf{R})=c\,\mathrm{rot}\{\mathbf{M}\delta(\mathbf{R})\}=-c[\mathbf{M}\times\nabla\delta(\mathbf{R})] \tag{2.18}$$

(see, e.g., Ref. 10, §29). In the simplest case when the moment $\mathbf{M}$ is parallel to the z-axis we find from (2.4) and (2.5)

$$E_x=-\frac{\partial\Psi^{(M)}}{\partial x}+\frac{\partial\Phi}{\partial y}+M\frac{\omega^3}{c^3}\eta\left(\varepsilon\frac{\partial}{\partial y}-ig\frac{\partial}{\partial x}\right)G, \tag{2.19}$$

$$E_y=-\frac{\partial\Psi^{(M)}}{\partial y}-\frac{\partial\Phi}{\partial x}-M\frac{\omega^3}{c^3}\eta\left(\varepsilon\frac{\partial}{\partial x}+ig\frac{\partial}{\partial y}\right)G, \tag{2.20}$$

$$E_z=-\frac{\partial\Psi^{(M)}}{\partial z}, \tag{2.21}$$

where

$$\Psi^{(M)}=igM\frac{\omega}{c}\Delta_\perp G, \tag{2.22}$$

$$\Phi=M\frac{\omega}{c}\left(\varepsilon\Delta_\perp+\eta\frac{\partial^2}{\partial z^2}\right)G, \tag{2.23}$$

$$\Delta_\perp=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{2.24}$$

In cylindrical coordinates Eqs. (2.19) and (2.20) give

$$E_r=-\frac{\partial\Psi^{(M)}}{\partial r}-ig\eta M\frac{\omega^3}{c^3}\frac{\partial G}{\partial r}, \tag{2.24}$$

$$E_\varphi=-\frac{\partial\Phi}{\partial r}-\varepsilon\eta M\frac{\omega^3}{c^3}\frac{\partial G}{\partial r}. \tag{2.25}$$

The corresponding expressions for other orientation directions of the electrical and magnetic dipoles are considered in Refs. 8 and 11.

3. STRUCTURE OF THE SINGULARITIES OF THE FIELD OF ELECTRICAL AND MAGNETIC DIPOLES. REGION OF APPLICABILITY OF THE QUASI-STATIC APPROXIMATION

We use the formulae obtained above to find the field near the dipoles and the resonance cones.

It is well known that resonance cones arise when the quantities ε and η (see (2.3)) in the dielectric tensor have different signs. We shall everywhere in what follows assume, to be definite, that $\varepsilon>0$ and $\eta<0$ and we introduce the parameter

$$\gamma^2 = -\varepsilon/\eta. \tag{3.1}$$

We consider the representation (2.10) for the function $G(\mathbf{R})$. We study the asymptotic form of its integrand for large wave numbers $n_\perp$. It follows from (2.9) that the roots $n_{1,2}(n_\perp)$ of the polynomial $P(n_\parallel, n_\perp)$ can be written in the form

$$n_{1,2} = \tilde{n}_{1,2}(n_\perp) + \chi_{1,2}(n_\perp), \tag{3.2}$$

where

$$\tilde{n}_1^2 = \gamma^2(\varkappa_1^2 + n_\perp^2), \quad \tilde{n}_2^2 = \varkappa_2^2 - n_\perp^2, \tag{3.3}$$

$$\varkappa_1^2 = -\eta\left[1 + \frac{\gamma^2 g^2}{\varepsilon^2(1+\gamma^2)}\right], \quad \varkappa_2^2 = \varepsilon\left[1 - \frac{\gamma^2 g^2}{\varepsilon^2(1+\gamma^2)}\right], \tag{3.4}$$

and the functions $\chi_{1,2}(n_\perp)$ for large $n_\perp$ behave as $n_\perp^{-3}$. For $\tilde{n}_{1,2}$ in (3.2) we choose those values of the square root from Eqs. (3.3) which, starting from some sufficiently large value of $n_\perp$, lie in the upper half-plane. We then assume that Re $\gamma^2 > 0$, Im $\gamma^2 > 0$; moreover, let Re $\varkappa_2^2 > 0$.

Using (3.2) we write the function $G(r,z)$ from (2.10) in the form

$$G(r,z) = \tilde{G}(r,z) + \frac{c}{\omega\eta}\int_0^\infty f(n_\perp, z) J_0\left(\frac{\omega}{c}n_\perp r\right) n_\perp\, dn_\perp, \tag{3.5}$$

where the tilde indicates that in the integrand in (2.10) we use $\tilde{n}_{1,2}$ instead of the exact values of the roots $n_{1,2}$. Using the asymptotic form of the functions $\chi_{1,2}(n_\perp)$ we can show that the function $f(n_\perp, z)$ behaves like $O(|z|n_\perp^{-6}, n_\perp^{-7})\exp(i\omega\tilde{n}_{1,2}|z|/c)$ as $n_\perp \to \infty$. Therefore, the second term in (3.5) and all its spatial derivatives up to and including the fourth are bounded. Neglecting such terms in what follows we shall thus consider only those regions of space where the fields reach appreciable values, i.e., the regions not too far from the singularities. We illustrate this approach using as an example dipoles directed along the z-axis.

We consider first of all an electrical dipole. We start from the exact expressions (2.12)–(2.17) for the field, in which we put $G(r,z) \approx \tilde{G}(r,z)$. We note first that the function $\tilde{G}(r,z)$ [like $G(r,z)$] is bounded in the whole of space. This follows from the integral representation (2.10) if we use the fact that $n_{1,2} \sim \tilde{n}_{1,2} \sim n_\perp$ as $n_\perp \to \infty$. To calculate the singular part of $\mathbf{E}$, we must thus know $\Delta\tilde{G}$, $\partial^2\tilde{G}/\partial z^2$ and $\partial^2\tilde{G}/\partial z\partial r$. Using (3.3), (3.4) we can bring the expressions for these quantities, after a few transformations, to the following form:

$$\Delta\tilde{G} = \frac{\omega}{c}\frac{\gamma^2}{\varepsilon}\int_0^\infty \frac{n_\perp}{\tilde{n}_1} J_0\left(\frac{\omega}{c}n_\perp r\right) e^{i\omega\tilde{n}_1|z|/c}\, dn_\perp - \frac{\omega^2}{c^2}\varkappa_2^2\tilde{G}, \tag{3.6}$$

$$\frac{\partial^2\tilde{G}}{\partial z^2} = \frac{\omega}{c}\frac{\gamma^2}{\varepsilon(1+\gamma^2)}\left[\gamma^2\int_0^\infty \frac{n_\perp}{\tilde{n}_1} J_0\left(\frac{\omega}{c}n_\perp r\right) e^{i\omega\tilde{n}_1|z|/c}\, dn_\perp\right.$$
$$\left. + \int_0^\infty \frac{n_\perp}{\tilde{n}_2} J_0\left(\frac{\omega}{c}n_\perp r\right) e^{i\omega\tilde{n}_2|z|/c}\, dn_\perp\right]$$
$$- \frac{\omega^2}{c^2}\frac{\gamma^2}{1+\gamma^2}(\varkappa_1^2 + \varkappa_2^2))\tilde{G}, \tag{3.7}$$

$$\frac{\partial^2\tilde{G}}{\partial z\partial r} = \frac{\gamma^2}{\varepsilon(1+\gamma^2)}\frac{\partial}{\partial z}\int_0^\infty J_1\left(\frac{\omega}{c}n_\perp r\right)$$

$$\times\left(\frac{e^{i\omega\tilde{n}_1|z|/c}}{\tilde{n}_1} - \frac{e^{i\omega\tilde{n}_2|z|/c}}{\tilde{n}_2}\right) dn_\perp$$

$$- \frac{2g^2\gamma^4}{\varepsilon^2(1+\gamma^2)^2}\frac{\partial}{\partial z}\int_0^\infty \frac{J_1(\omega n_\perp r/c)}{\tilde{n}_1^2 - \tilde{n}_2^2}\left(\frac{e^{i\omega\tilde{n}_1|z|/c}}{\tilde{n}_1} - \frac{e^{i\omega\tilde{n}_2|z|/c}}{\tilde{n}_2}\right) dn_\perp. \tag{3.8}$$

Evaluating the integrals in (3.6),(3.7) (Ref. 12) we get

$$\Delta\tilde{G} = i\frac{\gamma}{\varepsilon}\frac{e^{i\omega\varkappa_1 Q/c}}{Q} - \frac{\omega^2}{c^2}\varkappa_2^2\tilde{G}, \tag{3.9}$$

$$\frac{\partial^2\tilde{G}}{\partial z^2} = \frac{i\gamma^2}{\varepsilon(1+\gamma^2)}\left(\frac{\gamma e^{i\omega\varkappa_1 Q/c}}{Q} - \frac{e^{i\omega\varkappa_2 R/c}}{R}\right)$$
$$- \frac{\omega^2}{c^2}\frac{\gamma^2}{1+\gamma^2}(\varkappa_1^2 + \varkappa_2^2)\tilde{G}, \tag{3.10}$$

where

$$R = (r^2 + z^2)^{1/2}, \quad Q = (\gamma^2 z^2 - r^2)^{1/2}, \tag{3.11}$$

implying here and everywhere in what follows that

$$\arg Q = \begin{cases} 0 & \text{when} \quad |\gamma z| > r \\ \dfrac{\pi}{2} & \text{when} \quad |\gamma z| < r \end{cases} \tag{3.12}$$

For $\varkappa_{1,2}$ we have everywhere taken the arithmetic value of the square root of $\varkappa_{1,2}^2$. Using (3.9) we find for the potential (2.15)

$$\Psi^{(p)} \approx -p\frac{\gamma}{\varepsilon}\frac{\partial}{\partial z}\frac{e^{i\omega\varkappa_1 Q/c}}{Q} + ip\frac{\omega^2}{c^2}\frac{g^2\gamma^2}{\varepsilon(1+\gamma^2)}\frac{\partial\tilde{G}}{\partial z}. \tag{3.13}$$

Substituting (3.9) and (3.13) into (2.14) and then using (3.10) we get (apart from bounded terms)

$$E_z \approx p\frac{\gamma}{\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\varepsilon\right)\frac{e^{i\omega\varkappa_1 Q/c}}{Q}$$
$$- ip\frac{\omega^2}{c^2}\frac{g^2\gamma^2}{\varepsilon(1+\gamma^2)}\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\eta\right)\tilde{G}$$
$$\approx p\frac{\gamma^3}{\varepsilon}\left[3\gamma^2\frac{z^2}{Q^5} - \frac{1}{Q^3} + \frac{\omega^2\varkappa_1^2}{2c^2}\frac{r^2}{Q^3} + \frac{\omega^2\varepsilon}{c^2\gamma^2}\frac{1}{Q} + \frac{\omega^4\varkappa_1^4\gamma^2}{8c^4}\frac{z^2}{Q}\right]$$
$$+ p\frac{\omega^2}{c^2}\frac{g^2\gamma^4}{\varepsilon^2(1+\gamma^2)^2}\left(\frac{\gamma}{Q} - \frac{1}{R}\right) \quad r^2 = \gamma^2 z^2 - Q^2. \tag{3.14}$$

To calculate the components E_r and E_φ we consider Eq. (3.8). The second term in it is bounded in the whole of space. Neglecting this term and evaluating the first integral, using Ref. 12, we have near the singularities

$$\frac{\partial^2\tilde{G}}{\partial z\partial r} \approx \frac{c}{\omega}\frac{\gamma^2}{\varepsilon(1+\gamma^2)}\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{\gamma\varkappa_1}(e^{i\omega\gamma\varkappa_1|z|/c} - e^{i\omega\varkappa_1 Q/c})\right.$$
$$\left. - \frac{1}{\varkappa_2}(e^{i\omega\varkappa_2|z|/c} - e^{i\omega\varkappa_2 R/c})\right] \approx -\frac{i\gamma^2}{\varepsilon(1+\gamma^2)}\frac{z}{r}\left(\frac{\gamma}{Q} - \frac{1}{R}\right). \tag{3.15}$$

Substituting (3.13) and (3.15) in (2.16) and (2.17) we get

$$E_r \approx \frac{p\gamma}{\varepsilon}\frac{\partial^2}{\partial r\partial z}\frac{e^{i\omega\varkappa_1 Q/c}}{Q} - ip\frac{\omega^2}{c^2}\frac{g^2\gamma^2}{\varepsilon(1+\gamma^2)}\frac{\partial^2\tilde{G}}{\partial r\partial z}$$
$$\approx -\frac{3p\gamma^3}{\varepsilon}rz\left[\frac{1}{Q^5} + \frac{\omega^2\varkappa_1^2}{6c^2}\frac{1}{Q^3} + \frac{\omega^4\varkappa_1^4}{24c^4}\frac{1}{Q}\right]$$
$$- p\frac{\omega^2}{c^2}\frac{g^2\gamma^4}{\varepsilon^2(1+\gamma^2)^2}\frac{z}{r}\left(\frac{\gamma}{Q} - \frac{1}{R}\right), \tag{3.16}$$

$$E_\varphi \approx ip\frac{\omega^2}{c^2}\frac{g\gamma^2}{\varepsilon(1+\gamma^2)}\frac{z}{r}\left(\frac{\gamma}{Q} - \frac{1}{R}\right). \tag{3.17}$$

The first two terms in the square brackets of Eq. (3.14) and the term $\sim rz/Q^5$ in (3.16) correspond to the quasi-static approximation of Ref. 4. The additional terms obtained here take into account weaker singularities of the field at the origin and in the resonance cones. Equations (3.14), (3.16), and (3.17) thus fully describe the structure of the singularities of the field of an infinitesimal electrical dipole which is oriented along the z axis.

We now consider a magnetic dipole. Using (3.9) and (3.10) to obtain an expression for $\Delta_1 \widetilde{G}$ and substituting it into (2.22) and (2.23) we shall have

$$\Psi^{(M)} \approx -M \frac{\omega}{c} \frac{g\gamma^2}{\varepsilon(1+\gamma^2)} \left(\frac{e^{i\omega\varkappa_1 Q/c}}{\gamma Q} + \frac{e^{i\omega\varkappa_2 R/c}}{R} \right)$$
$$+ 2iM \frac{\omega^3}{c^3} \frac{g^3\gamma^2}{\varepsilon(1+\gamma^2)^2} \widetilde{G}, \tag{3.18}$$

$$\Phi \approx iM \frac{\omega}{c} \frac{e^{i\omega\varkappa_2 R/c}}{R} + M \frac{\omega^3}{c^3} \varepsilon\varkappa_1^2 \widetilde{G}. \tag{3.19}$$

The derivative $\partial\widetilde{G}/\partial r$ is bounded in the whole of space since its integrand contains a first order Bessel function that vanishes as $r \to 0$. This is clear from the expressions under the derivative sign in (3.8) and (3.15). The boundedness of $\partial\widetilde{G}/\partial z$ follows from an analysis of the convergence of the two integrals corresponding to the two terms in (2.10): the logarithmic singularities corresponding to them cancel one another. In the approximation considered we can thus neglect the second terms in (2.24), (2.25), (3.18), and (3.19). As a result we get (apart from bounded terms)

$$E_z \approx -M \frac{\omega}{c} \frac{g\gamma^2}{\varepsilon(1+\gamma^2)} z \left(\frac{\gamma}{Q^3} + \frac{1}{R^3} + \frac{\omega^2\varkappa_1^2}{2c^2} \frac{\gamma}{Q} \right), \tag{3.20}$$

$$E_r \approx M \frac{\omega}{c} \frac{g\gamma^2}{\varepsilon(1+\gamma^2)} r \left(\frac{1}{\gamma Q^3} - \frac{1}{R^3} + \frac{\omega^2\varkappa_1^2}{2c^2\gamma} \frac{1}{Q} \right), \tag{3.21}$$

$$E_\varphi \approx i \frac{\omega}{c} M \frac{r}{R^3}. \tag{3.22}$$

The first two terms in the brackets in (3.20) and (3.21) and also expression (3.22) correspond to the quasi-static approximation.[4–6] The terms proportional to $1/Q$ describe an additional weak square-root singularity of the components E_z and E_r on the resonance cone. Formulae (3.20)–(3.22) describe completely the field of an infinitesimal magnetic dipole oriented along the z axis.

We note that in principle one can also carry out the procedure for selecting the singular terms using spherical coordinates in wave number space, as was proposed in Ref. 13. However, one obtains then in the simplest manner only the quasi-static terms.[14,8] The selection of the next terms entails an analysis of complicated double integrals of unbounded rapidly oscillating functions over angles.

The results obtained here for electrical and magnetic dipoles enable us to consider the problem of the region of applicability of the quasi-static approximation. We found the solution in the form $\mathbf{E} = \mathbf{E}_{qs} + \Delta\mathbf{E}$, where $\mathbf{E}_{qs}$ is the field in the quasi-static approximation, and $\Delta\mathbf{E}$ the additional terms describing the weaker singularities of the field. Therefore, the necessary and sufficient condition for the applicability of the quasi-static approximation is

$$|\Delta\mathbf{E}| \ll |\mathbf{E}_{qs}|. \tag{3.23}$$

For fixed values of the parameters of the problem this condi-

tion can be assumed to be the definition of the quasi-static zone as a region of space.

Away from the singularities, the relative contribution of $\Delta\mathbf{E}$ increases. The region of space where

$$|\Delta\mathbf{E}| \sim |\mathbf{E}_{qs}|, \tag{3.24}$$

may be called the transition zone. Here the transition to the spatial oscillations of the field of the wave zone starts.

We give some simple conditions which are sufficient to satisfy (3.23). For an electrical dipole they have the form of a set of two equations:

$$\frac{\omega^2}{c^2} |Q|^2 \ll \varkappa_1^{-2}, \tag{3.25}$$

$$\frac{\omega^2}{c^2} R^2 \ll \min\left(\frac{\gamma}{|g|}, \frac{\varepsilon(1+\gamma^2)}{g^2\gamma} \right) \left| \frac{R}{Q} \right|^3 \left| 1 + \gamma \frac{R}{Q} \right|. \tag{3.26}$$

Indeed, when (3.25) is satisfied the main terms are those in the square brackets in Eqs. (3.14) and (3.16) which describe the field of the quasi-static zone. Furthermore, comparing these terms with the last terms in (3.14) and (3.16), respectively, we are led to two conditions the second of which turns out to be the strongest. This can be easily shown by using the inequalities $-1 \leqslant Q^2/R^2 \geqslant \gamma^2$ and $0 < \gamma^2 < \infty$. Combining this condition with the condition $|E_\varphi| \ll |E_r|$ we arrive at (3.26).

For a magnetic dipole (3.23) is satisfied when (3.25) is satisfied.

4. THE WEAK GYROTROPY CASE

When the gyrotropy parameter g is a small magnitude it turns out that it is possible to obtain for the fields explicit expressions valid in the whole of space. Indeed, it is clear from (2.5) that the function $G(\mathbf{R})$ of (2.6) contains g^2 rather than g. This enables us to write the components of the Green tensor $T_{ik} = \hat{A}_{ik} G$ to first order in g in the form

$$T_{ik} \approx T_{ik}^{(0)} + \delta T_{ik}, \tag{4.1}$$

where

$$T_{ik}^{(0)} = T_{ki}^{(0)} = \hat{A}_{ik}^{(0)} G^{(0)}, \tag{4.2}$$

and the index (0) labels quantities for $g = 0$ while the corrections δT_{ik} are, in accordance with (2.5), given by the expressions

$$\delta T_{12} = -\delta T_{21} = i \frac{\omega^2}{c^2} g \left(\Delta_\perp + \frac{\omega^2}{c^2} \eta \right) G^{(0)},$$

$$\delta T_{13} = -\delta T_{31} = i \frac{\omega^2}{c^2} g \frac{\partial^2 G^{(0)}}{\partial y\, \partial z},$$

$$\delta T_{23} = -\delta T_{32} = -i \frac{\omega^2}{c^2} g \frac{\partial^2 G^{(0)}}{\partial x\, \partial z}, \tag{4.3}$$

$$\delta T_{11} = \delta T_{22} = \delta T_{33} = 0.$$

The components $T_{ik}^{(0)}$ are evaluated explicitly in Ref. 8.

As an example we consider again electrical and magnetic dipoles oriented along the z axis. Substituting (2.11) into (2.4) and using (4.1), (4.3) and the expressions for $T_{ik}^{(0)}$ from Ref. 8 we shall have after changing to cylindrical coordinates

$$E_r \approx \frac{p\gamma}{\varepsilon} \frac{\partial^2}{\partial r\, \partial z} \frac{\exp[i\omega(-\eta)^{1/2}Q/c]}{Q},$$

$$E_\varphi \approx p\,\frac{\omega}{c}\,\frac{g\gamma^2}{\varepsilon^{3/2}(1+\gamma^2)}\,\frac{1}{r}\,\frac{\partial}{\partial z}\{\exp[i\omega(-\eta)^{1/2}Q/c]$$

$$-\exp(i\omega\varepsilon^{1/2}R/c)\},\qquad (4.4)$$

$$E_z = \frac{p\gamma}{\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c^2}\varepsilon\right)\frac{\exp[i\omega(-\eta)^{1/2}Q/c]}{Q}.$$

For the magnetic dipole (2.18) we have similarly

$$E_r \approx M\,\frac{\omega}{c}\,\frac{g\gamma^2}{\varepsilon(1+\gamma^2)}\left[\frac{\partial}{\partial r}\left\{\frac{\exp[i\omega(-\eta)^{1/2}Q/c]}{\gamma Q}\right.\right.$$

$$\left.+\frac{\exp(i\omega\varepsilon^{1/2}R/c)}{R}\right\}$$

$$-i\,\frac{\omega}{c}\,\frac{\varepsilon^{1/2}}{\gamma^2}\,\frac{1}{r}\{\exp[i\omega(-\eta)^{1/2}Q/c]-\exp(i\omega\varepsilon^{1/2}R/c)\}\,],$$

$$E_\varphi \approx -\frac{iM\omega}{c}\,\frac{\partial}{\partial r}\,\frac{\exp(i\omega\varepsilon^{1/2}R/c)}{R},\qquad (4.5)$$

$$E_z \approx M\,\frac{\omega}{c}\,\frac{g\gamma^2}{\varepsilon(1+\gamma^2)}\,\frac{\partial}{\partial z}\left\{\frac{\exp[i\omega(-\eta)^{1/2}Q/c]}{\gamma Q}\right.$$

$$\left.+\frac{\exp(i\omega\varepsilon^{1/2}R/c)}{R}\right\}.$$

For $g=0$ the results of Ref. 15 follow from (4.4) and (4.5). We note also that in the case of the electrical dipole the presence of gyrotropy destroys the shadow region ($r>|\gamma z|$), as is clear from (4.4).

Considering (4.4) and (4.5) near the singularities we obtain for the field the expression obtained in the previous section, if we neglect terms in the latter of order g^2 and, in particular, put $\varkappa_1^2 \approx -\eta$, $\varkappa_2^2 \approx \varepsilon$. On the other hand, comparing, for instance, (3.14), (3.16), (3.17) with (4.4), we see that away from the dipole and the resonance cones the terms containing singularities start to acquire on oscillating structure in space. The expressions obtained for the fields in the previous section thus give also a qualitative idea about the structure of the transition zone of dipole antennae.

We finally consider the region of applicability of the results of the present section. It follows from (2.7) and also from the general expressions for the T_{ik} that the corresponding condition has the form

$$(g/\varepsilon)^2 \ll 1.\qquad (4.6)$$

In particular, (4.6) is realized for a two-component cold plasma when

$$(\omega/\omega_{c_i})^2 \ll 1,\qquad (4.7)$$

i.e., in the band of the gyromagnetic oscillations (ω_{c_i} is the ion cyclotron frequency). As to the electron oscillations one must here first bear in mind that the conditions $\varepsilon>0$, $\eta<0$ assumed above are satisfied when

$$\max(\omega_{p_i},\,(\omega_{c_e}\omega_{c_i})^{1/2}) \ll \omega < \min(\omega_{p_e},\omega_{c_e}),\qquad (4.8)$$

where ω_{p_i} is the ion plasma frequency and ω_{p_e} and ω_{ce} are the electron plasma and cyclotron frequencies. In this range the quantity

$$\frac{g^2}{\varepsilon^2} \approx \frac{\omega_{ce}^2\omega_{p_e}^4}{\omega^2\,(\omega_{ce}^2+\omega_{p_e}^2-\omega^2)^2}$$

is small only when the additional conditions

$$(\omega_{p_e}/\omega_{c_e})^2 \ll (\omega/\omega_{p_e})^2 < 1\qquad (4.9)$$

are satisfied. Combining (4.8) with (4.9) we get finally

$$\max(\omega_{p_e}^2/\omega_{c_e}^2,\ \omega_{c_e}\omega_{c_i}/\omega_{p_e}^2) \ll (\omega/\omega_{p_e})^2 < 1.\qquad (4.10)$$

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[1] In the present paper we study the vicinities of only those singular points where the field strength tends to infinity. The term "singularity" is to be understood just in that sense.

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Translated by D. ter Haar