Beam instability regimes in a plasma

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(Submitted 3 April 1987)
Zh. Eksp. Teor. Fiz. **93**, 1714–1724 (November 1987)

In this paper we solve analytically and numerically the problem of the interaction between a magnetized electron beam and a magnetized plasma in a waveguide system of an arbitrary shape. We study the non-linear dynamics of the main regimes of the beam-plasma interaction—the single-particle Cherenkov effect and the anomalous Doppler effect—for strong and weak dispersions of the plasma waves. We determine the energy contributions of the beam to the excitation of plasma oscillations. We show that the realization of any interaction regime is easily accomplished through the choice of the densities and geometries of the beam and the plasma. This gives us the possibility to control the beam-plasma interaction for the purpose of solving many experimental and applied problems.

1. It is well known that when electron beams interact with a plasma many processes are observed which are interesting both from a practical and from a theoretical point of view. Depending on the geometric parameters of the systems of interacting particles and their densities, one observes such effects as the single-particle Cherenkov effect,[1] the anomalous Doppler effect,[1,2] or an instability in a medium with a negative permittivity.[3] The study of these processes enables us to solve the problem of controlling the beam-plasma interaction which is important for applications to microwave generation, for collective methods for modulating and accelerating beams, in the study of beam-plasma discharges, and so on. In the single important case of a beam and a plasma, which are "thin" in the transverse cross-section and completely magnetized, various regimes of the beam-plasma interaction can be studied in a waveguide in quite some detail. We note that thin beams and plasmas have recently been studied also experimentally.[4]

The electromagnetic properties of thin, completely magnetized beams and plasmas are determined in the potential approximation from the following set of non-linear equations:[5]

$$\left(\Delta_\perp+\frac{\partial^2}{\partial z^2}\right)\Phi=-4\pi e\sum_{\alpha=p,b} S_\alpha n_\alpha \delta(\mathbf{r}_\perp-\mathbf{r}_\alpha)\times\int\delta[z-z_\alpha(t,z_0)]dz_0,$$
$$\frac{d^2z_\alpha}{dt^2}=-\frac{e}{m}\frac{\partial\Phi}{\partial z}. \qquad (1)$$

Here $\Phi=\Phi(r_\perp,z,t)$ is the electrostatic potential, $r_\perp$ the coordinate in the transverse cross-section of the waveguide, $\Delta_\perp$ the transverse part of the Laplace operator, α the kind of particle ($\alpha=p$: plasma electrons, $\alpha=b$: beam electrons), S_α the area of the transverse cross-section of the system of particles of kind α, n_α their unperturbed density, $\mathbf{r}_\alpha$ the coordinates in the transverse cross-section of the waveguide, and z_α the longitudinal coordinates of particles of the kind α. The potential Φ vanishes at the metallic wall of the waveguide and, moreover, satisfies some boundary conditions with respect to the z coordinate. As such we choose these to be the periodicity conditions

$$\Phi(\mathbf{r}_\perp,0,t)=\Phi(\mathbf{r}_\perp,L,t), \qquad (2)$$

where $L=2\pi/k_\parallel$ is the period of the perturbations of the system. On the other hand, we define the initial conditions for Eqs. (1) as follows:

$$\Phi|_{t\to-\infty}=0,\quad z_\alpha|_{t\to-\infty}=z_0,$$
$$\dot z_p|_{t\to-\infty}=0,\quad \dot z_b|_{t\to-\infty}=u, \qquad (3)$$

where u is the unperturbed velocity of the beam electrons. Problem (1)–(3) clearly corresponds to the initial problem of the evolution of a perturbation which is adiabatically switched on in the infinite past. Just this problem will be considered in what follows. Our main attention will be focused on an explanation of how the interaction between the beam and the plasma depends on their geometry and density, and on finding analytical solutions of the problem.

2. We transform the set (1) to a form which is more convenient for what follows. To do this we expand the potential Φ using (2) in the following double series:

$$\Phi={}^1/_2\sum_{n=1}^{\infty}\varphi_n(\mathbf{r}_\perp)\left(\sum_{l=1}^{\infty}A_{nl}(t)\exp(ilk_\parallel z)+\text{c.c.}\right), \qquad (4)$$

where $\varphi_n(\mathbf{r}_\perp)$ are the eigenfunctions of the waveguide. Expressing the coefficients A_{nl} from the first equation of the set (1) and substituting the expansion (4) into the equation for z_α we get a set of the form

$$\frac{d^2y_b}{dt^2}+\frac{1}{2}i\Omega_b^2\sum_{l=1}^{\infty}\frac{1}{l}R_{bl}[\rho_{bl}\exp(ily_b)-\text{c.c.}]=-\frac{1}{2}i\Omega_p^2\sum_{l=1}^{\infty}\frac{1}{l}G_l\times[\rho_{pl}\exp(ily_b)-\text{c.c.}],$$
$$\frac{d^2y_p}{dt^2}+\frac{1}{2}i\Omega_p^2\sum_{l=1}^{\infty}\frac{1}{l}R_{pl}[\rho_{pl}\exp(ily_p)-\text{c.c.}]=-\frac{1}{2}i\Omega_b^2\sum_{l=1}^{\infty}\frac{1}{l}G_l[\rho_{bl}\exp(ily_p)-\text{c.c.}]. \qquad (5)$$

Here

$$\Omega_\alpha=\left(\frac{4\pi e^2n_\alpha}{m}\frac{S_\alpha}{S_W}\right)^{1/2},$$

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S_W is the area of the transverse cross-section of the waveguide,

$$R_{\alpha l}=S_W\sum_{n=1}^{\infty}\frac{l^2k_{\|}^2}{k_{\perp n}^2+l^2k_{\|}^2}\frac{\varphi_n^2(\mathbf{r}_\alpha)}{\|\varphi_n\|^2}\tag{6}$$

are geometric factors determining the frequencies of the plasma oscillations of particles of the kind α, $\|\varphi_n\|$ are the norms of the eigenfunctions, $k_{\perp n}^2$ are the transverse eigenwavenumbers of the waveguide, while

$$G_l=S_W\sum_{n=1}^{\infty}\frac{l^2k_{\|}^2}{k_{\perp n}^2+l^2k_{\|}^2}\frac{\varphi_n(\mathbf{r}_b)\varphi_n(\mathbf{r}_p)}{\|\varphi_n\|^2}\tag{7}$$

are factors determining the coupling of the beam and plasma waves, $y_\alpha=k_{\|}z_\alpha$,

$$\rho_{\alpha l}=\frac{1}{\pi}\int_0^{2\pi}\exp\{-ily_\alpha(t,y_0)\}dy_0\tag{8}$$

are the amplitudes of the Fourier components of the density perturbations of particles of the kind α (made dimensionless by the unperturbed density), and $y_0=k_{\|}z_0$. It is convenient to introduce also the following quantities:

$$\tilde{\Omega}_\alpha^2=\Omega_\alpha^2R_{\alpha 1},\qquad \tilde{G}=G_1(R_{b1}R_{p1})^{-1/2}.\tag{9}$$

The functions (6), (7), and (9) were obtained and studied in Ref.6. It is important to note $\tilde{G}\leqslant 1$, with equality reached when $\mathbf{r}_b=\mathbf{r}_p$. With increasing distance from the beam to the plasma $\tilde{G}$ decreases monotonically to zero, i.e., this quantity can be a small parameter. We give a few more results of the linear analysis of Eq. (5) which are important for what follows.

The spectra of the beam and plasma waves are determined from the dispersion equation

$$[(\omega-k_{\|}u)^2-\tilde{\Omega}_b^2](\omega^2-\tilde{\Omega}_p^2)=\tilde{G}^2\tilde{\Omega}_b^2\tilde{\Omega}_p^2,\tag{10}$$

whence one can obtain at $\tilde{G}=0$ the spectra when the beam and the plasma are not interacting. Writing down the dispersion equation in the form (10) gives a perfectly clear physical meaning to the quantities (6), (7), and (9). In the point where the plasma and the slow beam waves are in synchronism, i.e., when $k_{\|}$ satisfies the equation $\tilde{\Omega}=k_{\|}u-\tilde{\Omega}_b$, one gets easily from (10) an equation for the growth rate $\delta\omega$. In the case when $\tilde{\Omega}_b^2\ll\tilde{\Omega}_p^2$ and $\tilde{G}\sim 1$, the instability growth rate

$$\delta\omega={}^1/_2(-1+i\sqrt{3})\,({}^1/_2\tilde{G}^2\tilde{\Omega}_b^2\tilde{\Omega}_p)^{1/3}\tag{11}$$

satisfies the inequalities

$$\tilde{\Omega}_b\ll|\delta\omega|\ll\tilde{\Omega}_p.\tag{12}$$

The instability with the growth rate (11) is caused by the single-particle induced Cherenkov effect. However, if the factor $\tilde{G}$ is small, the instability growth rate

$$\delta\omega={}^1/_2i(\tilde{G}^2\tilde{\Omega}_b\tilde{\Omega}_p)^{1/2}\tag{13}$$

satisfies the inequalities

$$|\delta\omega|\ll\tilde{\Omega}_b,\ \tilde{\Omega}_p.\tag{14}$$

The instability with the growth rate (13) is caused by the collective Cherenkov effect or by the anomalous Doppler effect. In what follows we consider the case (12) and, mainly, the case (14).[1)]

We give two more integrals of Eq. (5), which reflect respectively the momentum and energy conservation laws:

$$\frac{1}{2\pi}\Omega_b^2\int_0^{2\pi}\dot{y}_b\,dy_0+\frac{1}{2\pi}\Omega_p^2\int_0^{2\pi}\dot{y}_p\,dy_0=\Omega_b^2k_{\|}u,$$

$$\Omega_b^2\mathscr{E}_b+\Omega_p^2\mathscr{E}_p+\Omega_b^2\Omega_p^2\mathscr{E}_{bp}={}^1/_2\Omega_b^2(k_{\|}u)^2.\tag{15}$$

Here

$$\Omega_\alpha^2\mathscr{E}_\alpha=\Omega_\alpha^2\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\dot{y}_\alpha^2dy_0+\frac{1}{4}\Omega_\alpha^2\sum_{l=1}^{\infty}\frac{1}{l^2}R_{\alpha l}|\rho_{\alpha l}|^2\right)\tag{16}$$

is the (kinetic + electrostatic interaction) energy of the particles of kind α, and

$$\Omega_b^2\Omega_p^2\mathscr{E}_{bp}=\frac{1}{4}\Omega_b^2\Omega_p^2\sum_{l=1}^{\infty}\frac{1}{l^2}G_l(\rho_{pl}\rho_{bl}^*+\rho_{pl}^*\rho_{bl})\tag{17}$$

is the energy of the electrostatic interaction with particles of another kind. We note that in obtaining (15) and (16) we used the initial conditions (3).

3. We start the analysis of Eq. (5) with the case of the single-particle Cherenkov effect. Under the conditions (12) we can assume that the plasma electrons are linear (see below) and change to the slow amplitudes of the plasma wave:

$$\rho_{pl}(t)=\rho_l(t)\exp(-il\tilde{\Omega}_pt),\qquad |\dot{\rho}_l|\ll l\tilde{\Omega}_p|\rho_l|.\tag{18}$$

In that case, substituting (18) into (5), performing the substitution $y_b=k_{\|}ut+y$, and using the condition for synchronism between the plasma wave and the slow beam wave we get, after linearizing with respect to the plasma electrons and changing to the dimensionless variables

$$\tau=|\delta\omega|t,\qquad \rho=i\rho_1(G_1\Omega_p^2/|\delta\omega|^2)\tag{19}$$

the following set of equations:

$$\frac{d\rho}{d\tau}=\rho_{b1}e^{-i\nu\tau},$$
$$\frac{d^2y}{d\tau^2}+\frac{1}{2}i\nu^2Q=-\frac{1}{2}(\rho e^{iy+i\nu\tau}+\text{c.c.}),\tag{20}$$

where $\nu=\tilde{\Omega}|\delta\omega|^{-1}$, while

$$Q=(\rho_{b1}e^{iy}-\text{c.c.})+\sum_{l=2}^{\infty}\frac{1}{lR_{b1}}\left(R_{bl}+\frac{G_l^2\Omega_p^2}{l^2\tilde{\Omega}_p^2-\Omega_p^2R_{pl}}\right)$$
$$\times\ (\rho_{bl}e^{ily}-\text{c.c.})\tag{21}$$

determines the force of the hf electron beam space charge. The infinite sum in (21) plays then the role of a non-linear correction to the frequency of the beam Langmuir wave. This correction arose as the result of the induced modulation of the plasma at higher harmonics and the reaction of the harmonics of the plasma wave on the beam. This non-linear effect is negligibly small. Since the parameter ν is small by virtue of the left-hand inequality (12), the set (20) reduces to well known equations describing the capture of beam electrons by the field of a plasma wave[7]:

$$\frac{d\rho}{d\tau}=\rho_{b1},\quad \frac{d^2y}{d\tau^2}=-\frac{1}{2}(\rho e^{iy}+\text{c.c.}).\tag{22}$$

It follows from a numerical analysis of Eq. (22) that the maximum value of the amplitude ρ is of the order of unity. Therefore the criterion for the linearity of the plasma $|\rho_1| \ll 1$ (we bear in mind that the quantities (8) are normalized by the unperturbed density) reduces to the inequality

$$|\delta\omega|^2 \ll G_1\Omega_p{}^2 = \tilde{G}\tilde{\Omega}_p{}^2 (R_{b1}/R_{p1})^{1/2}, \tag{23}$$

which is the same as the right-hand inequality (12). We note also that Eqs. (22), in contrast to the initial Eqs. (5), have only a single first integral. The fact is that under the conditions (12) the general integrals (15) and (16) are equivalent. Indeed, when $|\delta\omega| \ll \tilde{\Omega}_p \approx k_\parallel u$ the relative change in the beam electron velocity is small and, hence, the change in their momentum is proportional to the change in their kinetic energy. Moreover, when $\tilde{\Omega}_b \ll |\delta\omega|$ the electrostatic energies $\mathscr{E}_b$ and $\mathscr{E}_{bp}$ are small.

4. Equations (20), like (22), are correct only if the dispersion of the plasma oscillations is strong, i.e., $l^2\tilde{\Omega}_p^2 - \Omega_p^2 R_{pl} \neq 0$ (or $l^2 R_{pl} \neq R_{pl}$). In the opposite case there appears a divergence in the second equation of the set (20). The case of weak dispersion requires a separate consideration, since account must be taken of the resonance excitation of the harmonics of the plasma wave. We consider the case of weak dispersion assuming in that case that inequalities (12) are satisfied. Using the substitution (18) and changing to the dimensionless variables (19) we reduce Eqs. (5) to the following form:

$$\begin{gathered}\frac{d\rho_l}{d\tau} + i\eta_l\rho_l = l\rho_{bl},\\ \frac{d^2y}{d\tau^2} = -\frac{1}{2}\sum_l l(\rho_l e^{ily} + \text{c.c.}),\end{gathered} \tag{24}$$

where

$$\eta_l = \frac{1}{2}\frac{\tilde{\Omega}_p}{|\delta\omega|}\frac{(R_{pl} - l^2 R_{p1})}{lR_{p1}} \tag{25}$$

is the detuning from the Cherenkov synchronism and takes into account the dispersion of the plasma waves. To obtain (24) we put $G_l \approx l^2 G_1$ which is admissible for weak dispersion. The first integral of Eqs. (24)

$$\frac{1}{2\pi}\int_0^{2\pi} \dot{y}\,dy_0 + \frac{1}{4}\sum_l |\rho_l|^2 = 0 \tag{26}$$

reflects the momentum (and energy) conservation law.

Linearizing Eqs. (24) one can show that the growth rate of the lth mode of the plasma oscillations (when $|\eta_l| \ll 1$) is proportional to l, i.e., the instability spectrum has in its linear stage a tendency to shift to the high-frequency region. The harmonics for which $|\eta| \gtrsim l$ are weakly excited and it makes no sense to take them into account. We consider the results of a numerical integration of Eqs. (24).

We show in Fig. 1 the result of the solution of the equations for the case when ten harmonics of the fundamental mode ($l \leqslant 10$) are excited, under the following initial conditions: $\rho_1 = 0.01$ and $\rho_l = 0$ ($l = 2, 3, \ldots$) at $\tau = 0$ and the beam is not perturbed. In the initial stage the fundamental basic harmonic ρ_1 grows and the higher harmonics are infintesimally small and then the set (24) is equivalent to (22). However, already for $\tau \approx 2$ [long before saturation of the solutions of the single-mode Eqs. (22) are reached] higher harmonics up to the tenth one are excited by non-linearity. As the growth rate of the tenth harmonic is a maximum, it grows strongly already at $\tau \approx 3$ and reaches a saturation level, causing thereby growth of all lower harmonics. At a later stage, all harmonics are excited to one degree or other and their time dependence is very irregular. Recall that in the case of the set (22) the amplitude $|\rho|$ varies regularly after it is saturated, and the saturation itself is reached at $\tau \approx 8$ (when $\rho|_{\tau=0} = 0.01$), i.e., considerably later than in the many-mode regime.

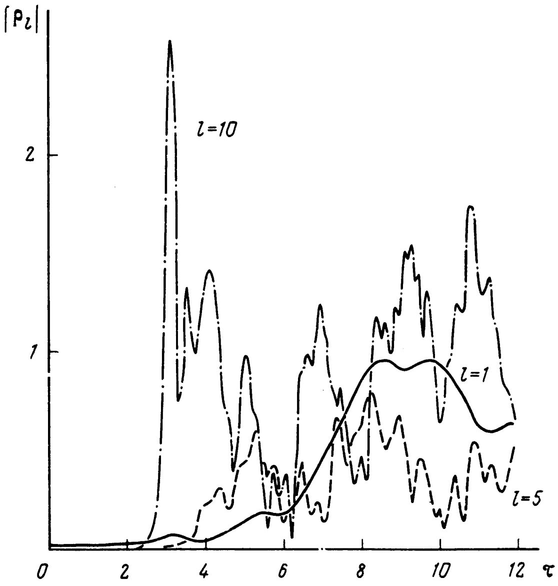

FIG. 1. Temporal evolution of the harmonics of the plasma wave $[\rho_l(\tau=0)] = 0, l = 2, 3, \ldots)$.

Figure 2 corresponds to the case of changed initial conditions: at $\tau = 0$ all ρ_l up to the tenth one are equal to 0.01. Here the tenth harmonic already grows right from the start and reaches saturation at $\tau \sim 1$. At later stages all the re-

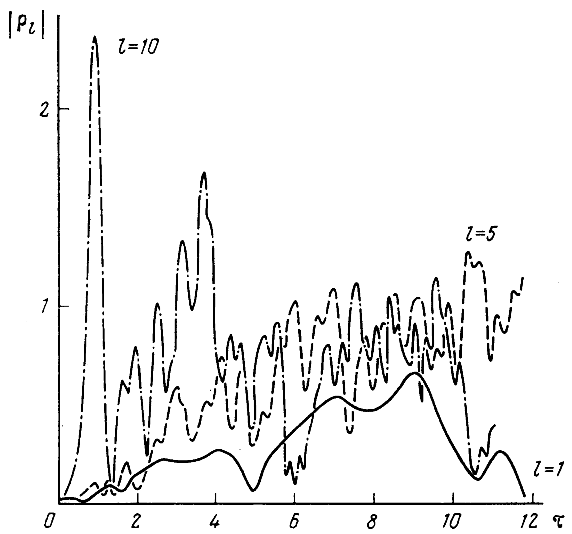

FIG. 2. Temporal evolution of the harmonics of the plasma wave $[\rho_l(\tau=0) \neq 0]$.

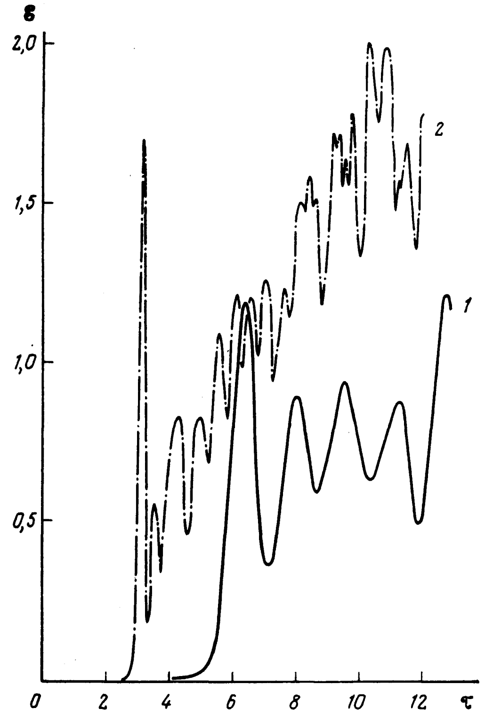

FIG. 3. Energy contribution of the beam to the plasma: 1: $l_{max}=3$; 2: $l_{max}=10$.

maining harmonics also grow, i.e., a broad spectrum of plasma oscillations is excited in the system.

When the oscillation spectrum is broadenened the energy contribution from the beam to the plasma grows. This contribution is proportional to the quantity $\mathscr{E}=\frac{1}{4}\sum_l|\rho_l|^2$ [see (26)] which is shown in Fig. 3 for the two cases $l_{max}=3$ and 10. It is clear that in the second case the energy contribution increases almost by a factor two. We note that the choice of l_{max} is not an artificial cutoff of the infinite sum over l in (24). The cutoff is automatically due to the increase of the detuning η_l when l increases. An analysis of actual spectra shows that the detuning can be small up to rather large numbers l. As to order of magnitude, the upper bound of the spectrum is determined by the equation [see (25) and (11)]

$$1-\frac{1}{l^2}\frac{R_{pl}}{R_{p1}}\approx\left(\frac{\Omega_b}{\Omega_p}\right)^{1/3}.$$

We note that for all real spectra the quantity $l^{-2}R_{pl}/R_{p1}$ changes from unity for $l=1$ to zero as $l\to\infty$.

5. We now turn to an analysis of the beam-plasma instability under the anomalous-Doppler-effect conditions when inequalities (14) are satisfied. We shall, however, now assume that the ratio of $\widetilde{\Omega}_b$ and $\widetilde{\Omega}_p$ is arbitrary. When the inequalities (14) are satisfied the beam and the plasma are equivalent to high-Q coupled oscillating systems. The interaction between these systems is efficient only if the resonance condition is very accurately satisfied. On the other hand, the latter can be violated due to a number of non-linear factors.[8,9] Among such factors we must reckon the change in the mean velocity of the particles and the dependence of the plasma frequencies on the amplitudes of the waves. If the resonance of the waves is destroyed even at a small depth of the modulation of the particles with respect to the density, it is possible to solve the problem analytically. We shall look for it by the method of "expansion in particle trajectories,"[5] the more so because Eqs. (5) are already directly suitable for using this method.

We write the solutions of Eqs. (5) in the form

$$y_\alpha=y_0+W_\alpha+{}^1/_2\sum_{l=1}^{\infty}(a_{\alpha l}e^{ily_0}+\text{c.c.}),\tag{27}$$

where W_α and $a_{\alpha l}$ are functions of the time and describe, respectively, the change in the mean velocity of the system of particles and the excitation in its field. We assume the following hierarchy of smallness of the various quantities: $a_{\alpha 1}$ is of first order of smallness, $a_{\alpha 2}$ of second order, and so on, and we restrict ourselves in considering in Eqs. (5) all non-linearities up to the cubic ones. It is then sufficient to take into account in Eqs. (5) only the first two terms in the sums over l. In such an approximation we get, after some rather complicated calculations,[5] from (5) and (8) the following:

$$\begin{aligned}
&d^2a_{b1}/dt^2+\widetilde{\Omega}_b^2a_{b1}=i\Omega_b^2(R_{b1}-R_{b2})a_{b1}^*a_{b2}\\
&+{}^1/_2\Omega_b^2(R_{b1}-R_{b2})|a_{b1}|^2a_{b1}-\Omega_p^2G_1(a_{p1}-{}^1/_2ia_{p1}^*a_{p2}\\
&-{}^1/_8|a_{p1}|^2a_{p1}-{}^1/_4|a_{b1}|^2a_{p1})\exp[i(W_b-W_p)]\\
&+{}^1/_2\Omega_p^2G_1(ia_{p1}^*a_{b2}+{}^1/_4a_{p1}^*a_{b1}^2)\exp[-i(W_b-W_p)]\\
&-\Omega_p^2G_2(ia_{p2}+{}^1/_2a_{p1}^2)a_{b1}^*\exp[2i(W_b-W_p)],\\
&d^2a_{b2}/dt^2+\Omega_b^2R_{b2}a_{b2}={}^1/_2i\Omega_b^2(R_{b2}-R_{b1})a_{b1}^2\\
&-{}^1/_2i\Omega_p^2G_1a_{b1}a_{p1}\exp[i(W_b-W_p)]\\
&+\Omega_p^2G_2({}^1/_2ia_{p1}^2-a_{p2})\exp[2i(W_b-W_p)],
\end{aligned}\tag{28}$$

$$\begin{aligned}
&d^2W_b/dt^2=-{}^1/_4i[(a_{b1}^*a_{p1}+{}^1/_2ia_{p1}a_{b1}a_{b2}^*-{}^1/_2ia_{p1}^*a_{b1}^*a_{p2}\\
&-{}^1/_8|a_{b1}|^2a_{b1}^*a_{p1}-{}^1/_8|a_{p1}|^2a_{b1}^*a_{p1})\Omega_p^2G_1\exp[i(W_b-W_p)]\\
&+{}^1/_2\Omega_p^2G_2(4a_{p2}a_{b2}^*+2ia_{p2}a_{b1}^{*2}-2ia_{p1}^2a_{b2}^*+a_{p1}^2a_{b1}^{*2})\\
&\times\ \exp[2i(W_b-W_p)]-\text{c.c.}],
\end{aligned}$$

$$\begin{aligned}
&\rho_{b1}=[-i(1-{}^1/_8|a_{b1}|^2)a_{b1}-{}^1/_2a_{b1}^*a_{b2}]\exp(-iW_b),\\
&\qquad\rho_{b2}=-(2ia_{b2}+a_{b1}^2)\exp(-2iW_b).
\end{aligned}\tag{29}$$

The equations for $a_{p1,2}$ and W_p and the expressions for $\rho_{p1,2}$ are obtained from (28) and (29) through a simple permutation of the subscripts (replacing b by p and vice versa). One can show that (28) and (29) contain without exception all non-linearities up to the cubic ones.[5] An obvious condition for the applicability of Eqs. (28) are the inequalities

$$|a_{b1}|\ll1,\quad |a_{p1}|\ll1,\tag{30}$$

the explicit form of which is given below.

We introduce in the formulae the slow wave amplitudes $\tilde{a}_{\alpha1,2}$ [slow in the sense of inequality (14)]:

$$\begin{aligned}
&a_{p1}=\tilde{a}_{p1}\exp(-i\widetilde{\Omega}_pt),\quad a_{p2}=\tilde{a}_{p2}\exp(-2i\widetilde{\Omega}_pt),\\
&a_{b1}=\tilde{a}_{b1}\exp(i\widetilde{\Omega}_bt),\quad a_{b2}=\tilde{a}_{b2}\exp(2i\widetilde{\Omega}_bt),
\end{aligned}\tag{31}$$

we make the substitution $W_b\to k_\parallel ut+W_b$, and use the synchronism condition. Moreover, we neglect in Eqs. (28) all terms containing the interaction parameter $\widetilde{G}$ in powers higher than $\frac{3}{2}$.[21] We can then, using the first-order differential equations for the amplitudes $\tilde{a}_{p1}$ and $\tilde{a}_{b1}$ obtained as the result of this procedure and also the initial conditions (3), integrate the equations for the W_α once:

$$dW_b/dt=-{}^1/_2\tilde\Omega_b|\tilde a_{b1}|^2,\quad dW_p/dt={}^1/_2\tilde\Omega_p|\tilde a_{p1}|^2, \tag{32}$$

after which one can show easily that the equations for $\tilde a_{p1}$ and $\tilde a_{b1}$ themselves have a first integral

$$\Omega_b^2\tilde\Omega_b|\tilde a_{b1}|^2-\Omega_p^2\tilde\Omega_p|\tilde a_{p1}|^2=0. \tag{33}$$

One can obtain from (32) and (33) the momentum conservation law in the form of the first Eq. (15). As to the integral (33) itself, one can obtain it also independently from the energy conservation law in the form of the second Eq. (15). We note that (33) does not have the meaning of an energy conservation law (although it is a consequence of it), but that it is a Manley-Rowe type of relation.

Introducing the new functions

$$b=\tilde a_{b1}\exp[i(W_p-W_b)],\quad p=\tilde a_{p1} \tag{34}$$

and using the integrals (32) and (33) we get the following set of equations for the amplitudes of the interacting waves:

$$\begin{gathered}dp/dt=-{}^1/_4 i\tilde\Omega_p C_p|p|^2p-{}^1/_2 i\Omega_b^2\tilde\Omega_p^{-1}G_1 b,\\ db/dt={}^1/_4 i\tilde\Omega_b C_b|b|^2b+{}^1/_2 i(\tilde\Omega_p|p|^2+\tilde\Omega_b|b|^2)b+{}^1/_2 i\Omega_p^2\tilde\Omega_b^{-1}G_1 p,\end{gathered} \tag{35}$$

where

$$C_\alpha=3(R_{\alpha1}-R_{\alpha2})/(R_{\alpha2}-4R_{\alpha1}). \tag{36}$$

The method for solving equations such as (35) with a cubic non-linearity is well known.[10] We elucidate here the meaning of the non-linear terms in Eqs. (35): the terms proportional to C_α determine the non-linear frequency shift caused by the generation of a second harmonic of the Langmuir waves of particles of the kind α (in the quasi-one-dimensional limit, when $k_\parallel\to\infty$ and $R_{\alpha1}=R_{\alpha2}$, this frequency shift does not occur, as should be the case[5,11]); the non-linear term which is independent of C_α is due to the acceleration of the plasma electrons and the retardation of the beam electrons.

Omitting the standard procedure, we write down the solutions of Eqs. (35):

$$|b|^2=\frac{|b_{max}|^2}{\operatorname{ch}(2|\delta\omega|t)},\quad |p|^2=\frac{\Omega_b^2\tilde\Omega_b}{\Omega_p^2\tilde\Omega_p}|b|^2,\quad -\infty<t<\infty. \tag{37}$$

The maximum values of the amplitudes of the beam and plasma waves are reached at $t=0$ and are given by the expression

$$|\alpha_{max}|=4\left\{\frac{|\delta\omega|}{\tilde\Omega_\alpha}\left[2+C_\alpha+\frac{\Omega_\alpha^2}{\Omega_{\alpha'}^2}(2+C_{\alpha'})\right]^{-1}\right\}^{1/2}, \tag{38}$$

in which $\alpha=b$ when $\alpha'=p$ and vice versa, while $\delta\omega$ is the growth rate determined by Eq. (13). One can see easily that the main criterion for the applicability of the results (37) and (38)—inequality (30)—reduces to (14) or, what amounts to the same, to the condition $\tilde G\ll 1$ or, more precisely, $\tilde G^2\ll 4\min\{\tilde\Omega_b/\tilde\Omega_p,\ \tilde\Omega_p/\tilde\Omega_b.\}$

The instability is thus stabilized in the anomalous Doppler effect regime by the non-linear frequency shift, as is confirmed by the numerical integration of the exact Eqs. (5). We note that one can see from (6) and (36) that the quantities $2+C_\alpha$ never vanish, i.e., the non-linear frequency shifts of different kinds do not cancel one another.

6. The results (37) and (38) are no longer applicable for weak dispersion of the beam and plasma waves, when $R_{\alpha2}=4R_{\alpha1}$. We consider the anomalous Doppler effect for this case, using the method of expansion in trajectories and restricting ourselves to the non-linear terms up to and including the third order. Putting $R_{\alpha2}=4R_{\alpha1}$, using (14) and the substitution (31) and

$$\begin{gathered}b_1=\tilde a_{p1},\quad b_2=\tilde a_{p2},\\ a_1=\tilde a_{b1}\exp[i(W_p-W_b)],\quad a_2=\tilde a_{b2}\exp[2i(W_p-W_b)],\end{gathered} \tag{39}$$

we transform (28) to the following form:

$$\begin{aligned}\frac{da_1}{dt}&=\frac{1}{2}i\tilde\Omega_b\left(1+\frac{\Omega_b^2}{\Omega_p^2}\right)(|a_1|^2+4|a_2|^2)a_1-\frac{3}{2}\tilde\Omega_b a_1^* a_2\\&\quad+\frac{3}{4}i\tilde\Omega_b|a_1|^2a_1+\frac{1}{2}i\Omega_p^2\tilde\Omega_b^{-1}G_1b_1,\\ \frac{da_2}{dt}&=i\tilde\Omega_b\left(1+\frac{\Omega_b^2}{\Omega_p^2}\right)(|a_1|^2+4|a_2|^2)a_2+\frac{3}{8}\tilde\Omega_b a_1^2,\\ \frac{db_1}{dt}&=\frac{3}{2}\tilde\Omega_p b_1^* b_2-\frac{3}{4}i\tilde\Omega_p|b_1|^2b_1-\frac{1}{2}i\Omega_b^2\tilde\Omega_p^{-1}G_1a_1,\\ \frac{db_2}{dt}&=-\frac{3}{8}\tilde\Omega_p b_1^2.\end{aligned} \tag{40}$$

Equations (40) have only one first integral

$$(\Omega_b^2\tilde\Omega_b/\Omega_p^2\tilde\Omega_p)(|a_1|^2+4|a_2|^2)-(|b_1|^2+4|b_2|^2)=0, \tag{41}$$

and can thus not be solved analytically. However, one can easily give a qualitative study of their solutions. Indeed, the time for the development of the instability is of the order $T_{inst}\sim|\delta\omega|^{-1}$. However, on the background of the general (due to the instability) growth of the wave amplitudes there proceeds an energy transfer between the harmonics of the plasma ($b_1\rightleftarrows b_2$) and beam ($a_1\rightleftarrows a_2$) waves. One can show that the characteristic transfer times[5] are waves determined for the beam and plasma, respectively by the expressions

$$T_b\sim\frac{1}{\tilde\Omega_b|a_1|_{max}},\quad T_p\sim\frac{1}{\tilde\Omega_p|b_1|_{max}}.$$

Using now (38) and the inequality (14) we get

$$\frac{T_{inst}}{T_b}\sim\frac{\tilde\Omega_b|a_1|_{max}}{|\delta\omega|}\sim\left(\frac{\tilde\Omega_b}{|\delta\omega|}\right)^{1/2}\gg 1,\quad \frac{T_{inst}}{T_p}\sim\left(\frac{\tilde\Omega_p}{|\delta\omega|}\right)^{1/2}\gg 1.$$

Hence, during the time for the development and saturation of the instability there are multiple energy transfers between the harmonics of the waves. This is also confirmed by the numerical integration of equations such as (40).[5]

It is appropriate to turn here to the single-particle Cherenkov effect, or more precisely, to Eqs. (24) and to compare their structure with Eqs. (40). It is clear that Eqs. (24) take into account the excitation of the harmonics of the plasma wave by the non-linear beam wave, but the interaction between the harmonics of the plasma wave is neglected. To justify this we estimate the ratio T_{inst}/T_p. Here, as before, $T_p\sim\tilde\Omega_p^{-1}|\rho_{p1}|_{max}^{-1}$ and $T_{inst}\sim|\delta\omega|^{-1}$. We then have, using the second Eq. (19)

$$\frac{T_{inst}}{T_p}\sim\frac{\tilde\Omega_p|\delta\omega|^2}{\Omega_p^2|\delta\omega|}\ll 1,$$

since (12) is satisfied for the single-particle Cherenkov effect. The energy transfer between the harmonics of the plasma wave is thus in the Cherenkov effect a negligibly slow process.

7. We formulate the main conclusions. The beam-plasma instability mechanism and its non-linear dynamics depend significantly on the parameters of the system. The main instability regimes studied in the present paper are: the single-particle Cherenkov effect for strong dispersion of the plasma waves; the single-particle Cherenkov effect (with resonance harmonics generation) in the weak dispersion case; the anomalous Doppler effect for strong disperion of the beam and plasma waves; the anomalous Doppler effect (with resonance interaction between the wave harmonics) in the weak dispersion case. All these effects lend themselves completely to a theoretical study, either analytically or on the basis of universal (not containing free parameters) equations such as (22) and (24). Finally, in the framework of the present paper we are not able to elucidate a whole number of specific features of the beam-plasma interaction. However, on the level of a general discussion this is hardly expedient. A detailed analysis is necessary only for a consideration of each actual system and experimental situation. It is clear from this paper that such an analysis is completely realizable at the present time, at least in the case of thin beams and plasmas.

[1)]The case $\widetilde{G}\simeq 1$ and $\widetilde{\Omega}_b \sim \widetilde{\Omega}_p$ can be studied only numerically, using, e.g., the very complicated Eqs. (5). As to the instability for $\widetilde{G}\sim 1$ and $\widetilde{\Omega}_p \ll \widetilde{\Omega}_b$, it can be discussed in the same way as in the case (12).

[2)]From Eqs. (37) and (38) below it follows that the wave amplitudes are $\propto \widetilde{G}^{1/2}$ [more precisely, they are $\propto |\delta\omega/\widetilde{\Omega}_\alpha|^{1/2}$, where $\delta\omega$ is the growth rate (13)] so that the expansion of the solutions in the small parameter $\widetilde{G}^{1/2}$ is natural and easily realizable.

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Translated by D. ter Haar