Vortex rings in excitable media

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An approximate analytic theory of vortex rings in excitable active media is developed. The main parameters, shapes, and temporal evolution of vortex rings are calculated within the framework of this semiphenomenological theory.

The study of formation and dynamics of structures in strongly chaotic media is becoming an important branch of the physics of nonlinear phenomena. The major role of autostructures in the formation of hydrodynamic turbulence is not the only reason (see Ref. 1). The onset of striations, "hot spots," and spatial lamination influences strongly the operation of distributed semiconductor systems.[2,3] It is suggested that the variegated properties of self-organizing structures can be used for information processing in a new class of microelectronic devices (see Refs. 4–6).

A special type of regular structure in two-dimensional active media is a rotating spiral wave. Similar to vortices in a superconductor or in superfluid helium, it has a topological charge, is stable, and can be treated as an elementary structure (the analog of the elementary excitation) or an active medium. In three dimensions, the counterparts of spiral waves are vortex rings (see Fig. 1). Although spiral waves and vortex rings were observed so far only in experiments with the Belousov-Zhabotinskiĭ chemical reaction,[7] and also in some biological systems,[8] even a very simple analysis shows that the same autowave structures can be found in their entirety in solid active media based on semiconductors, ferroelectrics, or magnetic superconductors.[9]

We develop here, for the first time, an approximate analytic theory that describes the main properties of vortex rings and their evolution in time. We shall show that, in contrast to spiral waves, vortex rings are unstable. Depending on the medium they either collapse or spread out with time. At large ring dimensions, however, the contraction or expansion rate is low, so that vortex rings can have long lifetimes.

The analysis set forth here is based on a kinematic description of autowave structures, previously proposed by us in Refs. 10–13 and used by us to calculate the main parameters of spiral waves in excitable media.

1. KINEMATICS OF AUTOWAVES

We consider here active media whose individual elements are related by diffusion or heat conduction. We assume that physical or chemical reactions take place in each separately considered physically small element of the medium. These reactions are accompanied by a change of the concentrations of the reacting particles and by release or absorption of heat. From the standpoint of the semiphenomenological description proposed in this paper, the nature of the reacting particles is immaterial. They can be molecules of some chemical compounds, electrons and holes in semiconductors, and so on. It is important only that definite assumptions be satisfied regarding the character of the wave processes in the media of interest. Principal among these assumptions are the ability of the medium to support propagation of solitary waves, after the passage of which the medium returns to the initial state. These are precisely the media referred to as excitable (details follow).

To describe a distributed reacting system it is necessary to specify the distributions of the concentrations of the reacting particles $x_1, x_2, \ldots, x_k$, and the temperature field T. If we formally define a vector $\mathbf{u}$ whose components are $x_1, x_2, \ldots, x_k$, and T, the most general form of the equations of such an active medium is

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}) + \hat{D}\Delta\mathbf{u}, \tag{1}$$

where $\mathbf{F}(\mathbf{u})$ are nonlinear functions describing the reactions and their thermal effects. The elements of the matrix D are the diffusion and heat-conduction coefficients.

The general equations (1) describe an extensive group of various effects. In particular, they characterize the propagation of a flame in a burning medium. This is a classical problem, investigated in detail many times (see Ref. 14). We emphasize right away that the excitable media of interest to us belong to a different class, not typical of combustion physics.

It is more convenient to present an exact definition of an excitable medium not in terms of some special assumptions concerning the specific form of the linear functions $\mathbf{F}(\mathbf{u})$ in Eqs. (1), but starting with some properties of the solutions of these equations. We regard an active medium as excitable if it meets the following three conditions:

a) There exist only one homogeneous stationary "rest" state, which is stable to small perturbation. It is always possible to define $\mathbf{u}$ such that $\mathbf{u} = 0$ in the state of rest.

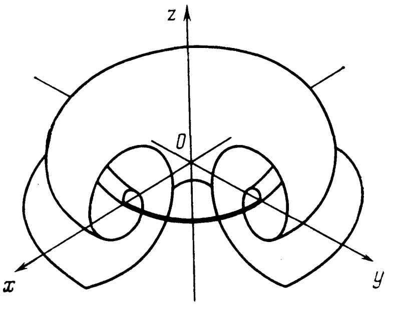

FIG. 1.

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b) There exists only one one-dimensional solution, stable to small perturbations, in the form of a solitary traveling pulse

$$\mathbf{u}=\mathbf{u}_0(\xi), \quad \xi=x-V_0t, \quad \mathbf{u}_0(\xi)\rightarrow 0, \quad \xi\rightarrow\pm\infty. \tag{2}$$

The profile of the pulse and its velocity are uniquely determined by the properties of the medium.

c) The collision of two such excitation pulses results in their annihilation, i.e., in complete mutual extinction.

Specific examples of active media in which such conditions are met are given, in particular, in Ref. 15. They include a medium with a Belousov-Zhabotinskiĭ reaction, semiconductor systems,[2–6] and also the tissue of the cardiac muscle[16] and current-carrying magnetic superconductors (note that in the last two cases Eqs. (1) have a different interpretation, but this does not change the character of the phenomena). It precisely in excitable active medium that one observes the complicated autowave regimes referred to at the beginning of the article. A direct analytic calculation of autowave regimes in excitable media is possible only in a number of very simple cases, under special simplifying assumptions concerning the forms of the nonlinear functions $\mathbf{F}(\mathbf{u})$. At the same time many numerical calculations were made for autowave processes in excitable media (see Ref. 10) and many interesting effects that require a theoretical explanation were observed. Complicated autowave structures were observed also in experiment, in a medium with the Belousov-Zhabotinskiĭ reaction.

We have previously proposed a semiphenomenological kinematic approach that permits (under certain additional assumptions concerning the properties of the medium) analytic calculation of the main parameters and of the form of the spiral waves. It makes possible also an investigation of nonstationary regime such as the establishment of the shape and of the rotation frequency of a spiral wave, resonance of spiral waves when the properties of the excitable medium are periodically varied, the drift of spiral waves in inhomogeneous media, and others. This kinematic approach is formulated in the language of wavefront dynamics and is similar to some degree to the geometric-optic approximation in the theory of linear waves.

Any semiphenomenological approach is based on description of the medium using a small number of parameters (such as the refractive index), the calculation of which is a separate problem and can be based on macroscopic equations. It is somewhat unusual that the role of "microscopic" equations in the kinematics of autowaves in excitable media is played by Eqs. (1). The Appendix contains a general procedure of calculating the kinematic parameters of the waves by using Eqs. (1), but the actual calculation of these parameters for different excitable media is not part of our present problem. Assuming also that the parameters are known, we analyze the properties and the evolution of vortex rings in the corresponding medium. Note that even in the employed phenomenological description it is possible to carry out such an analysis only by making a number of additional assumptions concerning the relations between the parameters of the excitable medium.

We examine first the construction of the kinematic description for two-dimensional excitable media. It is obvious first of all that plane solitary waves can propagate in such media. These waves are characterized by two parameters, velocity V_0 and width a of the wave, the latter meaning the dimension of this region along the propagation direction, where the medium is taken out of the quiescent state. In addition to the plane waves there should exist also bent waves. The velocity of a wave with a curved front is variable. At low curvatures this effect can be taken into account by perturbation theory (see the Appendix), and a linear dependence of the velocity on the curvature k, $V=V_0-Dk$, can be obtained. For a plane wave to be stable it is necessary that the coefficient D be positive. The dependence of the velocity of flame propagation waves on the front curvature was obtained in Ref. 17.

An excitable medium is nonlinear, and the autowaves interact with one another. Two colliding waves annihilate each other; if, however, one wave moves behind the other, it does not "feel" the wave ahead if the distance between them is considerably larger than the wave width a. We confine ourselves in this paper to regimes in which the distance between each wave and the one behind it is much larger than a. We can then neglect the width of a separate solitary wave and assume it to be completely specified by the oriented curve of its front.

Since the state of the medium before and after the passage of the excitation pulse remains the same, the front of the autowave can break off inside the medium, and the wave can have a free end. In addition to displacement in the normal direction, the free end of the wave can "expand" or shrink in the course of time. The rate of this tangential displacement C depends on the curvature k_0 of the front as it approaches the break point. Let C_0 be the rate of expansion of the free end of the half-wave. At sufficiently low curvatures k_0, the expansion rate C of the bent wave can then be calculated by perturbation theory (see the Appendix), and we get $C=C_0-\gamma k_0$. For the known examples of excitable media, the expansion rate decreases with increase of the curvature on the free end (i.e., $\gamma>0$).

Numerical simulation of processes in excitable media[18,19] shows that by varying the parameters of the medium it is possible to control the value of C_0 and even reverse its sign. Note that at a negative C_0 the medium does not admit of regimes in the form of spiral waves or vortex rings. We assume therefore that C_0 is positive. In addition, we assume that C_0 is small enough and the linear dependence of C on k is preserved up to a free-end curvature $k_{cr}=C_0/\gamma$, at which the expansion rate vanishes. We assume also that $Dk_{cr}\ll V_0$. These last two assumptions simplify greatly the analysis, in view of the appearance of an additional small parameter Dk_{cr}/V_0. They are met for weakly excitable active media with a so-called anomalous regime of circulation of the spiral wave.[20] If these conditions are not met, the kinematic description remains in force [see the remark concerning Eq. (5)], but the analytic calculation encounters a number of difficulties.

Thus, within the kinematic framework, the mathematical description of autowaves in a two-dimensional excited medium evolves as follows. We describe the wave only by specifying the curve of its front, using for this purpose the natural equation $k=k(l)$ of this curve. This equation establishes the connection between the curve-arc length l (which is best measured from the free end of the front) and the curvature of the front at the corresponding point; it is known that the natural equation specifies the curve accurate to its location on the plane. If the shape of the curve varies with time, we use $k = k(l,t)$.

An equation for the function $k(l,t)$ can be easily obtained by recognizing that: a) each section of the front is displaced with time in a direction normal to itself, at a rate

$$V = V_0 - Dk \tag{3}$$

and b) the free end of the wave is furthermore contracted ($C < 0$) or expanded in the tangential direction at a rate

$$C = \gamma(k_{cr} - k_0), \tag{4}$$

where $k_0 = \lim_{l\to 0} k(l)$. It takes the form (see Ref. 12)[1)]

$$\frac{\partial k}{\partial t} + \frac{\partial k}{\partial l}\left[\int_0^l kV\,dl + C\right] = -k^2 V - \frac{\partial^2 V}{\partial l^2}. \tag{5}$$

The stationary solution of (5) with the boundary conditions $k(0,t) = k_{cr}$ and $k \to 0$ as $l \to \infty$ yields a stationary regime in the form of a rotating spiral wave. When the condition $Dk_{cr} \ll V_0$ is met, the frequency of the stationary rotation is

$$\omega_0 = \xi (DV_0)^{1/2} k_{cr}^{3/2}, \quad \xi \approx 0.69. \tag{6}$$

The equality $k(0,t) = k_{cr}$ is always maintained on the free end of the stationary spiral wave, and this end therefore neither contracts nor expands, but moves along a circle that constitutes the boundary of the core of the spiral wave. A quiescent state is preserved inside the core (its radius is $R_0 \approx V_0/\omega_0$); the following equations hold on its boundary:

$$\left.\frac{\partial V}{\partial l}\right|_{l=0} = -D\left.\frac{\partial k}{\partial l}\right|_{l=0} = \omega_0. \tag{7}$$

The front of the spiral wave has the form of the evolvement of a circle—the core boundary, except for a small transition region (boundary layer) near the core itself, with a width on the order of

$$l_0 = \frac{1}{k_{cr}}\left(\frac{Dk_{cr}}{V_0}\right)^{1/2}. \tag{8}$$

In this narrow layer ($l_0 \ll R_0$) k is linear in l.

Equation (5) describes also the establishment of a stationary circulation of the spiral wave. In many studies of nonstationary effects, however, it is useful to resort to an additional simplification. Namely, the wave evolution can be quasistationary, so that it is possible to separate the expansion effects from the establishment of the shape of the front of the spiral wave near the core, since their time scales differ greatly [$\tau_1 = D/\gamma\omega_0$ and $\tau_2 = (k_{cr}V_0)^{-1}$], respectively. To have $\tau_1 \gg \tau_2$, it is necessary to satisfy the condition

$$\gamma/D \ll (V_0/Dk_{cr})^{1/2}, \tag{9}$$

which determines the region of the existence of the quasistationary regime (recall that V_0/Dk_{cr} is a large parameter of the problem). In this regime the form of the front at the core attunes itself adiabatically to the instantaneous value of the curvature $k(0,t)$ on the free end of the wave, and this curvature, in turn, varies slowly on account of the expansion or contraction, in accordance with the equation

$$\left.\frac{\partial k}{\partial t}\right|_{l=0} = -C\left.\frac{\partial k}{\partial l}\right|_{l=0}. \tag{10}$$

The derivative $(\partial k/\partial l)_{l=0}$ is determined by Eq. (7), in which ω_0 is obtained from (6) by replacing k_{cr} with k_0. Taking these remarks into account, the curvature on the free end varies with time and is subject to the following ordinary differential equation:

$$\dot{k}_0 = -\xi\gamma(V_0/D)^{1/2} k_0^{3/2}(k_0 - k_{cr}). \tag{11}$$

The kinematic description developed for two-dimensional excitable media admits of a generalization to include the three-dimensional case. Just as before, we assume that the autowave is completely described by specifying its oriented surface of the front. Any surface in three-dimensional space has at each of its point two principal curvature radii, R_1 and R_2. It is shown in the Appendix that the rate of the normal displacement of a section of the frontal surface depends only on the sum of the two principal curvatures, i.e., on double the average curvature $2H = 1/R_1 + 1/R_2$, in accordance with the formula

$$V = V_0 - 2DH. \tag{12}$$

On the line of its break, the frontal surface can expand or contract. The rate C of this tangential displacement has the following dependence (see the Appendix) on the average curvature H of the front as the break is approached at the given point, and on the tangential (or geodesic) curvature $\varkappa$ of the break line:

$$C = -\gamma_1(2H - k_{cr}) - \gamma_2\varkappa, \tag{13}$$

where γ_1 and γ_2 are certain positive coefficients; we take the curvature of the break line to be positive if it is convex to the front at the given point.

2. EVOLUTION OF VORTEX RING

The simplest three-dimensional autowave structure is a vortex in the form of a rotating scroll. It can be obtained by continuing the spiral wave on a plane along a straight line perpendicular to this plane. The core in the case of a right-angle scroll is played by a cylinder whose axial line is called the filament of the vortex. The vortex filament can be bent, and in particular closed to form a circle. As a result we obtain the vortex ring shown in Fig. 1. Computer calculations[21] show that, depending on the parameters of the active medium, a vortex ring can either be compressed (collapse) or expand, and more simultaneously in a direction perpendicular to the plane of the filament. The rates of these displacements are proportional to $1/R$, where R is the radius of the ring filament. Therefore, if the ring filament has a small curvature its motion is slow, and the ring remains observable for a sufficiently long time.

We use the kinematic model constructed above to investigate the evolution of a vortex ring. We choose a cylindrical coordinate frame (z,ρ,φ) with the ring axisymmetric about z. In view of the cylindrical symmetry of the problem, it suffices to consider the evolution of the line produced by the intersection of the ring with the plane $\varphi = \text{const}$, i.e., the meridian of the vortex ring (see Fig. 2). Since this is a line in a plane, the kinematic equation that describes its evolution is obviously identical with (5). In contrast, however, to the case of a spiral wave on a plane, the dependences of the velocities V and C on the curvature k of the meridian of a vortex ring is given by relations (12) and (13). The average curvature H for a vortex ring is easily calculated and is equal to

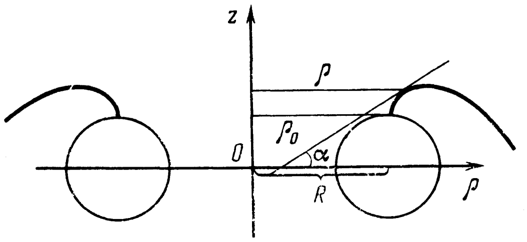

FIG. 2.

$$H=(k-\rho^{-1}\sin\alpha)/2, \tag{14}$$

while the tangential curvature of wave-front break line is of the form

$$\varkappa=-\rho_0^{-1}\cos\alpha_0, \tag{15}$$

where ρ is the distance from the z axis to the corresponding section of the surface, α is the angle between the tangent to the meridian of the ring on this section and the ρ axis (see Fig. 2), α_0 is the angle between the tangent to the ρ axis at the approach to the free end of the meridian, and ρ_0 is the distance from the z axis to the free end. From the definition of the curvature of a line it follows that[22]

$$\alpha=\alpha_0-\int_0^l k\,dl. \tag{16}$$

Since, as emphasized above, the angular velocity and the stationary form of the spiral wave are determined by its curvature near the free end, we must investigate the motion of only those sections of the vortex-ring surface which are directly adjacent to the core.

The motion of the free end of the vortex-ring meridian is defined by the following equations:

$$d\rho_0/dt=-V(0)\sin\alpha_0-C\cos\alpha_0,$$

$$dz_0/dt=V(0)\cos\alpha_0-C\sin\alpha_0. \tag{17}$$

We assume that the ring filament has small curvature, i.e., its radius R is much larger than the radius of the core. In the calculations that follow we then confine ourselves to first order in $1/R$. It follows from (12), (14), and (16) that

$$\frac{\partial V}{\partial l}=-D\frac{\partial k}{\partial l}-Dk\frac{\cos\alpha}{\rho}. \tag{18}$$

Substitution of (12), (14), and (18) in (5) leads to the equation

$$\frac{\partial}{\partial l}\left\{k\int_0^l k(V_0-Dk)\,dl-D\frac{\partial k}{\partial l}\right\}$$

$$=-\left[\frac{\partial k}{\partial t}+\left(C-D\frac{\cos\alpha_0}{R}\right)\frac{\partial k}{\partial l}\right]. \tag{19}$$

Examination of (19) shows that in the case of a vortex ring Eq. (10), which describes the quasistationary regime in a two-dimensional medium, becomes modified and takes the form

$$\frac{dk_0}{dt}=-\left(C-D\frac{\cos\alpha_0}{R}\right)\frac{\partial k}{\partial l}\bigg|_{l=0}, \tag{20}$$

with

$$\frac{d\alpha_0}{dt}=\left(\frac{\partial V}{\partial l}\right)\bigg|_{l=0}+Ck_0. \tag{21}$$

Taking (6), (7), (13)–(15), and (18) into account, Eqs. (20) and (21) can be written in the form

$$\dot{k}_0=\xi(V_0/D)^{1/2}[\gamma_1(k_{cr}-k_0+R^{-1}\sin\alpha_0)$$

$$+R^{-1}(\gamma_2-D)\cos\alpha_0]k_0^{1/2},$$

$$\dot{\alpha}_0=\xi(V_0D)^{1/2}k_0^{3/2}-Dk_0R^{-1}\cos\alpha_0+k_0[\gamma_2R^{-1}\cos\alpha_0$$

$$+\gamma_1(k_{cr}-k_0+R^{-1}\sin\alpha_0)]. \tag{22}$$

The system (22) determines completely the time dependences of α_0 and k_0 under conditions of the quasistationary regime. Solving this system in first order in $1/R$ and substituting the solution in (17), we get, averaging over the period of the circulation, an equation that describes the evolution of the vortex-ring filament:

$$\frac{dR}{dt}=\frac{3}{4}\frac{V_0}{k_{cr}}\frac{(\gamma_1/D)^2+(\gamma_2-D)/D}{(\gamma_1/D)^2+1}\frac{1}{R}-\frac{D}{R}, \tag{23}$$

$$\frac{dz_0}{dt}=-\frac{3}{4}\frac{V_0}{k_{cr}}\frac{\gamma_1}{D}\frac{(\gamma_2-D)/D-1}{(\gamma_1/D)^2+1}\frac{1}{R}$$

$$-\frac{(\gamma_2-D)}{2\xi}\left(\frac{V_0}{Dk_{cr}}\right)^{1/2}\frac{1}{R}. \tag{24}$$

We have left out of (23) terms that are small as a result of the inequality $Dk_{cr}\ll V_0$ and of the quasistationarity condition $(\gamma_1/D)\ll(V_0/Dk_{cr})^{1/2}$. For the particular case of a two-component active medium with identical diffusion coefficients, we obtain [$\gamma_2=D$, $\gamma_1=0$ (see the Appendix)]:

$$dR/dt=-D/R, \quad dz/dt=0, \tag{25}$$

which agrees with the computer calculations and with the theoretical considerations of this case [21].

3. CONCLUSION

It follows from (13) that, depending on the parameters that characterize the excitable medium, a vortex ring either contracts or expands with time. Contraction of the ring at a rate inversely proportional to the radius R of its filament means that any vortex ring in such a medium has a finite lifetime after which it vanishes without trace (this does not contradict the topological-charge conservation law, since the charge of a vortex ring is zero). It might seem at first glance that in the opposite case, when the ring tends to expand, it should turn eventually into a right cylindrical scroll. Actually, however, the situation is more complicated.

Note that although we consider in this article directly only the evolution of a circular vortex ring, the results permit a qualitative assessment of the time behavior of any ring whose filament is arbitrarily (but not too strongly) deformed. In fact, if a small section of such a vortex is considered, it can be regarded as part of a vortex ring of some appropriate angle.

Assume that we have produced a small local deformation in an initially straight vortex (a cylindrical scroll) (Fig. 3a). If the parameters of the medium are such that any vortex ring is compressed, the deformation will tend to become smaller in the course of time. In other words, the vortex filament is in this case elastic and tends to contract with time. In the opposite case, when the vortex ring expands, any "bulge" on the filament swells with time—the vortex fila-

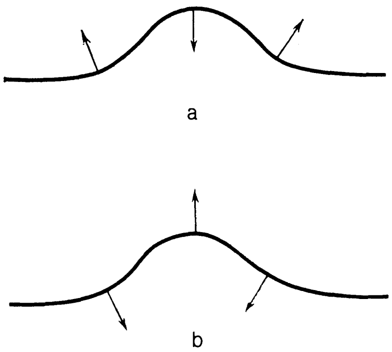

FIG. 3.

ment tends to lengthen (Fig. 3b). As a result, a straight cylindrical scroll is unstable to small deformations of its filament. This should lead to formation of very complicated (and possibly also randomly organized) structures in infinite media. This question calls for a separate attentive examination.

APPENDIX

We present a mathematical derivation of relations (3), (4), (12), and (13) on which the kinematic description is based, and demonstrate methods of calculating the parameters D, γ_1, and γ_2 in these equations, by starting directly from Eqs. (1).

1. We consider first small perturbation of a one-dimensional traveling pulse, transforming first to a coordinate frame (ξ,t) in which this pulse is at rest. Let $\mathbf{u}=\mathbf{u}_0(\xi)+\delta\mathbf{u}(\xi,t)$. Linearizing Eqs. (1) we get

$$\delta\dot{\mathbf{u}}=\hat{\Gamma}\delta\mathbf{u}, \tag{A.1}$$

where the linear differential operator $\hat{\Gamma}$ is specified by the matrix of its elements

$$(\hat{\Gamma})_{ij}=A_{ij}(\xi)+V_0\frac{\partial}{\partial\xi}\delta_{ij}+D_{ij}\frac{\partial^2}{\partial\xi^2}, \tag{A.2}$$

where

$$A_{ij}(\xi)=\left.\frac{\partial F_i}{\partial u_j}\right|_{\mathbf{u}=\mathbf{u}_0(\xi)}. \tag{A.3}$$

Since the stationary solution $\mathbf{u}=\mathbf{u}_0(\xi)$ is degenerate with respect to any shift, one of the eigenvalues λ_l of the operator $\hat{\Gamma}$,

$$\hat{\Gamma}\Phi_l=\lambda_l\Phi_l, \tag{A.4}$$

should be zero ($\lambda_0=0$) and corresponds to the shift mode $\Phi_0=\partial\mathbf{u}_0/\partial\xi$.

The linear operator $\hat{\Gamma}$ is not Hermitian, so that its eigenfunctions $\Phi_l(\xi)$ need not be orthogonal. If, however, we construct also eigenfunctions $\Phi^l(\xi)$ of the Hermitian-adjoint operator $\hat{\Gamma}^+$,

$$\hat{\Gamma}^+\Phi^l=\lambda_l^*\Phi^l, \tag{A.5}$$

it is easy to show that they will be orthogonal to $\Phi_l(\xi)$, i.e., $(\Phi^l,\Phi_{l'})=\delta_{ll'}$. We shall use this property presently.

2. Let a curved wave, but sufficiently small curvature, propagate on a plane. We consider a section of the wave on which it has a curvature k and is convex in the propagation direction. We introduce a polar coordinate frame (r,φ) with a center that coincides with the local center of curvature of this section of the front. In terms of these coordinates, Eqs. (1) take the form

$$\dot{\mathbf{u}}=\mathbf{F}+\frac{\tilde{D}}{r}\frac{\partial\mathbf{u}}{\partial r}+\tilde{D}\frac{\partial^2\mathbf{u}}{\partial r^2}. \tag{A.6}$$

We note now that the variables u change only in a region of width on the order of a (the dimension of the excitation pulse) near the front. If the curvature radius $R=1/k$ is large compared with a, we can put approximately $r=R$ in the second term of the right-hand side of (A.6) and obtain

$$\dot{\mathbf{u}}=\mathbf{F}(\mathbf{u})+\tilde{D}\frac{\partial^2\mathbf{u}}{\partial r^2}+\tilde{D}k\frac{\partial\mathbf{u}}{\partial r}. \tag{A.7}$$

Assuming the curvature k to be small, we seek the solution of (A.7) in the form

$$\mathbf{u}(r,t)=\mathbf{u}_0(\xi)+\delta\mathbf{u}(\xi,t),\quad \xi=r-Vt \tag{A.8}$$

with a velocity V still unknown, and require only that the expansion of δu in terms of the eigenfunctions Φ_l contain no shift mode (see Ref. 23). This condition yields

$$V=V_0-Dk, \tag{A.9}$$

where the coefficient D is defined as

$$D=(\Phi^0,\hat{D}\Phi_0)=\sum_{ij}D_{ij}\int\Phi_i^0(\xi)\Phi_{0j}(\xi)d\xi. \tag{A.10}$$

If all the diffusion coefficients are the same and there is no cross diffusion ($D_{ij}=\tilde{D}\delta_{ij}$), then $D=\tilde{D}$, since $(\Phi^0,\Phi_0)=1$.

3. Assume the existence of a self-similar solution in the form of a plane wave with a break; the wave moves along the normal at constant velocity V_0 and expands tangentially at a rate C_0:

$$\mathbf{u}=\mathbf{u}_1(\xi,\eta),\quad \xi=x-V_0t,\quad \eta=y-C_0t,$$

$$\mathbf{u}_1(\xi,\eta)\to 0 \text{ as } \xi\to\pm\infty,\quad \mathbf{u}_1(\xi,\eta)\to\text{const as }\eta\to+\infty,$$

$$\mathbf{u}_1(\xi,\eta)\to 0 \text{ as } \eta\to-\infty. \tag{A.11}$$

Consider small perturbations of this self-similar solution: $\mathbf{u}=\mathbf{u}_1(\xi,\eta)+\delta\mathbf{u}(\xi,\eta,t)$. Linearizing Eqs. (1) with respect to the perturbation, we get

$$\delta\dot{\mathbf{u}}=\hat{\Lambda}\delta\mathbf{u}. \tag{A.12}$$

The linear operator $\hat{\Lambda}$ is of the form

$$(\hat{\Lambda}_{ij})=B_{ij}(\xi,\eta)+V_0\frac{\partial}{\partial\xi}+C_0\frac{\partial}{\partial\eta}+D_{ij}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right), \tag{A.13}$$

where

$$B_{ij}=\left.\frac{\partial F_i}{\partial u_j}\right|_{u=u_1(\xi,\eta)} \tag{A.14}$$

Since the self-similar solution is degenerate with respect to horizontal and vertical shifts ($\xi\to\xi+\text{const}$, $\eta\to\eta+\text{const}$), the operator $\hat{\Lambda}$ should have a twofold degenerate zero eigenvalue corresponding to two shift modes:

$$\Psi_{01}=\frac{\partial \mathbf{u}_1}{\partial \xi}, \quad \Psi_{02}=\frac{\partial \mathbf{u}_1}{\partial \eta}. \qquad (A.15)$$

Since the operator $\hat{\Lambda}$ is not Hermitian, its eigenfunctions (including Ψ_{01} and Ψ_{02}), need not be orthogonal. We consider, however, the Hermitian-adjoint operator $\hat{\Lambda}^+$. Its eigenvalues are the complex-conjugate eigenvalues of the operator $\hat{\Lambda}$,

$$\hat{\Lambda}^+\Psi^l=\lambda_l^*\Psi^l. \qquad (A.16)$$

The eigenvalue $\lambda_0=0$ is twofold degenerate. In the corresponding invariant subspace it is always possible to choose two vectors Ψ_1^0 and Ψ_2^0 such that they are orthogonal to all the eigenvectors Ψ_l of the operator $\hat{\Lambda}$, including Ψ_{01} and Ψ_{02}, i.e., so as to satisfy the conditions

$$(\Psi_1^0, \Psi_l)=\delta_{0l}, \quad (\Psi_2^0, \Psi_l)=\delta_{0l}. \qquad (A.17)$$

4. Assume that on approaching the free end the wave front has a small curvature and is convex in the direction of motion. By the same reasoning as used to derive Eq. (A.9) we can show then that, with account taken of the curvature, the normal-displacement velocity V and the tangential "expansion" rate C on the free end are equal to

$$V=V_0-(\Psi_1^0, D\Psi_{01})k_0, \qquad (A.18)$$

$$C=C_0-(\Psi_2^0, D\Psi_{01})k_0. \qquad (A.19)$$

The coefficient γ in the equation $C=C_0-\gamma k_0$ is thus given by

$$\gamma=\sum_{ij} D_{ij}\iint \Psi_{2i}^0(\xi,\eta)\Psi_{01j}(\xi,\eta)\,d\xi\,d\eta. \qquad (A.20)$$

Note that if all the diffusion coefficients are equal and there is no crossing diffusion ($D_{ij}=D\delta_{ij}$), then $\gamma=0$ and $V=V_0-Dk_0$.

5. We consider now a three-dimensional excitable medium. An arbitrary surface is characterized at each of its point by two curvatures k_1 and k_2 or by their combinations, the Gaussian $G=k_1k_2$ and average $H=(k_1+k_2)/2$ surface curvatures. The flat continuous front, for which $k_1=k_2=0$, moves obviously with the same velocity V_0 as a single pulse along a straight line. If the surface curvatures k_1 and k_2 are small, their influence on the velocity can be taken into account by perturbation theory. When a section of the surface has the form of a right cylinder ($k_1\neq 0$, $k_2=0$), it moves along the normal to itself at the same velocity $V=V_0-Dk$ as the section of the front on the surface with curvature k_1. Simple reasoning shows that in the first (linear) perturbation-theory approximation the effects of the two curvatures k_1 and k_2 are additive. This means that in the general case, when the section of the surface is characterized by two nonzero principal curvatures, its velocity along the normal is[2)]

$$V=V_0-D(k_1+k_2)=V_0-2DH. \qquad (A.21)$$

6. Consider now the motion of the break of a surface in a three-dimensional excitable medium. A section of a surface with a break is characterized by three parameters: two principal curvatures k_1 and k_2 and the geodesic curvature $\varkappa$ of the break line. If all these curvatures are small compared with the reciprocal width $1/a$ of the pulse their effect can be treated by perturbation theory. It can be shown that in the linear approximation the contributions from k_1, k_2, and $\varkappa$ to the normal displacement velocity V and to the tangential rate of expansion on the break are additive and independent.[3)] Thus, the general case can be obtained by considering beforehand the motion of a straight break of a cylindrical surface ($k_1\neq 0$, $k_2=\varkappa=0$) and the expansion of a plane-surface section in the form of a circle ($k_1=k_2=0$, $\varkappa\neq 0$). As a result we get

$$C=-\gamma(2H-k_{cr})-\gamma_2\varkappa, \qquad (A.22)$$

$$V=V_0-2\alpha H-\beta\varkappa, \qquad (A.23)$$

where

$$\gamma_2=(\Psi_2^0, \hat{D}\Psi_{02})=\sum_{ij} D_{ij}\iint \Psi_{2i}^0(\xi,\eta)\Psi_{02j}(\xi,\eta)\,d\xi\,d\eta, \qquad (A.24)$$

$$\alpha=(\Psi_1^0, \hat{D}\Psi_{01}), \quad \beta=(\Psi_1^0, \hat{D}\Psi_{02}), \qquad (A.25)$$

and γ is given by (A.20).

Note that if the diffusion coefficients of all components are equal and there is no cross diffusion ($D_{ij}=\tilde{D}\delta_{ij}$) we have

$$\beta=\tilde{D}(\Psi_1^0, \Psi_{02})=0 \text{ and } \gamma_2=\tilde{D}(\Psi_2^0, \Psi_{02})=\tilde{D}.$$

7. It does not follow from the foregoing analysis that the normal velocity of the front section near the break should be equal to the velocity of the normal displacement of the section of the continuous front [see Eqs. (A.21) and (A.23)]. Only at $D_{ij}=\tilde{D}\delta_{ij}$ do the coefficients α and D coincide ($\alpha=D=\tilde{D}$), and the coefficient β vanishes). This means in practice that, as a rule, the sections near the break will be displaced somewhat more slowly along the normal than the sections of a continuous front, and therefore even an initially planar circular front with a break will be deformed with time (and initiate a vortex ring).

This effect can be taken into account within the framework of the kinematic approach developed above. Its inclusion, however, makes the calculations much more cumbersome without changing significantly the final results. Since the coefficients D, γ, γ_2, α and β should be regarded in the kinematics equations as independent prescribed parameters, we have confined ourselves in the present article to the case when $\alpha=D$ and $\beta=0$. This simplifying assumption does not mean, for example, that no spiral wave can be created from a break of a straight wave. It is easy to verify that even under this assumption the self-similar solution (A.11) in the form of a straight half-wave is absolutely unstable in the kinematic treatment: any small deformation near the free end grows with time and we reach ultimately a stationary regime in the form of a stationarily rotating spiral wave.

Note added in proof (6 October 1987): The results of computer experiments performed by us most recently on a model of an excitable medium show that compression of a vortical ring may also not cause it to vanish. It is possible to choose the excitable-medium parameters such that at a sufficiently small radius of the vortex ring its compression is stopped and it is only shifted in a direction to the plane of the filament. This effect is observed, for example, in a two-component model of type (1) proposed in Ref. 10, for the following coefficient values: $k_f=1.7$, $k_g=2$, $a=0.1$, $\sigma=0.1$,

$\varepsilon = 0.18$, $k_\varepsilon = 6$. The existence of vortex ring of constant size does not contradict the foregoing kinematic-analysis result, which were obtained for large values of R. However, the formation of stable vortex rings with small radius R, having a long lifetime, calls for a special analysis which is outside the scope of the present paper.

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[1)]We emphasize that in the derivation of Eq. (5) the dependences of V and C on k, l, and t were not specified and could be arbitrary.

[2)]We note in particular that the velocity V does not depend on the angle between the directions of the two principal curvatures. Such a dependence is an effect of the next order of smallness in k_1 and k_2.

[3)]Consequently, these quantities cannot depend on the angles between the break line and the directions of the principal curvatures.

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