Shear distortions of the structure of cholesterics

V. G. Kamenskiĭ and E. I. Kats

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR
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An analysis is made of the behavior of a cholesteric liquid crystal under the influence of shear deformation. It is shown that such deformation creates structural distortions in a cholesteric. A study is made of the time dependence of such distortions. The results are compared with experiments reported by Barbero *et al.* [Z. Naturforsch. Teil A **39**, 1195 (1984)] and by Scaramuzza *et al.* [Phys. Rev. A **32**, 1134 (1985)].

The behavior of liquid crystals under the influence of shear deformation is of interest because a relatively simple experimental technique makes it possible to determine a number of parameters of liquid crystal systems and to provide some information on the mechanisms of interaction between the structure and flows.

Relatively recently it has been found[1,2] that brief shear deformation of a cholesteric liquid crystal with planar ordering creates a stable distortion of the plane structure of cholesteric layers. This effect is interpreted on the assumption that shear induces an inhomogeneous (over the thickness of a sample) inclination of the director relative to the axis of the helix (in an unperturbed cholesteric the director is orthogonal to this axis). This hypothesis was checked independently by measurements of the positions of Bragg reflection maxima in a somewhat different experimental situation.[3]

It follows therefore that in the case of a cholesteric (in contrast to layer liquid crystals of the smectic type), in which layers may glide relative to one another, there is a definite correlation between the layers (although the shear modulus is naturally zero). In fact, if relative shear occurs between layers in smectics, there are no changes in the macroscopic state and in this sense we can assume that the layers in smectics are uncorrelated. However, the experimental results reported in Refs. 1–3 show that the situation in cholesterics is different. Shear creates an inhomogeneous distribution of the director, which means a change in the macroscopic state of a cholesteric, i.e., there is a correlation between layers exactly as in solids. Therefore, a cholesteric can be regarded as the most "solid" among liquid crystals.

The description provided in Ref. 2 applies to an already established deformed state of a cholesteric a fairly long time after a shear perturbation. However, the process of establishment of such a state is equally interesting. Moreover, although the explanation proposed in Ref. 2 is correct, the choice of the boundary conditions and equations of motion made in Ref. 2 is unsuitable for obtaining correct quantitative results. These are the topics which will be addressed in the present paper.

We consider a cholesteric liquid crystal with planar ordering. In the unperturbed state the axis of the helix is perpendicular to the surfaces bounding the liquid crystal. We choose this direction to be the z axis. In an unperturbed state the components of the director are:

$$n_x^0=\cos q_0 z,\quad n_y^0=\sin q_0 z,\quad n_z^0=0, \tag{1}$$

where $q_0 = 2\pi/p_0$ and p_0 is the equilibrium pitch of the helix.

For simplicity we assume that the size of a sample d in the direction of the z axis is a multiple of the equilibrium pitch of the helix. We shall see later that this hypothesis does not affect significantly the results but determine only the form of the boundary conditions.

We also postulate that a shear perturbation is applied[1,2] to the upper plate bounding a sample for a fairly brief time interval t_0. During this interval the plate travels a certain distance S along the x axis. This shear distorts the distribution of the director given by Eq. (1), which then becomes

$$n_x=\cos\varphi\cos\psi,\quad n_y=\sin\varphi\cos\psi,\quad n_z=\sin\psi. \tag{2}$$

The establishment of a distorted state of Eq. (2) is described by the equations of hydrodynamics of cholesterics which—on the assumption that a liquid crystal is incompressible and thermostatted (which is practically always true)—are as follows:

$$\frac{\partial\psi}{\partial t}+v_x\frac{\partial\psi}{\partial x}+v_z\frac{\partial\psi}{\partial z}=\frac{h_z}{\gamma_1\cos\psi}-\frac{1+\nu}{2}\left(\sin\varphi\frac{\partial v_y}{\partial z}+\cos\varphi\frac{\partial v_x}{\partial z}\right)$$
$$-\nu\,\mathrm{tg}\,\psi\frac{\partial v_z}{\partial z}+\frac{1-\nu}{2}\cos\varphi\frac{\partial v_z}{\partial x}, \tag{3}$$

$$\frac{\partial\varphi}{\partial t}+v_x\frac{\partial\varphi}{\partial x}+v_z\frac{\partial\varphi}{\partial z}$$
$$=\frac{h_y\cos\varphi-h_x\sin\varphi}{\gamma_1\cos\psi}+\frac{1-\nu}{2}\mathrm{tg}\,\psi\left(\cos\varphi\frac{\partial v_y}{\partial z}\right.$$
$$\left.-\sin\varphi\frac{\partial v_x}{\partial z}\right)-\frac{1}{2}(1-\nu\cos 2\varphi)\frac{\partial v_y}{\partial x}$$
$$+\frac{\nu}{2}\sin 2\varphi\frac{\partial v_x}{\partial x}+\frac{1+\nu}{2}\sin\varphi\,\mathrm{tg}\,\psi\frac{\partial v_z}{\partial x}, \tag{4}$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0, \tag{5}$$

$$\rho\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial\sigma_{ji}}{\partial x_j}+\frac{\partial\Sigma_{ji}}{\partial x_j}, \tag{6}$$

where P is the pressure; Σ_{ji} is the viscosity tensor; $\nu=\gamma_2/\gamma_1$; γ_1 and γ_2 are the viscosity coefficients; σ_{ji} is the reactive stress tensor

$$\sigma_{ji}=-\frac{\partial n_k}{\partial x_i}\frac{\partial F}{\partial(\partial n_k/\partial x_j)};\qquad h_i=\frac{\partial}{\partial x_k}\left(\frac{\partial F}{\partial(\partial n_i/\partial x_k)}\right)-\frac{\partial F}{\partial n_i};$$

F is the density of the free energy of a cholesteric:

$$F={}^1/_2K_1(\mathrm{div}\,\mathbf{n})^2+{}^1/_2K_2(\mathbf{n}\,\mathrm{rot}\,\mathbf{n}+q_0)^2+{}^1/_2K_3[\mathbf{n}\,\mathrm{rot}\,\mathbf{n}]^2. \tag{7}$$

In Eqs. (3)–(5) we have allowed for the geometry of the experiments, which postulates homogeneity of the sys-

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tem in the y direction and, consequently, the absence of the corresponding derivatives. On the other hand, the complete equations of motion for the velocities (6) are very cumbersome, so that we shall consider only their general form.

Obviously, it would be very difficult to obtain a consistent and complete solution of the system of equations (3)–(6), so that we shall make several simplifying assumptions. First of all, we shall postulate that the quantities S and t_0 are such that the deviations of the angles ψ and φ from the initial equilibrium state are small (such an assumption is in full agreement with the experimental situation in Ref. 2). Moreover, using estimates of the characteristic times in Eqs. (3), (4), and (6), we shall assume that at the moment of completion of the motion of the upper plate a steady-state distribution of the velocities and angles is established in the system. This assumption makes it possible to bypass the description of the initial stage of formation of distortions in a cholesteric structure and to obtain directly the form of the structure at the moment when the shear deformation is completed (or at least to obtain estimates of the maximum attainable characteristic quantities). Obviously, a steady-state pattern, because of its homogeneity along the x axis, contains only a dependence on z. It follows from the equation of continuity and the boundary conditions that the velocity is $v_z = 0$. We can find the quantities φ, ψ, v_x, and v_y of interest to us from the remaining four equations (the equation for v_z contains the pressure P which is of no interest to us) and these are of the form:

$$\alpha_3\left(\cos\varphi\frac{\partial v_x}{\partial z}+\sin\varphi\frac{\partial v_y}{\partial z}\right)$$

$$=K_1\frac{\partial^2\psi}{\partial z^2}+2(K_3-K_1)\psi\left(\frac{\partial\psi}{\partial z}\right)^2-K_3\psi\left(\frac{\partial\varphi}{\partial z}\right)^2, \qquad (8)$$

$$\alpha_2\left(\cos\varphi\frac{\partial v_y}{\partial z}-\sin\varphi\frac{\partial v_x}{\partial z}\right)\psi$$

$$=K_2\frac{\partial^2\varphi}{\partial z^2}+2K_2q_0\psi\frac{\partial\psi}{\partial z}+2(K_3-2K_2)\psi\frac{\partial\psi}{\partial z}\frac{\partial\varphi}{\partial z}, \qquad (9)$$

$$\frac{\partial}{\partial z}\left[(\alpha_4+\mu\cos^2\varphi)\frac{\partial v_x}{\partial z}+\frac{\mu}{2}\sin 2\varphi\frac{\partial v_y}{\partial z}\right]=0, \qquad (10)$$

$$\frac{\partial}{\partial z}\left[\frac{\mu}{2}\sin 2\varphi\frac{\partial v_x}{\partial z}+(\alpha_4+\mu\sin^2\varphi)\frac{\partial v_y}{\partial z}\right]=0, \qquad (11)$$

where $\mu = \alpha_3 + \alpha_6$, and α_i are the corresponding Leslie coefficients. In deriving Eqs. (8)–(11) we used the condition $\psi \ll 1$ and also omitted from Eqs. (10) and (11) the contributions proportional to α_1, because usually $\alpha_1 \ll \mu$.

We can easily see that Eqs. (10) and (11) yield

$$\frac{\partial v_x}{\partial z}=\frac{C(\alpha_4+\mu\sin^2\varphi)-D\mu\sin\varphi\cos\varphi}{\alpha_4(\alpha_4+\mu)}, \qquad (12)$$

$$\frac{\partial v_y}{\partial z}=\frac{D(\alpha_4+\mu\cos^2\varphi)-C\mu\sin\varphi\cos\varphi}{\alpha_4(\alpha_4+\mu)}, \qquad (13)$$

where C and D are constants whose meaning becomes clear if we assume that the boundary conditions imposed on the angle φ in the form $\varphi(0) = \varphi(d) = 0$ are satisfied (this denotes a rigid anchoring of the director to the orienting surfaces relative to rotation in the xy plane). We then have

$$\left.\frac{\partial v_x}{\partial z}\right|_{z=0,d}=\frac{C}{\alpha_4+\mu},\quad \left.\frac{\partial v_y}{\partial z}\right|_{z=0,d}=\frac{D}{\alpha_4}.$$

An analysis of Eqs. (12) and (13) shows that the constant D should be identically equal to zero, because otherwise the velocity v_y does not vanish on the upper plate, which is in conflict with the formulation of the problem. Therefore, the velocities v_x and v_y have finally periodic components which depend on the pitch of the helix. Moreover, v_x has a component linear in z and unrelated to the presence of a liquid-crystal structure. Consequently, in the case of v_x and v_y we obtain the following expressions:

$$v_x=v_x^0[z-\mu\sin 2q_0z/2q_0(2\alpha_4+\mu)]/d, \qquad (14)$$

$$v_y=-v_x^0\mu\sin^2 q_0z/dq_0(2\alpha_4+\mu), \qquad (15)$$

where $v_x^0 = S/t_0$ is the velocity of motion of the upper plate (it is reported in Ref. 2 that this velocity is $v_x^0 \sim 0.1$ cm/sec). Substituting now Eqs. (14) and (15) into Eqs. (8) and (9) and assuming that $\tilde{\varphi}$, $\psi \ll 1$ ($\tilde{\varphi} = \varphi - q_0z$) and also that $\tilde{\varphi} \ll q_0z$ (which is confirmed by the results of calculations), we obtain the following expressions which are valid in the leading approximation:

$$K_1\frac{\partial^2\psi}{\partial z^2}-K_3q_0^2\psi=\frac{2\alpha_3\alpha_4v_x^0}{(2\alpha_4+\mu)d}\cos q_0z, \qquad (16)$$

$$K_2\frac{\partial^2\tilde{\varphi}}{\partial z^2}+2(K_3-K_2)q_0\psi\frac{\partial\psi}{\partial z}=-\psi\frac{2\alpha_2(\alpha_4+\mu)v_x^0}{(2\alpha_4+\mu)d}\sin q_0z. \qquad (17)$$

Since we have gone over directly to the established steady state, we have to consider particularly the problem of the boundary conditions imposed on the variable ψ. If $\psi = 0$ is fixed rigidly at the boundaries (this is the limit of strong anchoring), the solution of Eq. (16) becomes

$$\psi=a\left[\cos q_0z-\frac{\operatorname{ch}\lambda(z-d/2)}{\operatorname{ch}(\lambda d/2)}\right],$$

where

$$a=-2\alpha_3\alpha_4v_x^0/(2\alpha_4+\mu)dq_0^2(K_1+K_3),$$

$$\lambda=q_0(K_3/K_1)^{1/2}\equiv q_0\gamma.$$

We can show that in this case there is no wave-like modulation in a cholesteric. Therefore, the very existence of a quasistationary wave-like modulation is an indication that in the course of shift the angle at the boundary should become $\psi \neq 0$. This in turn means that the experimental conditions of Refs. 1 and 2 correspond to the limit of weak anchoring or that in the range of small values of ψ of importance to us the anchoring energy depends weakly on the value of ψ.[1)] Then, under steady-state conditions, the value of ψ at the boundary is found simply by equating the elastic and viscous torques:

$$K_1\partial\psi/\partial z\approx\alpha_3v_x \quad \text{at} \quad z=0,\ d. \qquad (18)$$

In Ref. 2 the boundary conditions are as follows: a strong anchoring at $z = 0$ ($\psi = 0$) and a weak anchoring of Eq. (18) at $z = d$. Moreover, we shall ignore the contribution of the periodic velocity in the interior. Such a situation corresponds best to the intermediate stage of establishment of an equilibrium state, when an equilibrium of the torques already exists at the upper plate, but the entire perturbation is still concentrated near the upper boundary. Equation (16) subject to these boundary conditions gives

$$\psi=b\frac{\operatorname{sh}\lambda z}{\operatorname{ch}\lambda d}+a\left[\cos q_0z-\frac{\operatorname{ch}\lambda(z-d)}{\operatorname{ch}\lambda d}\right],$$

where $b=\alpha_3 v_x^0/q_0(K_1K_3)^{1/2}$. Since the parameters of the problem are such that $b \gg a$, this result is practically identical with that obtained in Ref. 2.

We shall finally consider the case when the boundary conditions at the upper and lower plates stipulate equality of the torques. This situation corresponds either to a complete absence of anchoring to the surface for a variable value of ψ or to long times for the establishment of a steady state and of such equality. Then, the angle

$$\psi=b\,\mathrm{ch}\,\lambda z/\mathrm{sh}\,\lambda d+a\cos q_0 z \tag{19}$$

while small at the lower boundary, does not vanish identically [the behavior of ψ near the upper plate ($z\simeq d$) is qualitatively similar to the preceding case, if we bear in mind that the experimental conditions indicate that $\lambda d \gg 1$]. We shall in future consider the situation described by Eq. (19).

Substituting Eq. (19) into Eq. (17), and using the boundary conditions $\tilde{\varphi}(0)=\tilde{\varphi}(d)=0$, we obtain

$$\tilde{\varphi}=\frac{z}{d}\left[\frac{\varepsilon b^2}{2\gamma K_2}\mathrm{cth}\,\lambda d-\frac{2\gamma bc}{(\gamma^2+1)^2}\right]-\frac{\varepsilon b^2}{4\gamma K_2}\frac{\mathrm{sh}\,2\lambda z}{\mathrm{sh}^2\lambda d}+\frac{ac}{8}\sin 2q_0 z+\frac{(1-\gamma^2)bc}{(1+\gamma^2)^2}\sin q_0 z\frac{\mathrm{ch}\,\lambda z}{\mathrm{sh}\,\lambda d}+\frac{2\gamma bc}{(1+\gamma^2)^2}\cos q_0 z\frac{\mathrm{sh}\,\lambda z}{\mathrm{sh}\,\lambda d}, \tag{20}$$

where

$$\varepsilon=K_3-K_2,\quad c=2\alpha_2(\alpha_4+\mu)v_x^0/(2\alpha_4+\mu)dq_0^2K_2.$$

It is clear from Eq. (20) that the corrections to the dependence $\varphi(z)=q_0 z$ are very small and, therefore, the assumptions made in the derivation of Eqs. (16) and (17) are fully justified.

The next stage in our analysis is a study of the behavior of the system after stoppage of the upper plate. Before considering details, we must make one comment. The viscous moment disappears after stoppage of the plate. However, in the case of weak anchoring a steady-state distribution of ψ is still in equilibrium. This is due to the fact that the structure of a cholesteric is invariant to two transformations commuting with one another: translations along the axis of the helix and rotations about this axis. According to the Noether theorem, it follows that there should be two integrals of variational equations first obtained in Ref. 5. These integrals are of the form

$$\mathscr{P}=\frac{\partial\psi}{\partial z}\frac{\partial F}{\partial(\partial\psi/\partial z)}+\frac{\partial\varphi}{\partial z}\frac{\partial F}{\partial(\partial\varphi/\partial z)}-F=\mathrm{const},\qquad M=\frac{\partial F}{\partial(\partial\varphi/\partial z)}=\mathrm{const}, \tag{21}$$

where $\mathscr{P}$ represents the "pressure" of the helix and M is the moment of the force. We can see from Eq. (21) that the distribution of ψ near the boundary obeying the law $d\psi/dz=\pm\lambda\psi$ is in equilibrium. This is the condition satisfied by our steady-state distribution subject to the boundary condition ψ_0 governed by the equality of the torques.

We shall now consider in greater detail the processes which occur in the system after the stoppage of the upper plate. The equations of motion (with the necessary precision) are

$$\gamma_1\frac{\partial\tilde{\varphi}}{\partial t}=K_2\frac{\partial^2\tilde{\varphi}}{\partial z^2}+2\varepsilon q_0\psi\frac{\partial\psi}{\partial z}-\alpha_2\psi\left(\cos\varphi\frac{\partial v_y}{\partial z}-\sin\varphi\frac{\partial v_x}{\partial z}\right), \tag{22}$$

$$\gamma_1\frac{\partial\psi}{\partial t}=K_1\frac{\partial^2\psi}{\partial z^2}-K_3q_0^2\psi-\alpha_3\left(\cos\varphi\frac{\partial v_x}{\partial z}+\sin\varphi\frac{\partial v_y}{\partial z}\right), \tag{23}$$

$$\rho\frac{\partial v_x}{\partial t}=\frac{\partial}{\partial z}\left[-\alpha_2\sin\varphi\frac{\partial\tilde{\varphi}}{\partial t}\psi+\alpha_3\cos\varphi\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial v_x}{\partial z}\times(\alpha_4+\mu\cos^2\varphi)+\frac{\mu}{4}\frac{\partial v_y}{\partial z}\sin 2\varphi\right], \tag{24}$$

$$\rho\frac{\partial v_y}{\partial t}=\frac{\partial}{\partial z}\left[\alpha_2\psi\cos\varphi\frac{\partial\tilde{\varphi}}{\partial t}+\alpha_3\sin\varphi\frac{\partial\psi}{\partial t}+\frac{\mu}{4}\frac{\partial v_x}{\partial z}\sin 2\varphi+\frac{1}{2}\frac{\partial v_y}{\partial z}(\alpha_4+\mu\sin^2\varphi)\right]. \tag{25}$$

Omitting from the equations for the velocities the terms proportional to $\delta\psi/\delta t$ and $\delta\tilde{\varphi}/\delta t$ (further calculations confirm the validity of this approximation), introducing an auxiliary function $w=v_x+iv_y$, and adopting dimensionless variables $\xi=\pi z/d$ and $\tau=t\,|\mu|\pi^2/4\rho d^2$, we obtain the following equation for w:

$$\frac{\partial w}{\partial\tau}=\frac{\partial}{\partial\xi}\left[L\frac{\partial w}{\partial\xi}-\exp(4iN\xi)\frac{\partial\bar{w}}{\partial\xi}\right], \tag{26}$$

where $L=(2\alpha_4+\mu)/|\mu|$; $N=d/p_0=q_0 d/2\pi$ is the number of itches of the helix; the bar marks a complex conjugate.

Since the problem has two characteristic scales (d and p_0), it is natural to seek the solution of Eq. (26) in the form $w=\varphi+f\exp(4iN\xi)$, where

$$\Phi=\Phi_0+N^{-1}\Phi_1+N^{-2}\Phi_2+\dots,\quad f=N^{-1}f_0+N^{-2}f_1+\dots, \tag{27}$$

and the characteristic scale of changes in the functions φ_i and f_i is d. Substitution of Eq. (27) into Eq. (26) and the subsequent solution of the resultant system of equations makes it possible to determine w with any precision in respect of the parameter N^{-1}.

Retaining only the first two terms of the expansion, we finally obtain

$$w=\Phi_0(\xi,\tau)+i(4LN)^{-1}[1-\exp(4iN\xi)]\partial\Phi_0/\partial\xi, \tag{28}$$

where the function $\varphi_0(\xi,\tau)$ should satisfy the heat conduction equation

$$\partial\Phi_0/\partial\tau=A\partial^2\Phi_0/\partial\xi^2,\quad A=L-L^{-1}=4\alpha_4(\alpha_4+\mu)/|\mu|(2\alpha_4+\mu). \tag{29}$$

The boundary and initial conditions for φ_0 are easily found if, using Eq. (28) and the definition of the function w, we write down the expressions for the velocities:

$$v_x(\xi,\tau)=\Phi_0+(4LN)^{-1}\sin(4N\xi)\partial\Phi_0/\partial\xi,\qquad v_y(\xi,\tau)=(2LN)^{-1}\sin^2(2N\xi)\partial\Phi_0/\partial\xi, \tag{30}$$

and compare them with the steady-state values given by Eqs. (14) and (15). Consequently, finding of the function $\varphi_0(\xi,\tau)$ reduces to solution of Eq. (29) subject to

$$\Phi_0(\xi,0)=v_x^0\xi/\pi,\quad \Phi_0(0,\tau)=0,\quad \Phi_0(\pi,\tau)=v_x(\pi,\tau).$$

It should also be noted that the very nature of the initial and boundary conditions ensures that $\varphi_0(\xi,\tau)$ is real.

Since we do not know the time dependence of the change in the velocity of the upper plate, we consider two model cases. We assume first that the plate stops instantaneously, i.e., that $v_x(\pi,\tau)=0$. When this condition is obeyed, we have

$$\Phi_0(\xi,\tau)=\frac{2v_x^0}{\pi}\sum_{n=1}^{\infty}\exp(-n^2A\tau)(-1)^{n-1}n^{-1}\sin n\xi. \quad (31)$$

Obviously, in this case the characteristic time for the vanishing of $\varphi_0(\xi,\tau)$ [and, consequently, on the basis of Eq. (30) the characteristic time of the vanishing of v_x and v_y] is $\tau_{\rm ch}^{(1)}\propto A^{-1}$. Using the dimensional quantities, we find that

$$t_{\rm ch}^{(1)}\sim(2\alpha_4+\mu)\rho d^2/\alpha_4(\alpha_4+\mu)\pi^2\sim10^{-5}\ \text{sec}.$$

If we postulate that the plate is decelerated in accordance with a linear law $v_x(\pi,\tau)=v_x^0[1-\tau/\tau_0]$ at $\tau\leqslant\tau_0$, then

$$\Phi_0(\xi,\tau)=\begin{cases}\dfrac{2v_x^0}{\pi}\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1}\sin n\xi\left[1-\frac{\tau}{\tau_0}+\frac{1-\exp(-n^2A\tau)}{n^2A\tau_0}\right], & \tau\leqslant\tau_0,\\[2ex] \dfrac{2v_x^0}{\pi\tau_0A}\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}n^{-3}[\exp(n^2A\tau_0)-1]\exp(-n^2A\tau)\sin n\xi, & \tau>\tau_0.\end{cases} \quad (32)$$

We can easily see that if $A\tau_0\ll1$, Eq. (32) reduces to Eq. (31). If $A\tau_0\gg1$, then at times $\tau\sim A^{-1}$ we have $\varphi_0(\xi,\tau)\sim\varphi_0(\xi,0)$ and the characteristic time for the relaxation of the function φ_0 is then $\tau_{\rm ch}^{(2)}\sim\tau_0\gg A^{-1}$. However, bearing in mind that the total time during which the plate moves is 10^{-2} sec in the experiments, it is reasonable to adopt an estimate $\tau_{\rm ch}^{(2)}\sim\tau_0\leqslant10^{-3}$–$10^{-4}$ sec.

We shall now find the time dependence of ψ. Differentiating Eq. (23) with respect to z and introducing $\mu=\partial\psi/\partial z$ we obtain the following equation for the function u in terms of dimensionless variables ξ and τ:

$$\frac{\partial u}{\partial\tau}=B\frac{\partial^2u}{\partial\xi^2}-Ru+Q\left[\Phi_0'\sin2N\xi-\frac{2\alpha_4-\mu}{4\alpha_4N}\times\Phi_0''\cos2N\xi+\frac{\mu}{8\alpha_4N^2}\Phi_0'''\sin2N\xi\right]. \quad (33)$$

We shall first consider the solution of Eq. (33) subject to the boundary conditions

$$u(0,\tau)=0,\quad u(\pi,\tau)=\alpha_3K_1^{-1}v_x(\pi,\tau),$$

and the initial conditions

$$u(\xi,0)=\gamma q_0b\,\mathrm{sh}(2\gamma N\xi)/\mathrm{sh}(2\gamma N\pi)-aq_0\sin2N\xi.$$

A function φ_0 occurring in Eq. (33) is governed by the conditions of stopping of the plate and it is described either by Eq. (31) or Eq. (32). The constants B, R, and Q are determined by the parameters of the problem and are described by the expressions

$$B=4\rho K_1/\gamma_1|\mu|,\quad R=16\rho K_3N^2/\gamma_1|\mu|,$$
$$Q=16\rho\alpha_3\alpha_4N/\gamma_1|\mu|(2\alpha_4+\mu).$$

In the case of instantaneous stopping of the plate, when φ_0 is described by Eq. (31) and we have $\mu(\pi,\tau)=0$, we obtain from Eq. (33)

$$\psi=\frac{\alpha_3dv_x^0}{\pi^2K_1}\sum_{n=1}^{\infty}(-1)^n\cos n\xi\left(2\theta^{-1}\exp(-\theta B\tau)+\left(1-\frac{n}{2LN}\right)\times\frac{\exp[-(2N+n)^2A\tau]-\exp(-\theta B\tau)}{\theta-G(2N+n)^2}+\left(1+\frac{n}{2LN}\right)\times\frac{\exp[-(2N-n)^2A\tau]-\exp(-\theta B\tau)}{\theta-G(2N-n)^2}\right\}+a\cos2N\xi, \quad (34)$$

where $\theta\equiv\theta(n,N)=n^2+4\gamma^2N^2$ and $G=A/B$. If $\tau=0$, this expression reduces exactly to Eq. (19).

An analysis of Eq. (34) shows that if

$$N^2\leqslant G=\alpha_4(\alpha_4+\mu)\gamma_1/(2\alpha_4+\mu)\rho K_1\sim10^6-10^7$$

we can ignore the last two terms of the sum for any value $n\neq2N$. Therefore, for characteristic velocity decay times $\tau_{\rm ch}^{(1)}\sim A^{-1}$ the value of ψ differs very little from the initial steady-state value. In the experiments reported in Ref. 2 we have $N\approx150$ and, therefore, the above condition is satisfied. The characteristic decay time of ψ is $\tau_{\rm ch}^{(3)}\sim(4\gamma^2N^2B)^{-1}$ ($t_{\rm ch}^{(3)}\sim\gamma_1/K_3q_0^2\sim10^{-4}$ sec).

If the stoppage of the plate is linear, then under the same assumptions about the relationships between the quantities N, A, and B in the case when $\tau>\tau_0$ the value of ψ is

$$\psi\approx a\cos2N\xi+\frac{4\alpha_3dv_x^0}{\pi^2K_1}\sum_{n=1}^{\infty}(-1)^n\theta^{-1}\cos n\xi\times\exp(-\theta B\tau)\left[\frac{\exp(-\theta B\tau_0)-1}{\theta B\tau_0}-\frac{1}{2}\right]. \quad (35)$$

It is clear from this formula that for the characteristic velocity decay time $\tau_{\rm ch}^{(2)}\sim\tau_0$ the arguments of the exponential functions are $\sim\theta(n,N)G^{-1}A\tau_0$.

Since G is large and the estimated experimental value of $A\tau_0$ does not exceed 10^1–10^2, it is clear that at least up to harmonics $n\sim10^2$–10^3 the exponential functions differ little from unity and, consequently, ψ differs little from its steady-state value. The characteristic decay time of ψ is $\tau_{\rm ch}^{(3)}$, exactly as in the preceding case.

Using the values of v_x, v_y, and ψ found in this way, we shall now determine the time dependence of $\tilde\varphi$ from Eq. (22). We shall consider only the case of instantaneous stoppage of the plate (because generalization to the linear law is self-evident) and for $\tau\sim\tau_{\rm ch}^{(3)}$ we then obtain

$$\tilde\varphi=\frac{4bc}{\pi(1+\gamma^2)^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{\pi nz}{d}\right)\exp(-n^2W\tau)\times\left[1-\frac{n^4}{\theta(2N+n)\theta(2N-n)}-\frac{8\alpha_3\alpha_4\varepsilon(1+\gamma^2)N^2}{\alpha_2(\alpha_4+\mu)K_1(16\gamma^2N^2+n^2)}\right]+4ac\sin2q_0z\exp(-16N^2W\tau), \quad (36)$$

where $W = 4\rho K_2/\gamma_1|\mu|$.

The above expression is derived bearing in mind that $\alpha_3 \ll \alpha_2$ and that the ratio $W/B = K_2/K_1$ is several times less than unity for the majority of known liquid crystals. It is clear from Eq. (36) that the characteristic decay time of $\tilde{\varphi}$ is $\tau_{\rm ch}^{(4)} \sim W^{-1}$ ($t_{\rm ch}^4 \sim 10$ sec).

We shall now consider the case when after stoppage of the plate a steady-state value is retained at the upper boundary $\psi = \psi_{\rm st}$ and we have $(\partial\psi/\partial z)_{z=d} = \lambda\psi_{\rm st}$. In the case of instantaneous stoppage of the plate in a time $\tau > \tau_{\rm ch}^{(1)}$, we have

$$\psi = \frac{2\alpha_3\, dv_x^0}{N\pi(K_1K_3)^{1/2}}\left(\operatorname{cth}\lambda d + \frac{a}{b}\right)\frac{\operatorname{ch}\lambda z}{\operatorname{sh}\lambda d} + \frac{2\alpha_3\, dv_x^0}{\pi^2 K_1 N}\left(1 - \operatorname{cth}\lambda d - \frac{a}{b}\right) \times \sum_{n=1}^{\infty}\frac{(-1)^n\cos(\pi nz/d)\exp(-\theta B\tau)}{n^2+4\gamma^2N^2} + a\cos q_0 z\exp[-4(1+\gamma^2)N^2B\tau]. \quad (37)$$

Since a/b is small, the first term in the above expression hardly differs from the first term of Eq. (19) for the steady-state case. The second term contains a small coefficient, because $\lambda d \gg 1$. At times $\tau > \tau_{\rm ch}^{(3)}$ the last two terms are exponentially small. This means that during such times the angle ψ does not contain the dependence on the scale p_0, remains constant, and is close to the steady-state value $\psi_{\rm st}$.

The time dependence of $\tilde{\varphi}$ is then described by

$$\tilde{\varphi} = \frac{\varepsilon b^2}{2\gamma K_2}\left(\frac{z}{d} - \frac{\operatorname{sh}2\lambda z}{\operatorname{sh}2\lambda d}\right) + 4ac\sin 2q_0 z\exp(-16N^2\tau/\tau_{\rm ch}^{(4)}) + \frac{4bc}{\pi(1+\gamma^2)^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{\pi nz}{d}\right)\exp\left(-\frac{n^2\tau}{\tau_{\rm ch}^{(4)}}\right) \times\left[1 - \frac{n^4}{\theta(2N+n)\theta(2N-n)} - \frac{8\alpha_3\alpha_4\varepsilon(1+\gamma^2)N^2}{\alpha_2(\alpha_4+\mu)K_1(16\gamma^2N^2+n^2)}\right] - \frac{16ab\varepsilon\gamma}{\pi K_2}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(\pi nz/d)N^2[n^2+4(\gamma^2+1)N^2]}{\theta(2N+n)\theta(2N-n)[n^2-4(\gamma^2+1)N^2K_1/K_2]} \times\exp[-\theta(2N)B\tau]. \quad (38)$$

If $\tau \gg \tau_{\rm ch}^{(3)}$, we can drop the last term from the above expression and for $\tau > \tau_{\rm ch}^{(4)}$ the value of $\tilde{\varphi}$ is altogether independent of time and of the scale p_0.

Conceptually simple, but fairly time-consuming analysis of Eqs. (24) and (25) carried out using the expressions obtained for ψ, $\tilde{\varphi}$, v_x, and v_y shows, to the same accuracy as that in the derivation of the function w in Eq. (28), that the above hypothesis of the smallest of the terms proportional to $\partial\psi/\partial t$ and $\partial\tilde{\varphi}/\partial t$ is justified.

We have thus shown that after stoppage of the plate in a sufficiently short time $\sim\tau_{\rm ch}^{(1)}$ or $\sim\tau_{\rm ch}^{(2)}$, the velocities of motion v_x and v_y vanish in the principal order. Then, the values of ψ and $\tilde{\varphi}$ are still practically equal to the steady-state values $\psi_{\rm st}$ and $\tilde{\varphi}_{\rm st}$ and remain such right up to times $\sim\tau_{\rm ch}^{(3)}$ and $\tau_{\rm ch}^{(4)}$ or even longer.

An inhomogeneous distribution of the angle ψ across the layer thickness can be described as the result of the action of some effective field, which is known[6] to create a wave-like modulation of the cholesteric layers. Such a deformation has indeed been observed[2] at times of the order of 10 sec.

The development of a wave-like modulation can be described by solving the system of equations (3)–(6) using the principal nonlinear terms and the initial conditions $v_x = v_y = 0$, $\psi = \psi_{\rm st}$, and $\tilde{\varphi} = \tilde{\varphi}_{\rm st}$. It is necessary to allow for the dependences of the variables on the coordinates x and z. Clearly, this is a very difficult task. However, since we are not interested in the nature of modulation over a small scaling length p_0, the problem can be reduced to an analogous problem for a smectic A (Ref. 6) by a large-scale approximation. The dynamics of appearance of a wave-like modulation in the case of a smectic A was discussed in detail in a paper by one of the present authors[7] and the results solve the problem formulated above if simple transformations of the parameters are adopted. The appearance of a wave-like modulation has a threshold and it depends on the value of ψ (i.e., on v_x^0). However, just above the threshold the characteristic time for the establishment of a modulated structure is long and can be considerably greater than $\tau_{\rm ch}^{(3)}$. In this case a modulated structure does not develop. Bearing this fact in mind, we conclude that the experiments of Ref. 2 are better described by the quasisteady case of Eq. (37).

Since the results of our analysis provide a complete interpretation of the experimental data of Ref. 2 and not only for the stage asymptotic in time, we shall consider one other feature of shear distortion of a cholesteric.

An inhomogeneous tilt of the director relative to the axis of the helix shifts the selective reflection by a cholesteric and corresponds to an effective reduction in the pitch of the helix, which can be estimated from[2]

$$\delta p/p_0 \approx -\varepsilon_a\psi^2/4\bar{\varepsilon}. \quad (39)$$

Here, δp is the change in the pitch because of the tilt of the director, ε_a is the permittivity anisotropy, and $\bar{\varepsilon}$ is the average permittivity.

In addition to this apparent reduction in the pitch of the helix [Eq. (39)], there is a real increase in the pitch described by Eq. (38). If $\tau > \tau_{\rm ch}^{(3)}$, then

$$\frac{\delta p}{p_0} \approx -\frac{1}{zq_0}\left[\frac{b^2(K_3-K_2)}{2\gamma K_2}\left(\frac{z}{d} - \frac{\operatorname{sh}2\lambda z}{\operatorname{sh}2\lambda d}\right) - \frac{4bc}{\pi(1+\gamma^2)^2}\sin\frac{\pi z}{d}\exp\left(-\frac{\tau}{\tau_{\rm ch}^{(4)}}\right)\right].$$

Inversion of the sign of the change in the pitch of the helix occurs at times

$$\tau_i \sim \tau_{\rm ch}^{(4)}\ln\left|\frac{\alpha_3(2\alpha_4+\mu)K_2N\pi^2}{2\alpha_2(\alpha_4+\mu)K_1}\left(\frac{K_3-K_2}{K_2} - \frac{\varepsilon_a\pi N}{\bar{\varepsilon}}\frac{\operatorname{ch}^2\lambda z}{\operatorname{sh}^2\lambda d}\right)\right|.$$

The time interval $t_i \sim 1$ sec is in qualitative agreement with the experimental data which is again an argument in support of the quasisteady state.

Although we considered and used the assumption that the number of pitches N is large, the main result remains valid also for "moderate values" $N \sim 10$. In this case the time $\tau_{\rm ch}^{(3)}$ increases considerably and a wave-like modulation may appear if the boundary conditions are other than those corresponding to the quasisteady case. Therefore, it would be of interest to investigate experimentally thinner samples and to determine the dependences and nature of the effect on their thickness.

[1)]It should be noted that there is independent evidence of this nature of the angular dependence of the surface energy.[4]

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