Galvanomagnetic properties of inhomogeneous media in a weak magnetic field

B. Ya. Balagurov

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A systematic theory of the galvanomagnetic properties of inhomogeneous media in a weak magnetic field **H** is given. By means of perturbation theory in powers of **H** the effective conductivity tensor $\hat{\sigma}_e$ of the medium is expressed in terms of the solution for the electrical conductivity at $\mathbf{H} = 0$. For isotropic two-component systems the dependence of $\hat{\sigma}_e$ on the galvanomagnetic characteristics of the components is found in the approximation quadratic in **H**. The coefficients of this expansion are two-parameter functions determined by the properties of the medium for $\mathbf{H} = 0$. Analysis of these functions shows that in the framework of the scaling hypothesis the Hall coefficient is described by one new critical exponent (new, in that it is not one of those describing the electrical conductivity). For a complete description of the critical behavior of the magnetoresistance it is sufficient to introduce four further exponents.

1. INTRODUCTION

The galvanomagnetic properties of inhomogeneous (in particular, two-component) media have been studied intensively,[1–8] because of their significance both in general physics and in applications.

In the two-dimensional case ($D = 2$, where D is the dimensionality of space) the problem of the galvanomagnetic properties of two-component systems has been solved completely. In the work of Dykhne[1] (see also Refs. 2–4) an exact expression for the effective conductivity tensor $\hat{\sigma}_e$ of a two-dimensional, randomly inhomogeneous two-component system placed in a transverse magnetic field **H** was found for the case of the critical concentration $p = 1/2$. The results of Ref. 1 gave the possibility of expressing the effective Hall coeffficient R_e in a weak magnetic field in terms of the effective electrical conductivity σ_e of the system (for $\mathbf{H} = 0$) for arbitrary concentrations[2] (see also Ref. 3). In papers by the author[6,7] an isomorphism had been established between the problems of the galvanomagnetic properties (for any $\mathbf{H} \neq 0$) and the electrical conductivity (for $\mathbf{H} = 0$) of arbitrary two-dimensional two-component systems. The isomorphism relations that were found in Refs. 6 and 7 make it possible to express the components of the tensor $\hat{\sigma}$ in terms of the galvanomagnetic characteristics of the components and in terms of the dimensionless effective conductivity f of the system (for $\mathbf{H} = 0$). In the vicinity of the metal-insulator phase-transition point the principal properties of the function f are known in the framework of the scaling hypothesis.[9,10] Therefore, in the two-dimensional case it is possible to give a complete (in the sense of the scaling hypothesis) description of the critical behavior both of the Hall coefficient in a weak magnetic field[2] and of the effective conductivity tensor $\hat{\sigma}_e$ for arbitrary magnetic fields, including strong ones—see Ref. 6. Tabulation of the function f by numerical methods makes it possible to give a complete quantitative description of the galvanomagnetic properties of two-dimensional two-component systems in the entire range of variation of the parameters of the problem.

The symmetry transformation proposed in Ref. 1 and used (for $D = 2$) in Refs. 1–4, 6, and 7 cannot be carried over to the three-dimensional case, since for $D = 3$ exact results of such a general character as in the two-dimensional case have not been obtained. At the same-time, the critical behavior of, e.g., the Hall coefficient R_e (for $\mathbf{H} \to 0$) in three-dimensional systems is highly nontrival (see Refs. 2 and 8), because of the many-parameter nature of the problem. The presence of additional (in comparison with the case $\mathbf{H} = 0$) parameters leads to the appearance of different types of critical behavior of R_e (Refs. 2, 8), depending on the values of these parameters—compare with the analogous situation in the case of thermoelectric phenomena.[7,11,12] An even more complicated critical behavior must be expected for the magnetoresistance. Therefore, a unified description (valid in the entire range of variation of the parameters) of the galvanomagnetic properties of three-dimensional two-component systems is a timely and rather difficult problem in the theory of inhomogeneous media.

In Ref. 2, Shklovskiĭ proposed a theory of the critical behavior of the Hall coefficient R_e (for $\mathbf{H} \to 0$) of three-dimensional two-component media on the basis of the so-called two-band model. This approach is not rigorous and does not permit a systematic description for R_e in the entire range of variation of the parameters. Nevertheless, for the neighborhood of the percolation threshold, a successful interpolation formula for R_e was proposed in Ref. 2 on the basis of physical and scaling arguments. The use of this formula made it possible to predict (see Ref. 2) practically all types of critical behavior of the Hall coefficient, and these were subsequently investigated by numerical methods.[5,8] Thus, the results of Ref. 2 give a qualitative description of R_e in the critical region, while at the same time leaving open the question of a rigorous (quantitative) approach to this problem.

A definite step in this direction was taken in the work of Skal[5] (see also Ref. 8). In Ref. 5 an exact formal expression, applicable for media whose properties depend in an arbitrary manner on the coordinates, was obtained for the Hall coefficient R_e in the approximation linear in **H**. However, this expression was not analyzed theoretically in the appropriate manner, and so it was not possible to give a systematic description of the critical behavior of R_e. (For example, in Ref. 8 it was concluded that in the framework of the scaling hypothesis the Hall coefficient is characterized by four independent critical exponents that are not related to the corresponding exponents of the electrical conductivity. However, as shown in the present paper, for R_e it is sufficient to intro-

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duce one new exponent.) In addition, in Refs. 5 and 8 a systematic scheme for seeking the next corrections in **H** was not given, and so it was not possible to consider the problem of the magnetoresistance by this method. At the same time, we note that the general expression obtained in Ref. 5 for the Hall coefficient was used successfully in Refs. 5 and 8 to investigate the critical behavior of R_e by numerical methods.

In the present paper the problem of the galvanomagnetic properties of inhomogeneous media in a weak magnetic field **H** is considered by a perturbation-theory method, i.e., by expansion in powers of **H**. It is shown that, for isotropic two-component systems, in the approximation quadratic in **H** the structure of the effective conductivity tensor $\hat{\sigma}_e$ can be elucidated using symmetry considerations. In this case the dependence of $\hat{\sigma}_e$ on the additional (in the present case, galvanomagnetic) parameters discussed above is established in explicit form. The coefficients of these parameters are determined by the properties of the medium for $\mathbf{H} = 0$ and are, in turn, two-parameter functions. Investigation of the behavior of these functions in the neighborhood of the percolation threshold makes it possible to give a description of them in the framework of the scaling hypothesis. As a result it is found that for the Hall coefficient in the critical region it is sufficient to introduce one new critical exponent (new, in the sense that it is not one of the exponents of the electrical conductivity), while for the magnetoresistance there are a further four exponents.

In the paper we also consider questions associated with the formal apparatus of the theory—certain relations and identities used in the calculation of the effective conductivity tensor $\hat{\sigma}_e$ by the method of expansion in powers of **H**. By this method we find an exact expression for $\hat{\sigma}_e$ to within terms $\sim H^2$ inclusive. In the case of two-component systems this approach makes it possible to find the relationship of the two-parameter functions (coefficients in the expansion of $\hat{\sigma}_e$) discussed above to characteristics of the problem of the conductivity for $\mathbf{H} = 0$. The relationship established makes it possible, in principle, to determine these functions by numerical methods for all values of the parameters. In this way, all the new critical exponents that arise in the problems of the Hall coefficient and the magnetoresistance can be found. Tabulation of these functions will give the possibility of describing the galvanomagnetic properties of a two-component medium (in the aprroximation quadratic in **H**) in the entire range of variation of the parameters (both fundamental and additional) appearing in the problem.

2. THE EFFECTIVE CHARACTERISTIC OF THE MEDIUM

We first discuss the formulation of the problem and introduce the necessary notation. To calculate the effective conductivity of an inhomogeneous medium it is necessary to solve the direct-current equations

$$\operatorname{rot} \mathbf{E}=0, \quad \operatorname{div} \mathbf{j}=0 \tag{1}$$

with the appropriate boundary conditions. Here **E** is the electric-field intensity and **j** is the current density. In the linear (in the field **E**) problem, **j** and **E** are related by Ohm's law

$$\mathbf{j}=\hat{\sigma}(\mathbf{r})\mathbf{E}, \tag{2}$$

where the tensor $\hat{\sigma}(\mathbf{r})$ describes the coordinate-dependent conductivity of the medium. The effective conductivity tensor $\hat{\sigma}_e$ is defined in the usual way:

$$\langle \mathbf{j} \rangle=\hat{\sigma}_e\langle \mathbf{E} \rangle, \tag{3}$$

where $\langle ... \rangle$ denotes averaging over the volume V of the sample ($V \to \infty$):

$$\langle (\ldots) \rangle=\frac{1}{V}\int_V (\ldots)\,d\mathbf{r}. \tag{4}$$

For an isotropic two-component medium in the absence of a magnetic field the conductivity $\sigma(\mathbf{r})$ is scalar and takes constant values σ_1 and σ_2, respectively in the first and second components. The effective electrical conductivity σ_e of such a system can be written in the form

$$\sigma_e=\sigma_1 f(p,h), \quad h=\sigma_2/\sigma_1, \tag{5}$$

where p is the concentration (fraction of the occupied volume) of the first component. As noted in Refs. 6, 7, and 13, and in other papers, the two-parameter function $f(p,h)$ (the dimensionless effective conductivity) plays a fundamental role in the whole theory of transport phenomena in isotropic two-component media. In this sense, the problem of the galvanomagnetic properties of three-dimensional two-component isotropic systems in a weak magnetic field is no exception.

The conductivity of an isotropic medium placed in a magnetic field **H** is described by the tensor

$$\hat{\sigma}=\begin{pmatrix} \sigma_x & \sigma_a & 0 \\ -\sigma_a & \sigma_x & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}, \tag{6}$$

where we take **H** to be directed along the z axis. With aim of simplifying the subsequent formulas, in (6) we have introduced the notation: $\sigma_x = \sigma_{xx} = \sigma_{yy}$, $\sigma_z = \sigma_{zz}$, and $\sigma_a = \sigma_{xy} = -\sigma_{yx}$ for the transverse, longitudinal, and Hall components, respectively, of the conductivity tensor. In a weak magnetic field ($\mathbf{H} \to 0$) the quantity σ_a is linear in **H** and the corrections to σ_x and σ_z are quadratic:

$$\sigma_x=\sigma+\gamma_x, \quad \sigma_z=\sigma+\gamma_z; \quad \gamma\propto H^2. \tag{7}$$

Here σ is the conductivity of the medium for $\mathbf{H} = 0$.

For two-component systems the conductivity tensor (6) takes constant values $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in the first and second components, respectively. The effective conductivity tensor $\hat{\sigma}_e$, defined by (3), depends in this case on seven arguments:

$$\hat{\sigma}_e=\hat{\sigma}_e(p;\ \sigma_{x1},\ \sigma_{z1},\ \sigma_{a1};\ \sigma_{x2},\ \sigma_{z2},\ \sigma_{a2}), \tag{8}$$

where p is the same as in (5). The calculation of the three many-parameter functions σ_{xe}, σ_{ze}, and σ_{ae} of the form (8) is the main task of the theory of the galvanomagnetic properties of two-component media.

3. PHENOMENOLOGICAL TREATMENT

In a weak magnetic field the structure of the effective conductivity tensor $\hat{\sigma}_e$ (i.e., its dependence on the galvanomagnetic parameters σ_{ai}, γ_{xi}, and γ_{zi}, where $i = 1,2$) can be established without solving Eqs. (1), and (2). Such an approach, based on certain rather general symmetry considerations, is naturally called the phenomenological approach.

The components of the effective conductivity tensor $\hat{\sigma}_e$ possess the same properties as the components of the tensor $\hat{\sigma}$ from (6): For $\mathbf{H}\to 0$, in the expansion of the quantity σ_{ae} in powers of $\mathbf{H}$ only terms odd in $\mathbf{H}$ appear, while in σ_{xe} and σ_{ze}, only even terms appear. For $\mathbf{H} = 0$ the conductivity of the medium is given by the expression (5). In the approximation linear in $\mathbf{H}$ the Hall component σ_{ae} should have the form

$$\sigma_{ae}=\sigma_{a1}\varphi_1+\sigma_{a2}\varphi_2. \tag{9}$$

Here the coefficients φ_1 and φ_2 depend only on the properties of the medium with $\mathbf{H} = 0$, i.e., they are, like the dimensionless conductivity f, functions of two arguments, p and h. It turns out that the quantities φ_1 and φ_2 are connected by a well-defined relation, which can be found as follows.

As noted in Ref. 7, the direct-current equations (1) retain their form under the symmetry transformation

$$\mathbf{E}=\mathbf{E}', \quad \mathbf{j}=\mathbf{j}'+\hat{C}\mathbf{E}', \tag{10}$$

where $\hat{C}$ is a coordinate-independent arbitrary antisymmetric tensor: $C_{\alpha\beta} = -C_{\beta\alpha}$. The conductivity tensor of the "primed" system has the form

$$\hat{\sigma}'(\mathbf{r})=\hat{\sigma}(\mathbf{r})-\hat{C}. \tag{11}$$

An analogous relation connects the effective conductivity tensors of the original system ($\hat{\sigma}_e$) and the primed system ($\hat{\sigma}'_e$):

$$\hat{\sigma}_e=\hat{\sigma}_e'+\hat{C}. \tag{12}$$

We shall consider a two-component medium and set $\hat{C} = \hat{\sigma}_{a2}$, where $\hat{\sigma}_{a2}$ is the antisynmetric part of the conductivity tensor of the second component. Then in the primed system the Hall component of the second component is equal to zero ($\sigma'_{a2} = 0$), while for the first component we have $\sigma'_{a1} = \sigma_{a1} - \sigma_{a2}$ Thus, the quantity $\hat{\sigma}'_e$ depends on σ_{a1} and σ_{a2} only through their difference $\sigma_{a1} - \sigma_{a2}$. According to (12), this means that in the original system the quantities σ_{xe}, σ_{ze}, and $\sigma_{ae} - \sigma_{a2}$ also depend only on $\sigma_{a1} - \sigma_{a2}$ (compare with Ref. 7).

The quantity σ_{ae} in (9) satisfies the requirement that stems from the symmetry considered above, if $\varphi_1 + \varphi_2 = 1$. Then, setting $\varphi_1 = \varphi$, we obtain from (9) a general expression for the Hall component σ_{ae} in the approximation linear in $\mathbf{H}$:

$$\sigma_{ae}=\sigma_{a2}+(\sigma_{a1}-\sigma_{a2})\varphi(p, h). \tag{13}$$

In formula (13) the galvanomagnetic characteristics of the components are separated out in explicit form, and the function $\varphi(p,h)$ is determined by the properties of the medium for $\mathbf{H} = 0$.

We write the quantities σ_{xe} and σ_{ze} in a form analogous to (7):

$$\sigma_{xe}=\sigma_e+\gamma_{xe}, \quad \sigma_{ze}=\sigma_e+\gamma_{ze}, \tag{14}$$

where σ_e is the same as in (5). In the approximation quadratic in $\mathbf{H}$ the general expressions for γ_{xe} and γ_{ze} (with allowance for the symmetry discussed above) will have the form

$$\gamma_{xe}=\gamma_{x1}\psi_x^{(1)}+\gamma_{z1}\psi_x^{(2)}+\gamma_{x2}\psi_x^{(3)}+\gamma_{z2}\psi_x^{(4)}+\frac{(\sigma_{a1}-\sigma_{a2})^2}{\sigma_1}\chi_x, \tag{15}$$

$$\gamma_{ze}=\gamma_{x1}\psi_z^{(1)}+\gamma_{z1}\psi_z^{(2)}+\gamma_{x2}\psi_z^{(3)}+\gamma_{z2}\psi_z^{(4)}+\frac{(\sigma_{a1}-\sigma_{a2})^2}{\sigma_1}\chi_z. \tag{16}$$

Here the dimensionless coefficients $\psi_x^{(a)},\psi_z^{(a)}$ ($a = 1,2,3,4$) and χ_x, χ_z depend only on the properties of the medium for $\mathbf{H} = 0$ and are functions of the arguments p and h.

For geometrically (structurally) isotropic media there are a number of relations between the quantities $\psi_x^{(a)}$ and $\psi_z^{(a)}$. To find them, we shall consider a system with "natural" ($\sigma_{ai} = 0$) anisotropy, and shall assume that $\gamma_{yi} \neq \gamma_{xi}$. As noted in Ref. 14, structurally isotropic two-component media (with anisotropic conductivities of the components) possess an additional symmetry. For example, the replacement $\gamma_{xi} \leftrightarrows \gamma_{yi}$ with a simultaneous rotation of the coordinate system through 90° about the z axis does not change the properties of the medium. This implies that the quantities γ_{xe} and γ_{ye} go over into each other under the replacement $\gamma_{xi} \leftrightarrows \gamma_{yi}$; here the quantity γ_{ze} should not change. The consequences of the structural isotropy under rotations about the x and y axes are considered in an analogous manner. The quantities γ_{xe}, γ_{ye}, and γ_{ze} satisfying these requirements have the following form.:

$$\begin{aligned}
\gamma_{xe}&=\gamma_{x1}g_1+\gamma_{y1}g_2+\gamma_{z1}g_2+\gamma_{x2}g_3+\gamma_{y2}g_4+\gamma_{z2}g_4,\\
\gamma_{ye}&=\gamma_{x1}g_2+\gamma_{y1}g_1+\gamma_{z1}g_2+\gamma_{x2}g_4+\gamma_{y2}g_3+\gamma_{z2}g_4,\\
\gamma_{ze}&=\gamma_{x1}g_2+\gamma_{y1}g_2+\gamma_{z1}g_1+\gamma_{x2}g_4+\gamma_{y2}g_4+\gamma_{z2}g_3,
\end{aligned} \tag{17}$$

where g_1, g_2, g_3, and g_4 are certain coefficients.

Setting $\gamma_{y1} = \gamma_{x1}$ and $\gamma_{y2} = \gamma_{x2}$ in (17) and comparing with (15), (16), we conclude that $\psi_x^{(1)} = g_1 + g_2$, $\psi_x^{(2)} = g_2$, etc. As a result, we find the desired relations:

$$\psi_z^{(1)}=2\psi_x^{(2)}, \quad \psi_z^{(2)}=\psi_x^{(1)}-\psi_x^{(2)}, \quad \psi_z^{(3)}=2\psi_x^{(4)},$$
$$\psi_z^{(4)}=\psi_x^{(3)}-\psi_x^{(4)}. \tag{18}$$

Consequently, all the functions $\psi_z^{(a)}$ can be expressed in terms of $\psi_x^{(a)}$ In turn, the four functions $\psi_x^{(a)}$ are not independent, and two relations, connecting them with $f(p,h)$, can be established between them. We set $\sigma_{ai} = 0$ and $\gamma_{x1} = \gamma_{z1} = \gamma_1$, $\gamma_{x2} = \gamma_{z2} = \gamma_2$. Such a medium is isotropic and differs from the original medium with $\mathbf{H} = 0$ only by the replacements $\sigma_1 \to \sigma_1 + \gamma_1$, $\sigma_2 \to \sigma_2 + \gamma_2$, and $\sigma_e \to \sigma_e + \gamma_e$. Making these replacements in (5) and performing the expansion in the γ_i up to the linear terms, we obtain

$$\psi_x^{(1)}+\psi_x^{(2)}=f-hf', \quad \psi_x^{(3)}+\psi_x^{(4)}=f'; \; f'\equiv\partial f(p, h)/\partial h. \tag{19}$$

Thus, of the eight functions $\psi_x^{(a)}$ and $\psi_z^{(a)}$ only two are independent (e.g., $\psi_x^{(1)}$ and $\psi_x^{(3)}$). Consequently, the quantities γ_{xe} and γ_{ze} are described by four (in addition to f) independent two-parameter functions: $\psi_x^{(1)},\psi_x^{(3)},\chi_x$, and χ_z. (More rigorous will be the statement that there are no more than four independent functions, since the possibility of the existence of further relations between them is not ruled out.)

The above phenomenological analysis shows that, in the approximation quadratic in $\mathbf{H}$, for a complete description of the galvanomagnetic characteristics of a three-dimensional isotropic two-component system it is sufficient to introduce five (besides f) functions, determined by the properties of the medium for $\mathbf{H} = 0$: φ, $\psi_x^{(1)},\psi_x^{(3)},\chi_x$, and χ_z. It is extremely important that here the original many-parameter quantities σ_{ae}, σ_{xe}, and σ_{ze} are reduced to the level of two-parameter functions that depend, like the dimensionless effective conductivity f, on the arguments p and h. The critical behavior of the two-parameter functions can be described in the spirit of the standard scaling hypothesis—compare with the analogous procedure in the case of the electrical conductivity. Application of this hypothesis to the functions φ, $\psi_x^{(a)}$, $\psi_z^{(a)}$, χ_x, and χ_z will make it possible to give a systematic description of the critical behavior of the quantities σ_{ae}, σ_{xe}, and σ_{ze}, and, consequently, of the Hall coefficient and magnetoresistance.

4. SOME RELATIONS

We shall consider certain identities and relations that will be needed in the calculation of the effective conductivity tensor $\hat{\sigma}_e$ from "first principles," i.e., by direct solution of Eqs. (1) and (2).

1. In the theory of transport phenomena in inhomogeneous media a number of useful identities of a general character can be established.

We denote by $\mathbf{E}^{(\nu)}(\mathbf{r})$ and $\mathbf{j}^{(\nu)}(\mathbf{r})$ the electric field and current density in the medium, determined for a given value of $\langle\mathbf{E}^{(\nu)}\rangle$, where the exponent ν denotes that the mean field is directed along the ν axis. It can be shown that the following identity (for $V\to\infty$) is valid:

$$\langle\mathbf{j}^{(\mu)}\mathbf{E}^{(\nu)}\rangle=\langle\mathbf{j}^{(\mu)}\rangle\langle\mathbf{E}^{(\nu)}\rangle, \tag{20}$$

where $\langle...\rangle$ is the same as in (4). The formula (20) is a generalization of a well-known identity (see, e.g., Ref. 15) and is proved as follows. We introduce the potential $\varphi^{(\nu)}$ through the equality $\mathbf{E}^{(\nu)}(\mathbf{r}) = \langle\mathbf{E}^{(\nu)}\rangle - \nabla\varphi^{(\nu)}(\mathbf{r})$, where $\langle\nabla\varphi^{(\nu)}\rangle = 0$. Then the difference $\langle\mathbf{j}^{(\mu)}\mathbf{E}^{(\nu)}\rangle - \langle\mathbf{j}^{(\mu)}\rangle\langle\mathbf{E}^{(\nu)}\rangle$ can be transformed, with the aid of the equation $\operatorname{div}\mathbf{j}^{(\nu)} = 0$, to a surface integral. In the limit $V\to\infty$ the surface effects vanish and we arrive at the equality (20). We note that for certain boundary conditions this surface integral is identically equal to zero, so that in these cases (20) is also valid for systems of finite size. We note also that one can arrive at the result (20) ina slightly different way—by introducing a vector potential by means of the equality $\mathbf{j}^{(\mu)} = \langle\mathbf{j}^{(\mu)}\rangle + \operatorname{curl}\mathbf{A}^{(\mu)}$, where $\langle\operatorname{curl}\mathbf{A}^{(\mu)}\rangle = 0$.

In an analogous way (by the introduction of a potential $\varphi^{(\mu)}$) one can prove the identity

$$\langle[\mathbf{E}^{(\mu)}\times\mathbf{E}^{(\nu)}]\rangle = [\langle\mathbf{E}^{(\mu)}\rangle\times\langle\mathbf{E}^{(\nu)}\rangle]. \tag{21}$$

It is not difficult to see that the more general identity

$$\langle(\mathbf{E}^{(\mu)}[\mathbf{E}^{(\nu)}\times\mathbf{E}^{(\lambda)}])\rangle = (\langle\mathbf{E}^{(\mu)}\rangle[\langle\mathbf{E}^{(\nu)}\rangle\times\langle\mathbf{E}^{(\lambda)}\rangle]) \tag{22}$$

is also valid. To prove (22) we write $\mathbf{E}^{(\mu)}$ in the form $\mathbf{E}^{(\mu)} = \langle\mathbf{E}^{(\mu)}\rangle - \nabla\varphi^{(\mu)}, \langle\nabla\varphi^{(\mu)}\rangle = 0$. The integral containing $\nabla\varphi^{(\mu)}$ obtained as a result of substituting for $\mathbf{E}^{(\mu)}$ can be transformed to a surface integral by virtue of the equality

$$\operatorname{div}[\mathbf{E}^{(\nu)}\mathbf{E}^{(\lambda)}]=\mathbf{E}^{(\lambda)}\operatorname{rot}\mathbf{E}^{(\nu)}-\mathbf{E}^{(\nu)}\operatorname{rot}\mathbf{E}^{(\lambda)}\equiv 0.$$

In the two-dimensional case, as well as (21) we can also prove the important identity

$$\langle[\mathbf{j}^{(\mu)}\times\mathbf{j}^{(\nu)}]_z\rangle = [\langle\mathbf{j}^{(\mu)}\rangle\times\langle\mathbf{j}^{(\nu)}\rangle]_z, \quad D=2, \tag{23}$$

(The quantity $\mathbf{j}$ has nonzero components only in the xy plane.) To prove (23) we set $\mathbf{j}^{(\mu)} = \langle\mathbf{j}^{(\mu)}\rangle + \operatorname{curl}\mathbf{A}^{(\mu)}$, where $\langle\operatorname{curl}\mathbf{A}^{(\mu)}\rangle = 0$ and the vector potential has the form $\mathbf{A}^{(\mu)} = \{0,0,A\}$. Then,

$$\langle[\mathbf{j}^{(\mu)}\mathbf{j}^{(\nu)}]_z\rangle-[\langle\mathbf{j}^{(\mu)}\rangle\langle\mathbf{j}^{(\nu)}\rangle]_z=\left\langle\frac{\partial A}{\partial x}j_x^{(\nu)}+\frac{\partial A}{\partial y}j_y^{(\nu)}\right\rangle.$$

With allowance for the condition $\operatorname{div}\mathbf{j}^{(\nu)} = 0$ we transform the integral in the right-hand side to an integral that vanishes as $V\to\infty$. We emphasize that the identity (23) is valid only for two-dimensional systems and has no analog in the three-dimensional case.

2. The identities proved make it possible to establish a number of exact relations that will be needed subsequently.

We suppose that for a certain system the quantities $\mathbf{j}^{(\mu)}(\mathbf{r})$ and $\mathbf{E}^{(\nu)}(\mathbf{r})$ are known. We change the conductivity of this system: $\sigma(\mathbf{r})\to\tilde{\sigma}(\mathbf{r})$. Correspondingly, we shall have $\tilde{\mathbf{j}}^{(\mu)}(\mathbf{r})$ and $\tilde{\mathbf{E}}^{(\mu)}(\mathbf{r})$. Then, in anaology with (20), we can prove the two identities

$$\langle\mathbf{j}^{(\mu)}\tilde{\mathbf{E}}^{(\nu)}\rangle=\langle\mathbf{j}^{(\mu)}\rangle\langle\tilde{\mathbf{E}}^{(\nu)}\rangle, \quad \langle\tilde{\mathbf{j}}^{(\nu)}\mathbf{E}^{(\mu)}\rangle=\langle\tilde{\mathbf{j}}^{(\nu)}\rangle\langle\mathbf{E}^{(\mu)}\rangle. \tag{24}$$

We consider an isotropic two-component medium, for which, according to (3) and (5),

$$\langle\mathbf{j}^{(\mu)}\rangle=\sigma_e\langle\mathbf{E}^{(\mu)}\rangle, \quad \sigma_e=\sigma_1 f(p,h), \quad h=\sigma_2/\sigma_1.$$

By the system indicated by the tilde we shall understand a medium of the same structure with changed conductivities:

$$\langle\tilde{\mathbf{j}}^{(\nu)}\rangle=\tilde{\sigma}_e\langle\tilde{\mathbf{E}}^{(\nu)}\rangle, \quad \tilde{\sigma}_e=\tilde{\sigma}_1 f(p,\tilde{h}), \quad \tilde{h}=\tilde{\sigma}_2/\tilde{\sigma}_1.$$

We write out $\langle(...)\rangle$ in the form of a sum over the components:

$$\langle(\ldots)\rangle=\langle(\ldots)\rangle^{(1)}+\langle(\ldots)\rangle^{(2)}, \quad \langle(\ldots)\rangle^{(i)}=\frac{1}{V}\int_{V_i}(\ldots)d\mathbf{r}, \tag{25}$$

where in $\langle(...)\rangle^{(i)}$ the integration is performed over the volume V_i of the i-th component. Then from (24) we have

$$\begin{aligned}&\sigma_1\langle\mathbf{E}^{(\mu)}\tilde{\mathbf{E}}^{(\nu)}\rangle^{(1)}+\sigma_2\langle\mathbf{E}^{(\mu)}\tilde{\mathbf{E}}^{(\nu)}\rangle^{(2)}=\sigma_1 f(p,h)\langle\mathbf{E}^{(\mu)}\rangle\langle\tilde{\mathbf{E}}^{(\nu)}\rangle,\\&\tilde{\sigma}_1\langle\mathbf{E}^{(\mu)}\tilde{\mathbf{E}}^{(\nu)}\rangle^{(1)}+\tilde{\sigma}_2\langle\mathbf{E}^{(\mu)}\tilde{\mathbf{E}}^{(\nu)}\rangle^{(2)}=\tilde{\sigma}_1 f(p,\tilde{h})\langle\mathbf{E}^{(\mu)}\rangle\langle\tilde{\mathbf{E}}^{(\nu)}\rangle.\end{aligned} \tag{26}$$

From (26) we find (for $\langle\mathbf{E}^{(\mu)}\rangle\langle\tilde{\mathbf{E}}^{(\nu)}\rangle\neq 0$)

$$\begin{aligned}&\langle\mathbf{E}^{(\mu)}\tilde{\mathbf{E}}^{(\nu)}\rangle^{(1)}/\langle\mathbf{E}^{(\mu)}\rangle\langle\tilde{\mathbf{E}}^{(\nu)}\rangle=[\tilde{h}f(p,h)-hf(p,\tilde{h})]/(\tilde{h}-h),\\&\langle\mathbf{E}^{(\mu)}\tilde{\mathbf{E}}^{(\nu)}\rangle^{(2)}/\langle\mathbf{E}^{(\mu)}\rangle\langle\tilde{\mathbf{E}}^{(\nu)}\rangle=[f(p,\tilde{h})-f(p,h)]/(\tilde{h}-h).\end{aligned} \tag{27}$$

From this, taking the limit $\tilde{h}\to h$, we obtain

$$\begin{aligned}&\langle\mathbf{E}^{(\mu)}\mathbf{E}^{(\nu)}\rangle^{(1)}/\langle\mathbf{E}^{(\mu)}\rangle\langle\mathbf{E}^{(\nu)}\rangle=f-hf',\\&\langle\mathbf{E}^{(\mu)}\mathbf{E}^{(\nu)}\rangle^{(2)}/\langle\mathbf{E}^{(\mu)}\rangle\langle\mathbf{E}^{(\nu)}\rangle=f'; \quad f'=\partial f(p,h)/\partial h.\end{aligned} \tag{28}$$

We note that, according to (28), determination of the mean square electric field in the second component by numerical methods makes it possible to find the derivative f' with the same accuracy as for the function f itself. Knowledge of the quantity f' is necessary, e.g., in problems concerning the low-frequency dispersion of the electrical conductivity,[13] the thermopower,[12] and the magnetoresistance (see the expressions (19) and Secs. 7 and 8 of the present paper).

The relations (28) give the possibility of finding the mean square fluctuations of the field and current:

$$\Delta_E^2=\{\langle\mathbf{E}^2\rangle-(\langle\mathbf{E}\rangle)^2\}/\langle\mathbf{E}\rangle^2, \quad \Delta_j^2=\{\langle\mathbf{j}^2\rangle-(\langle\mathbf{j}\rangle)^2\}/\langle\mathbf{j}\rangle^2. \tag{29}$$

From (28) and (29) we obtain

$$\Delta_E^2=f+(1-h)f'-1, \quad \Delta_j^2=[f(1-f)-h(1-h)f']f^{-2}. \tag{30}$$

It is not difficult to verify that the expressions (30) satisfy the exact relation

$$\Delta_j^2=\frac{(\sigma_1-\sigma_e)(\sigma_e-\sigma_2)}{\sigma_e^2}-\frac{\sigma_1\sigma_2}{\sigma_e^2}\Delta_E^2,$$

found in Ref. 3. For a two-dimensional, randomly inhomogeneous medium with $p=1/2$ we have $f=h^{1/2}$ (Ref. 15), so that from (30) we obtain

$$\Delta_E^2=\Delta_j^2={}^1/_2(h^{-1/4}-h^{1/4})^2,\quad p={}^1/_2\quad (D=2),$$

which coincides with the result of Ref. 15. For the mean values of the fields in the first and second components we have, respectively,

$$\langle\mathbf{E}^{(\nu)}\rangle^{(1)}=\frac{\sigma_e-\sigma_2}{\sigma_1-\sigma_2}\langle\mathbf{E}^{(\nu)}\rangle,\quad \langle\mathbf{E}^{(\nu)}\rangle^{(2)}=\frac{\sigma_1-\sigma_e}{\sigma_1-\sigma_2}\langle\mathbf{E}^{(\nu)}\rangle,\quad (31)$$

whence follow, for $p=1/2$, the expressions obtained in Ref. 15.

In the framework of the conductivity problem (for $\mathbf{H}=0$) it is of interest to study the more general characteristic $\langle E_\alpha^{(\mu)}E_\beta^{(\nu)}\rangle$. For an isotropic medium, taking the identity (21) into account, we have

$$\langle(E_\alpha^{(\mu)}-\langle E_\alpha^{(\mu)}\rangle)(E_\beta^{(\nu)}-\langle E_\beta^{(\nu)}\rangle)\rangle=a\delta_{\alpha\beta}\langle\mathbf{E}^{(\mu)}\rangle\langle\mathbf{E}^{(\nu)}\rangle +(b-1)(\langle E_\alpha^{(\mu)}\rangle\langle E_\beta^{(\nu)}\rangle+\langle E_\beta^{(\mu)}\rangle\langle E_\alpha^{(\nu)}\rangle).\quad (32)$$

Here a and b are dimensionless functions of the arguments p and h. Contracting on the indices α and β in (32) and comparing (for $\mu=\nu$) with (29), (30), we obtain the relationship between the coefficients a and b and the function f:

$$Da+2b=f+(1-h)f'+1,\quad (33)$$

where D is the dimensionality of space.

As will be clear from the following, the quantities $\langle E_\alpha^{(\mu)}E_\beta^{(\nu)}\rangle^{(i)}$, where $\langle\ldots\rangle^{(i)}$ is the same as in (25), are also of considerable interest. For an isotropic medium,

$$\langle E_\alpha^{(\mu)}E_\beta^{(\nu)}\rangle^{(i)}=a^{(i)}\delta_{\alpha\beta}\langle\mathbf{E}^{(\mu)}\rangle\langle\mathbf{E}^{(\nu)}\rangle+b^{(i)}\langle E_\alpha^{(\mu)}\rangle\langle E_\beta^{(\nu)}\rangle +c^{(i)}\langle E_\beta^{(\mu)}\rangle\langle E_\alpha^{(\nu)}\rangle,\quad (34)$$

where $a^{(i)}$, $b^{(i)}$, and $c^{(i)}$ are functions of p and h. Contraction on α and β with allowance for the equalities (28) gives

$$Da^{(1)}+b^{(1)}+c^{(1)}=f-hf',\quad Da^{(2)}+b^{(2)}+c^{(2)}=f'.\quad (35)$$

On the other hand, taking the sum over i of (34) and comparing with (32), we conclude that

$$a^{(1)}+a^{(2)}=a,\quad b^{(1)}+b^{(2)}=b,\quad c^{(1)}+c^{(2)}=b-1.\quad (36)$$

The study by numerical methods of the functions $a^{(i)}$, $b^{(i)}$, $c^{(i)}$, a, and b in the entire range of variation of the arguments p and h is of interest both for the problem of the galvanomagnetic properties (for $\mathbf{H}\to 0$) and for the problem of the conductivity of weakly anisotropic media (in the approximation linear in the anisotropy).

5. THE APPROXIMATION LINEAR IN H

To calculate the galvanomagnetic characteristics of an inhomogeneous medium from first principles it is necessary to solve Eqs. (1) and (2) with the conductivity tensor (6), which has the invariant form

$$\sigma_{\alpha\beta}=\sigma_x\delta_{\alpha\beta}+(\sigma_z-\sigma_x)n_\alpha n_\beta+\sigma_a e_{\alpha\beta\gamma}n_\gamma,\quad \mathbf{n}=\mathbf{H}/H.\quad (37)$$

Here $e_{\alpha\beta\gamma}$ is the antisymmetric unit tensor. In a weak magnetic field ($\mathbf{H}\to 0$) Eqs. (1), (2) can be solved of perturbation theory, i.e., by expanding in powers of $\mathbf{H}$. In this case the electric-field intensity $\mathbf{E}^{(\nu)}(\mathbf{r})$ and current density $\mathbf{j}^{(\nu)}(\mathbf{r})$ can be represented by series:

$$\mathbf{E}^{(\nu)}(\mathbf{r})=\mathbf{E}_0^{(\nu)}(\mathbf{r})+\mathbf{E}_1^{(\nu)}(\mathbf{r})+\ldots,$$
$$\mathbf{j}^{(\nu)}(\mathbf{r})=\mathbf{j}_0^{(\nu)}(\mathbf{r})+\mathbf{j}_1^{(\nu)}(\mathbf{r})+\ldots,\quad (38)$$

where $\mathbf{E}_n^{(\nu)}(\mathbf{r})$ and $\mathbf{j}_n^{(\nu)}(\mathbf{r})$ are the terms of nth order in $\mathbf{H}$. The quantities $\mathbf{E}_n^{(\nu)}(\mathbf{r})$ and $\mathbf{j}_n^{(\nu)}(\mathbf{r})$ satisfy the equations

$$\operatorname{rot}\mathbf{E}_n^{(\nu)}=0,\quad \operatorname{div}\mathbf{j}_n^{(\nu)}=0.\quad (39)$$

In the zeroth approximation in $\mathbf{H}$, for the potential $\varphi_0^{(\nu)}(\mathbf{r})$ defined by the equality $\mathbf{E}_0^{(\nu)}(\mathbf{r})=\langle\mathbf{E}_0^{(\nu)}\rangle-\nabla\varphi_0^{(\nu)}(\mathbf{r})$ ($\langle\nabla\varphi_0^{(\nu)}\rangle=0$), we obtain from $\operatorname{div}\mathbf{j}_0^{(\nu)}=\operatorname{div}\{\sigma(\mathbf{r})=\mathbf{E}_0^{(\nu)}\}=0$ the equation

$$\nabla\{\sigma(\mathbf{r})\nabla\varphi_0^{(\nu)}(\mathbf{r})\}=\nabla\{\sigma(\mathbf{r})\langle\mathbf{E}_0^{(\nu)}\rangle\}.\quad (40)$$

The formal solution of (40) can be found with the aid of the Green function $g(\mathbf{r},\mathbf{r}')$ satisfying the equation

$$\nabla_{\mathbf{r}}\{\sigma(\mathbf{r})\nabla_{\mathbf{r}}g(\mathbf{r},\mathbf{r}')\}=\delta(\mathbf{r}-\mathbf{r}'),\quad (41)$$

with $g(\mathbf{r},\mathbf{r}')=g(\mathbf{r}',\mathbf{r})$. We assume also that the Green function vanishes when $\mathbf{r}$ or $\mathbf{r}'$ lies on the boundary of the sample. Solving (40) with the aid of (41), for the field $\langle\mathbf{E}_0^{(\nu)}\rangle(\mathbf{r})$ (after integration by parts) we obtain

$$E_{0\alpha}^{(\nu)}(\mathbf{r})=\langle E_{0\beta}^{(\nu)}\rangle\left\{\delta_{\alpha\beta}+\int d\mathbf{r}'\,\sigma(\mathbf{r}')\frac{\partial^2 g(\mathbf{r},\mathbf{r}')}{\partial x_\alpha\,\partial x_\beta'}\right\}.\quad (42)$$

Calculating $\langle\mathbf{j}_0^{(\nu)}\rangle=\langle\sigma\mathbf{E}_0^{(\nu)}\rangle$, we obtain a formal exact expression for the effective electrical conductivity of an isotropic inhomogeneous medium:

$$\sigma_e=\langle\sigma\rangle+\frac{1}{D}\frac{1}{V}\int d\mathbf{r}\,\sigma(\mathbf{r})\int d\mathbf{r}'\,\sigma(\mathbf{r}')\frac{\partial^2 g(\mathbf{r},\mathbf{r}')}{\partial\mathbf{r}\,\partial\mathbf{r}'},\quad (43)$$

where D is the dimensionality of space and $V\to\infty$.

In the approximation linear in $\mathbf{H}$ we have

$$\mathbf{j}_1^{(\nu)}=\sigma(\mathbf{r})\mathbf{E}_1^{(\nu)}+\hat\sigma_a(\mathbf{r})\mathbf{E}_0^{(\nu)},\quad (\hat\sigma_a)_{\alpha\beta}=\sigma_a e_{\alpha\beta\gamma}n_\gamma.\quad (44)$$

Setting $\mathbf{E}_1^{(\nu)}(\mathbf{r})=\langle\mathbf{E}_1^{(\nu)}\rangle-\nabla\varphi_1^{(\nu)}(\mathbf{r})$ ($\langle\nabla\varphi_1^{(\nu)}\rangle=0$), from $\operatorname{div}\mathbf{j}_1^{(\nu)}=0$ we find an equation for the potential $\varphi_1^{(\nu)}$:

$$\nabla\{\sigma(\mathbf{r})\nabla\varphi_1^{(\nu)}(\mathbf{r})\}=\nabla\{\sigma(\mathbf{r})\langle\mathbf{E}_1^{(\nu)}\rangle\}+\nabla\{\hat\sigma_a(\mathbf{r})\mathbf{E}_0^{(\nu)}(\mathbf{r})\}.\quad (45)$$

Solving (45) with the aid of the Green function, for the field $\mathbf{E}_1^{(\nu)}(\mathbf{r})$, in analogy with (42), we obtain

$$E_{1\alpha}^{(\nu)}(\mathbf{r})=\langle E_{1\beta}^{(\nu)}\rangle\left\{\delta_{\alpha\beta}+\int d\mathbf{r}'\,\sigma(\mathbf{r}')\frac{\partial^2 g(\mathbf{r},\mathbf{r}')}{\partial x_\alpha\,\partial x_\beta'}\right\} +\int d\mathbf{r}'\,(\hat\sigma_a(\mathbf{r}'))_{\beta\gamma}\frac{\partial^2 g(\mathbf{r},\mathbf{r}')}{\partial x_\alpha\,\partial x_\beta'}E_{0\gamma}^{(\nu)}(\mathbf{r}').\quad (46)$$

The effective Hall component σ_{ae} is determined from the relation

$$\langle\mathbf{j}_1^{(\nu)}\rangle=\sigma_e\langle\mathbf{E}_1^{(\nu)}\rangle+\hat\sigma_{ae}\langle\mathbf{E}_0^{(\nu)}\rangle,\quad (\hat\sigma_{ae})_{\alpha\beta}=\sigma_{ae}e_{\alpha\beta\gamma}n_\gamma.\quad (47)$$

As a result, after certain transformations we can obtain for the quantity σ_{ae} an expression that does not contain the Green function—see (51). Below, the formula (51) will be derived by a simpler method.

A substantial simplification of the iteration procedure described above is achieved when identities of the type (20) are used. It is not difficult to see that, by virtue of Eqs. (39), the following identities are valid:

$$\langle \mathbf{E}_n^{(\mu)}\rangle\langle \mathbf{j}_m^{(\nu)}\rangle=\langle \mathbf{E}_n^{(\mu)}\mathbf{j}_m^{(\nu)}\rangle, \tag{48}$$

where n and m are arbitrary. In the approximation linear in $\mathbf{H}$, we have two equalities:

$$\langle \mathbf{E}_0^{(\mu)}\rangle\langle \mathbf{j}_1^{(\nu)}\rangle=\langle \mathbf{E}_0^{(\mu)}\mathbf{j}_1^{(\nu)}\rangle,\quad \langle \mathbf{E}_1^{(\nu)}\rangle\langle \mathbf{j}_0^{(\mu)}\rangle=\langle \mathbf{E}_1^{(\nu)}\mathbf{j}_0^{(\mu)}\rangle. \tag{49}$$

From the first identity (49), taking (44) and (47) into account we obtain

$$\langle \mathbf{E}_0^{(\mu)}\rangle\sigma_e\langle \mathbf{E}_1^{(\nu)}\rangle+\langle \mathbf{E}_0^{(\mu)}\rangle\hat{\sigma}_{ae}\langle \mathbf{E}_0^{(\nu)}\rangle$$
$$=\langle \mathbf{E}_0^{(\mu)}\sigma\mathbf{E}_1^{(\nu)}\rangle+\langle \mathbf{E}_0^{(\mu)}\hat{\sigma}_a\mathbf{E}_0^{(\nu)}\rangle. \tag{50}$$

Since $\sigma\mathbf{E}_0^{(\mu)}=\mathbf{j}_0^{(\mu)}$, the first terms in the left- and right-hand sides of (50) cancel by virtue of the second identity (49). As a result, from (50) we finally obtain

$$\sigma_{ae}=\langle\sigma_a([\mathbf{E}_0^{(\mu)},\ \mathbf{E}_0^{(\nu)}]\mathbf{n})\rangle/([\langle\mathbf{E}_0^{(\mu)}\rangle,\ \langle\mathbf{E}_0^{(\nu)}\rangle]\mathbf{n}),\quad \mathbf{n}=\mathbf{H}/H. \tag{51}$$

The formula (51) gives the desired general expression for the Hall component σ_{ae}, valid in the approximation linear in $\mathbf{H}$ for arbitrary inhomogeneous media (including, as is not difficult to see, anisotropic media). Substitution of $\mathbf{E}_0(\mathbf{r})$ from (42) into (51) makes it possible to obtain for σ_{ae} a formal exact expression of the type (43). However, the form (51) is convenient for numerical investigation of σ_{ae} since it permits the use of the results of the standard electrical-conduction problem.

We note that for a medium with a coordinate-independent Hall component σ_a it follows from (51), with allowance for the identity (21), that $\sigma_{ae}=\sigma_a$, which agrees with Ref. 16. But if the conductivity σ does not depend on the coordinates, then $\mathbf{E}_0(\mathbf{r})=\langle\mathbf{E}_0\rangle$, and then $\sigma_{ae}=\langle\sigma_a\rangle$. From this, for the Hall coefficient R_e, follows the expression $R_e=\langle R\rangle$ obtained in Ref. 2 for two-component systems. Finally, for media with an arbitrary dependence of σ and σ_a on the coordinates, from (51) we obtain for the Hall coefficient this general expression

$$R_e=\langle R([\mathbf{j}_0^{(\mu)},\ \mathbf{j}_0^{(\nu)}]\mathbf{n})\rangle/([\langle\mathbf{j}_0^{(\mu)}\rangle,\ \langle\mathbf{j}_0^{(\nu)}\rangle]\mathbf{n}), \tag{52}$$

which is equivalent (for $\mu=x$ and $\nu=y$) to the formula obtained in Refs. 5 and 8. A more direct way of deriving (52) is to use the identities (49) with Ohm's law in the form $\mathbf{E}=\hat{\rho}\mathbf{j}$, where $\hat{\rho}$ is the resistivity tensor.

We shall consider an isotropic two-component system. We write the quantity $\sigma_a(\mathbf{r})$ in the form

$$\sigma_a(\mathbf{r})=\sigma_{a2}+(\sigma_{a1}-\sigma_{a2})\theta_1(\mathbf{r}), \tag{53}$$

where $\theta_1(\mathbf{r})=1$ inside the first component and $\theta_1(\mathbf{r})=0$ outside it. Substitution of (53) into (51) leads, with allowance for the identity (21), to a formula of the form (13), where

$$\varphi(p,\ h)=\langle([\mathbf{E}_0^{(\mu)},\ \mathbf{E}_0^{(\nu)}]\mathbf{n})\rangle^{(1)}/([\langle\mathbf{E}_0^{(\mu)}\rangle,\ \langle\mathbf{E}_0^{(\nu)}\rangle]\mathbf{n}). \tag{54}$$

Here $\langle...\rangle^{(1)}$ is the same as in (25). The formula (54) gives an expression for the function $\varphi(p,h)$ in terms of the field $\mathbf{E}_0(\mathbf{r})$ in the medium, i.e., gives the solution of the problem of the electrical conductivity for $\mathbf{H}=0$. We note that multiplying (34) (for $i=1$) by $e_{\alpha\beta\gamma}n_\gamma$ makes it possible to relate φ to the functions $b^{(1)}$ and $c^{(1)}$: $\varphi=b^{(1)}-c^{(1)}$. We note also that for anisotropic media too the quantity φ is given by the expression (54), but is in this case a function of six dimensionless arguments: p, σ_{x1}/σ_{z1}, σ_{y1}/σ_{z1}, σ_{x2}/σ_{z1}, σ_{y2}/σ_{z1}, σ_{z2}/σ_{z1}.

6. THE CRITICAL BEHAVIOR OF THE HALL COEFFICIENT

For a system with dielectric ($\hat{\sigma}_2=0$) inclusions with the critical concentration $p=p_c$ (Refs. 9, 10) all components of the tensor $\hat{\sigma}_e$ vanish. Consequently, $\varphi(p,0)=0$ for $p\leqslant p_c$, and $\varphi(p,0)\neq0$ for $p>p_c$. As in the case of the electrical conductivity, we shall assume that for $p>p_c$ and $p\to p_c$ the function $\varphi(p,0)$ vanishes in a power-law manner:

$$p>p_c:\quad \varphi\sim\tau^l,\quad \tau=(p-p_c)/p_c, \tag{55}$$

where $l>0$. The physical reason for the decrease of $\varphi(p,0)$ as $p\to p_c$ is obvious—the electric field is expelled from the conducting (first) component. For $p<p_c$ the first (metallic) component forms unconnected finite clusters,[9,10] inside which the electric field is equal to zero, so that $\varphi(p,0)\equiv0$. If, however, the conductivity of the second ("dielectric") component is nonzero, then $\varphi(p,h)\neq0$ even for $p<p_c$, but φ is small for $h=\sigma_2/\sigma_1\ll1$. According to the first equality (31), the mean electric field in the first component for $p<p$ and $h\to0$ vanishes linearly with h. It may be supposed that this is also true for the field $\mathbf{E}(\mathbf{r})$ at any point of the first component, so that $\varphi\propto h^2$ for $p<p$ and $h\to0$ (the assumption of a dependence of the form $\varphi\propto h^2$ is confirmed in the two-dimensional case, when φ can be exprssed in terms of f; see Sec. 8). Then for $p<p_c$ and $p\to p_c$ outside the region of smearing (see below) we should expect

$$p<p_c:\quad \varphi\sim h^2/(-\tau)^r,\quad r>0. \tag{56}$$

Finally, for the critical concentration $p=p_c$ the quantity φ will evidently be a power function of h:

$$p=p_c:\quad \varphi\sim h^u,\quad u>0, \tag{57}$$

where u, generally speaking, is not an integer.

The dimensionless electrical conductivity $f(p,h)$ has behavior similar (except for the dependence on h for $p<p_c$) to (55)–(57) in the critical region $|\tau|\ll1$, $h\ll1$. In the framework of the scaling hypothesis for f we have[2,9,10]

$$|\tau|\ll1,\ h\ll1:\ f=h^sF(\tau/h^{s/t}) \tag{58}$$

with the following asymptotic forms:

$$\begin{aligned}\tau>0,\ \ \Delta_0\ll\tau\ll1:&\quad f=\tau^t\{A_0+A_1(h/\tau^{t/s})+\dots\},\\ |\tau|\ll\Delta_0:&\quad f=h^s\{a_0+a_1(\tau/h^{s/t})+\dots\},\end{aligned} \tag{59}$$

$$\tau<0,\quad \Delta_0\ll|\tau|\ll1:\quad f=\frac{h}{(-\tau)^q}\left\{B_0+B_1\frac{h}{(-\tau)^{t/s}}+\dots\right\},$$

$$\Delta_0=h^{s/t},\quad t/s=t+q.$$

Here Δ_0 is the size of the region of smearing[10] and the numerical coefficients are of the order of unity.

In an analogous manner, for the function φ in the critical region we shall have

$$|\tau|\ll1,\quad h\ll1:\quad \varphi=h^u\Phi(\tau/h^{u/l}),\quad l/u={}^1/_2(l+r). \tag{60}$$

The leading terms of the asymptotic forms of φ are given by the expressions (55)–(57). The relation in (60) between the critical indices can be obtained in the usual manner from the matching of (56) with (57) at $|\tau|\sim\Delta_1$, where $\Delta_1=h^{u/l}$ is the size of the region of smearing for the function φ. According the scaling hypothesis, all critical phenomena (for $\mathbf{H}=0$) are characterized by a single scale, so that we must require equality of Δ_1 and Δ_0. In this case, the following relation between the critical exponents should be fulfilled:

$$l/u=t/s. \tag{61}$$

Thus, for consistent application of the scaling hypothesis the function φ is characterized by one new (additional to the indices of the function f) independent critical exponent, e.g., l. The other two exponents r and u can be expressed in terms of l and t, q:

$$r=2(t+q)-l, \quad u=l/(t+q). \tag{62}$$

For the Hall coefficient ($R=H^{-1}\sigma_a/\sigma^2$) we obtain from (13)

$$R_e=h^2R_2f^{-2}+(R_1-h^2R_2)\mathscr{R}, \quad \mathscr{R}(p, h)=\varphi(p, h)/f^2(p, h). \tag{63}$$

We emphasize that in the approximation linear in $\mathbf{H}$ the formula (63) is exact and the parameters in it can take any values. For a system with a metal-insulator phase transition, in the critical region $|\tau|\ll 1$, $h\ll 1$ we obtain for the function $\mathscr{R}$ from (55)–(62)

$$\begin{gathered}\mathscr{R}=h^{-k}\mathscr{F}(\tau/h^{s/t}),\\ \tau>0,\quad \Delta_0\ll\tau\ll 1:\quad \mathscr{R}\sim\tau^{-g},\\ |\tau|\ll\Delta_0:\quad \mathscr{R}\sim h^{-k},\\ \tau<0,\quad \Delta_0\ll|\tau|\ll 1:\quad \mathscr{R}\sim(-\tau)^{-g},\end{gathered} \tag{64}$$

where

$$g=2t-l=r-2q, \quad k=2s-u=g/(t+q). \tag{65}$$

The equality of the critical exponents above ($\tau>0$, $\tau\gg\Delta_0$) and below ($\tau<0$, $|\tau|\gg\Delta_0$) the transition point is a consequence of the first relation (62). The expression (63), together with (58), (59) and (64), (65), gives a complete (in the sense of the scaling hypothesis) description of the critical behavior of R_e that is valid for arbitrary values of the galvanomagnetic characteristics of the components.

For the description of the Hall coefficient in the region of the metal–insulator phase transition, Shklovskiĭ proposed the following interpolation formula (in the notation of the present paper):

$$R_e=h^2R_2f^{-2}+R_1\mathscr{R}, \quad \mathscr{R}=(\tau^2+\Delta_0^2)^{-g/2} \tag{66}$$

with exponent $g=\nu$, where ν is the critical exponents of the correlation length. In its structure (66) differs from the exact expression only by the replacement $R_1\to R_1-h^2R_2$ in the second term. This difference is unimportant in the entire critical region, so that the approximate nature of the formula (66) consists mainly in the interpolation character of the description of the function $\mathscr{R}$. It should be noted that this interpolation is extremely successful, since it reproduces in order of magnitude all the asymptotic forms (64) and gives the exact relationship between the indices k and g: $k=gs/t=g/(t+q)$; compare with (65). Thus, the interpolation formula (66) gives a correct qualitative description of the Hall coefficient in the entire critical region. At the same time, it is evident that the critical exponent g must be determined by of a numerical experiment.

In the papers of Skal[5,8] the Hall coefficient of isotropic two-component systems was investigated by numerical methods, with the use of an expression of the type (52) for R_e. In the limit $R_2=0$ considered in Ref. 8, from (63) we have $R_e/R_1=\mathscr{R}$. In this case, according to Ref. 8, there is equality (within the error bars of the calculation) of the critical exponents indices above and below the transition point; the numerical value of the corresponding exponent is $g=0.6\pm0.1$ (Refs. 5, 8) (in Refs. 5 and 8 this exponent was denoted by f). In the second limiting case $R_1=0$ the term with $\mathscr{R}$ in (63) for $|\tau|\ll 1$ and $h\ll 1$ is small, and the critical behavior of the Hall coefficient is determined, primarily, by the electrical conductivity f. Correspondingly, in Ref. 8 it was found that within the error bars of the calculations the critical exponents of R_e (for $R_1=0$) coincide with the indices of the quantity f^{-2}.

Thus, the results of tbe numerical experiments of Refs. 5 and 8 agree with the conclusions of the present paper and make it possible to find the critical exponent g that is not determined in the theory: $g=0.6\pm0.1$. If for t and q we make use of the values given in Ref. 8 ($t=1.6\pm0.1$, $q=0.7\pm0.1$), then for the remaining indices we obtain

$$l=2.6\pm0.3;\quad r=2.0\pm0.3;\quad u\gtrsim 1,\quad k\approx 1/4.$$

It is of considerable interest to determine all these indices from independent results of a numerical investigation of the quantities R_e, φ, and f. Such a complex numerical experiment would serve as a serious check on the validity of the scaling hypothesis. We note also that a numerical investigation of the function $\varphi(p,h)$ (i.e., not only of $\mathscr{R}=\varphi/f^2$) in the entire range of variation of p and h is necessary, since φ also appears in the expression for the transverse magnetoresistance.

7. THE APPROXIMATION QUADRATIC IN H

In the approximation quadratic in $\mathbf{H}$, from the identity [see (48)]

$$\langle\mathbf{E}_0^{(\mu)}\rangle\langle\mathbf{j}_2^{(\nu)}\rangle=\langle\mathbf{E}_0^{(\mu)}\mathbf{j}_2^{(\nu)}\rangle,$$

where

$$\mathbf{j}_2^{(\nu)}=\sigma\mathbf{E}_2^{(\nu)}+\hat{\sigma}_a\mathbf{E}_1^{(\nu)}+\gamma_x\mathbf{E}_0^{(\nu)}+(\gamma_z-\gamma_x)\mathbf{n}(\mathbf{n}\mathbf{E}_0^{(\nu)}),$$

we obtain

$$\begin{aligned}&\langle\mathbf{E}_0^{(\mu)}\rangle\gamma_{xe}\langle\mathbf{E}_0^{(\nu)}\rangle+(\mathbf{n}\langle\mathbf{E}_0^{(\mu)}\rangle)(\gamma_{ze}-\gamma_{xe})(\mathbf{n}\langle\mathbf{E}_0^{(\nu)}\rangle)\\ &=\langle\mathbf{E}_0^{(\mu)}\gamma_x\mathbf{E}_0^{(\nu)}\rangle\\ &\quad+\langle(\mathbf{n}\mathbf{E}_0^{(\mu)})(\gamma_z-\gamma_x)(\mathbf{n}\mathbf{E}_0^{(\nu)})\rangle\\ &\quad+\langle\sigma_a([\mathbf{E}_0^{(\mu)},\mathbf{E}_1^{(\nu)}]\mathbf{n})\rangle,\quad \mathbf{n}=\mathbf{H}/H.\end{aligned} \tag{67}$$

In the derivation of (67) we have also used the identity $\langle\mathbf{E}_2^{(\nu)}\rangle\langle\mathbf{j}_0^{(\mu)}\rangle=\langle\mathbf{E}_2^{(\nu)}\mathbf{j}_0^{(\mu)}\rangle$, allowance for which leads to cancellation of the terms with $\mathbf{E}_2$. In addition, it is not difficult to verify oneself that all the terms containing $\langle\mathbf{E}_1\rangle$ drop out of the final expressions for γ_{xe} and γ_{ze}. This circumstance has been taken into account in (67), in which for $\mathbf{E}_1$ we must use the expression (46) with $\langle\mathbf{E}_1\rangle=0$, which in this case can also be written in the form

$$\mathbf{E}_1^{(\nu)}(\mathbf{r})=\nabla_{\mathbf{r}}\int d\mathbf{r}'\,\sigma_a(\mathbf{r}')([\nabla_{\mathbf{r}'},\mathbf{E}_0^{(\nu)}(\mathbf{r}')g(\mathbf{r},\mathbf{r}')]\mathbf{n}). \tag{68}$$

Let $\mathbf{n} = \mathbf{H}/H$ be directed, as in (6), along the z axis. Then, setting $\mu = \nu = x$ in (67) (so that $\langle E_{0\alpha}^{(x)}\rangle = \langle E_{0x}^{(x)}\rangle\delta_{\alpha x}$), we find a general expression for the quantity γ_{xe}. In the case of two-component systems, for γ_{xe} we obtain the formula (15) with the following coefficients $\psi_x^{(a)}$:

$$\begin{aligned}
\psi_x^{(1)} &= \langle (E_{0x}^{(x)})^2 + (E_{0y}^{(x)})^2\rangle^{(1)}/\langle E_{0x}^{(x)}\rangle^2,\\
\psi_x^{(2)} &= \langle (E_{0z}^{(x)})^2\rangle^{(1)}/\langle E_{0x}^{(x)}\rangle^2,\\
\psi_x^{(3)} &= \langle (E_{0x}^{(x)})^2 + (E_{0y}^{(x)})^2\rangle^{(2)}/\langle E_{0x}^{(x)}\rangle^2,\\
\psi_x^{(4)} &= \langle (E_{0z}^{(x)})^2\rangle^{(2)}/\langle E_{0x}^{(x)}\rangle^2,
\end{aligned} \tag{69}$$

where $\langle\ldots\rangle^{(i)}$ is the same as in (25). Taking into account the identity (21) and the condition $\langle \mathbf{E}_1\rangle = 0$, we can write the term with σ_a in (67) in the form

$$\langle \sigma_a([\mathbf{E}_0^{(\mu)}\mathbf{E}_1^{(\nu)}]\mathbf{n})\rangle = (\sigma_{a1}-\sigma_{a2})\langle([\mathbf{E}_0^{(\mu)}\mathbf{E}_1^{(\nu)}]\mathbf{n})\rangle^{(1)} \tag{70}$$

Next, the expression (68) for $\mathbf{E}_1(\mathbf{r})$ after substitution of (53) takes the form (with allowance for the condition $g(\mathbf{r},\mathbf{r}') = 0$ on the boundary of the sample)

$$\begin{aligned}
\mathbf{E}_1^{(\nu)}(\mathbf{r}) &= \frac{\sigma_{a1}-\sigma_{a2}}{\sigma_1}\mathbf{e}_1^{(\nu)}(\mathbf{r}),\\
\mathbf{e}_1^{(\nu)}(\mathbf{r}) &= \sigma_1\nabla_{\mathbf{r}}\int_{V_1} d\mathbf{r}'[\nabla_{\mathbf{r}'},\mathbf{E}_0^{(\nu)}(\mathbf{r}')g(\mathbf{r},\mathbf{r}')]_z.
\end{aligned} \tag{71}$$

Then for the function χ_x we obtain the following expression:

$$\chi_x = \langle[\mathbf{E}_0^{(x)}\mathbf{e}_1^{(x)}]_z\rangle^{(1)}/\langle E_{0x}^{(x)}\rangle^2. \tag{72}$$

We note that the quantities $\psi_x^{(a)}$ from (69) can be expressed in terms of the functions $a^{(i)}$, $b^{(i)}$, and $c^{(i)}$ defined in (34):

$$\psi_x^{(1)} = 2a^{(1)}+b^{(1)}+c^{(1)},\quad \psi_x^{(2)} = a^{(1)},\quad \psi_x^{(3)} = 2a^{(2)}+b^{(2)}+c^{(2)},\quad \psi_x^{(4)} = a^{(2)}. \tag{73}$$

From (73), when the equalities (35) (with $D = 3$) are taken into account, the relations (19) follow.

The quantity γ_{ze} is considered in an analogous manner. Putting $\mu = \nu = z$ in (67) (for $\mathbf{n}\|z$), for γ_{ze} in the case of two-component systems we obtain the expression (16) with coefficients

$$\begin{aligned}
\psi_z^{(1)} &= \langle (E_{0x}^{(z)})^2 + (E_{0y}^{(z)})^2\rangle^{(1)}/\langle E_{0z}^{(z)}\rangle^2,\\
\psi_z^{(2)} &= \langle (E_{0z}^{(z)})^2\rangle^{(1)}/\langle E_{0z}^{(z)}\rangle^2,\\
\psi_z^{(3)} &= \langle (E_{0x}^{(z)})^2 + (E_{0y}^{(z)})^2\rangle^{(2)}/\langle E_{0z}^{(z)}\rangle^2,\\
\psi_z^{(4)} &= \langle (E_{0z}^{(z)})^2\rangle^{(2)}/\langle E_{0z}^{(z)}\rangle^2,\\
\chi_z &= \langle[\mathbf{E}_0^{(z)},\mathbf{e}_1^{(z)}]_z\rangle^{(1)}/\langle E_{0z}^{(z)}\rangle^2,
\end{aligned} \tag{74}$$

where $\mathbf{e}_1^{(z)}$ is given by the formula (71) with $\nu = z$. The quantities $\psi_z^{(a)}$ can also be expressed in terms of the functions $a^{(i)}$, $b^{(i)}$, and $c^{(i)}$ defined in (34):

$$\begin{aligned}
&\psi_z^{(1)} = 2a^{(1)},\quad \psi_z^{(2)} = a^{(1)}+b^{(1)}+c^{(1)},\\
&\psi_z^{(3)} = 2a^{(2)},\quad \psi_z^{(4)} = a^{(2)}+b^{(2)}+c^{(2)}.
\end{aligned} \tag{75}$$

Comparison of (75) with (73) leads to the relations (18).

Because they are rather cumbersome, we shall not write out the easily derivable expressions for the transverse and longitudinal magnetoresistances in the approximation quadratic in $\mathbf{H}$. Moreover, a detailed study of the critical behavior of the magnetoresistance is possible only after a detailed investigation (first of all, numerical) of the properties of the functions $\psi_x^{(a)}$, $\psi_z^{(a)}$, χ_x and χ_z in the neighborhood of the percolation threshold. Such an investigation goes beyond the scope of the present article, and so we shall confine ourselves to the following remarks.

As already noted, the quantities γ_{xe} and γ_{ze} are described by not more than four new (additional to f and φ) functions, e.g., $\psi_x^{(1)}$ and $\psi_x^{(3)}$, χ_x and χ_z. Consistent application of the scaling hypothesis (similar to that performed for the function φ in the preceding section) to each of them gives one independent exponent. Therefore, according to the analysis performed here, in the framework of the scaling hypothesis the quantities γ_{xe} and γ_{ze} are described by not more than four new critical exponents. We note also that when one considers the properties of the functions $\psi_x^{(a)}$, $\psi_z^{(a)}$, χ_x, and χ_z "in the large", i.e., in the entire range of variation of the concentration, the results of solving the problem of the galvanomagnetic properties of two-component systems by the method of effective-medium theory may turn out to be useful—see, e.g., Ref. 17.

In a numerical experiment the functions $\psi_x^{(a)}$ and $\psi_z^{(a)}$ can be determined in the framework of the electrical-conductivity problem (for $\mathbf{H} = 0$) by using their explicit form—see (69) and (74). At the same time, the expressions for χ_x and χ_z contain the Green function $g(\mathbf{r},\mathbf{r}')$, and this substantially complicates the calculation of these quantities. Therefore, it is clear that the functions χ_x and χ_z must be determined by direct modeling of the galvanomagnetic phenomena (for $\gamma_{xi} = \gamma_{zi} = 0$) on lattices. Tabulation of $\psi_x^{(a)}$, $\psi_z^{(a)}$, χ_x, and χ_z (and also of f and φ) for all p and h will make it possible to give a description of the galvanomagnetic properties of isotropic two-component media in a weak magnetic field in the entire range of variation of the parameters of the problem.

8. THE TWO-DIMENSIONAL CASE

For two-dimensional two-component systems the problem of the galvanomagnetic properties has an exact solution for arbitrary magnetic fields—see Refs. 6 and 7. Therefore, an analysis of this problem (for $D = 2$) by means of the approach used in the present paper will make it possible to check the method proposed. In the two-dimensional case it is possible, in particular, to find the explicit form of the function χ (analogous to the functions χ_x and χ_z in the three-dimensional problem) by expressing it in terms of f. This gives the possibility of investigating the basic properties of χ for all p and h.

The function $\varphi(p,h)$, appearing in the quantity σ_{ae} and defined by (54), can be found for $D = 2$ as follows. We write out the identities (21) (with $\mathbf{E}$ replaced by $\mathbf{E}_0$) and (23) (for $\mathbf{j}\to\mathbf{j}_0$) in the form of a sum over the components—see (25). From these two equations we determine the quantity $\langle[\mathbf{E}_0^{(\mu)}\mathbf{E}_0^{(\nu)}]_z\rangle^{(1)}$, and thereby the function φ:

$$\varphi(p,h) = (\sigma_e^2-\sigma_2^2)/(\sigma_1^2-\sigma_2^2) = (f^2-h^2)/(1-h^2). \tag{76}$$

Substitution of (76) into (63) gives for the effective Hall coefficient the expression obtained in Refs. 2 and 3.

We set $\sigma_{xe} = \sigma_e + \gamma_e$ and $\sigma_{xi} = \sigma_i + \gamma_i$. Then in the approximation quadratic in $\mathbf{H}$ we shall have

$$\gamma_e=\gamma_1\psi_1+\gamma_2\psi_2+\frac{(\sigma_{a1}-\sigma_{a2})^2}{\sigma_1}\chi, \tag{77}$$

where

$$\psi_1=\langle(\mathbf{E}_0^{(x)})^2\rangle^{(1)}/\langle E_{0x}^{(x)}\rangle^2, \quad \psi_2=\langle(\mathbf{E}_0^{(x)})^2\rangle^{(2)}/\langle E_{0x}^{(x)}\rangle^2,$$

$$\frac{(\sigma_{a1}-\sigma_{a2})^2}{\sigma_1}\chi=\langle\sigma_a[\mathbf{E}_0^{(x)}\mathbf{E}_1^{(x)}]_z\rangle/\langle E_{0x}^{(x)}\rangle^2$$

$$=(\sigma_{a1}-\sigma_{a2})\langle[\mathbf{E}_0^{(x)}\mathbf{E}_1^{(x)}]_z\rangle^{(1)}/\langle E_{0x}^{(x)}\rangle^2. \tag{78}$$

Expressions for ψ_1 and ψ_2 follow directly from (28):

$$\psi_1=f-hf', \quad \psi_2=f'. \tag{79}$$

In order to find the function χ, in the identity (21) we set $\mathbf{E}^{(\mu)}=\mathbf{E}_0^{(x)}$ and $\mathbf{E}^{(\nu)}=\mathbf{E}_1^{(x)}$, and in (23) we set $\mathbf{j}^{(\mu)}=\mathbf{j}_0^{(x)}$ and $\mathbf{j}^{(\nu)}=\mathbf{j}_1^{(x)}$, where $\mathbf{j}_1$ is given in (44) and $\langle\mathbf{E}_1^{(x)}\rangle=0$. Proceeding as in the derivation of (76), for $\chi(p,h)$ we obtain

$$\chi=(f-hf'-f\varphi)/(1-h^2) \tag{80}$$

with φ from (76). The formulas (77), (79), and (80) coincide with the result of expanding the general expression for σ_{xe} from Ref. 6 to terms of order H^2 inclusive.

For the quantity $\Delta\rho_e/\rho_e=[\rho_{xe}(\mathbf{H})-\rho_{xe}(0)]/\rho_{xe}(0)$, where $\rho_x=\sigma_x/(\sigma_x^2+\sigma_a^2)$, in the approximation quadratic in **H** we have

$$\frac{\Delta\rho_e}{\rho_e}=-\left(\frac{\gamma_e}{\sigma_e}+\frac{\sigma_{ae}^2}{\sigma_e^2}\right). \tag{81}$$

Here σ_e is specified in (5), σ_{ae} in (13) and (76), and γ_e in (77), (79), and (80). Thus, for two-dimensional systems the magnetoresistance can be expressed in terms of the dimensionless electrical conductivity f and its derivative f'. Therefore, the use for f of formulas (58), (59) (with $s=1/2$) makes it possible in this case to give a complete (within the framework of the scaling hypothesis) description of the critical behavior of $\Delta\rho_e/\rho_e$. We note also that the quantity f' can be expressed in terms of the permittivity of the medium in a quasistationary electric field—see Ref. 13. Consequently, the relation (81) (with the definitions of the quantities γ_e and σ_{ae}) is subject to direct experimental verification by a simultaneous investigation of the Hall coefficient, the magnetoresistance, and the low-frequency dispersion of the electrical conductivity of thin films.

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