Calculation of the asymptotic form of the spectrum of locally isotropic turbulence in the viscous range

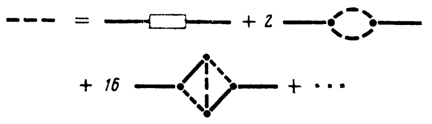
M. M. Dubovikov and V. I. Tatarskiĭ

Institute of Atmospheric Physics, Academy of Sciences of the USSR, Moscow
(Submitted 27 May 1987)
Zh. Eksp. Teor. Fiz. **93**, 1992–2001 (December 1987)

The turbulence spectrum in the dissipation range is investigated using a diagram technique. It is shown that for this interval the Chandrasekhar equation is approximately valid (in the coordinate representation). This equation, continued into the complex plane, has singularities, and analysis of the isolated singularity closest to the real axis makes it possible to determine the short-wavelength asymptotic form of the turbulence spectrum and show that it depends on one free parameter. Integral relations are given that make it possible to establish uniquely the dependence of this parameter on the constant appearing in the Kolmogorov-Obukhov "5/3 law", and thereby to determine the point at which the asymptotic form is joined with the power-law part of the spectrum. After this joining the spectrum obtained agrees well with the published experimental data over the entire range.

1. INTRODUCTION. BASIS OF THE MODEL

The fully-developed-turbulence spectrum predicted in the inertial range by the theory of Kolmogorov and Obukhov[1–4] has been studied in detail in experiments[5–7] (see also Ref. 8) both in the inertial range and in the dissipation range. However, as is well known, the problem of explaining its behavior on the basis of the equations of hydrodynamics, even assuming time-independence, homogeneity, and local isotropy, encounters difficulties that are typical for systems with strong coupling: It has not been possible to represent such a system in the form of a set of noninteracting (or weakly interacting) subsystems for each of which the problem posed could be solved exactly, thereby yielding a starting solution, albeit in the form of a series in a real small parameter. As a result, we can write down only a formal expression for the spectrum in the form of an infinite series, the combinatoric structure of which is described in the language of Wyld diagrams.[9] Such series are usually formal solutions of equations of the Dyson type. In particular, when the problem is posed with a random external force the Dyson equation acquires the form


(1)

In it there appear exact values of the spectral density (the thick dashed lines), exact Green's functions (the thick solid lines), exact vertices (the heavy points), and the spectral density of the external force (the rectangle); the series (not given here) corresponding to the exact vertices and exact Green's functions are also formal. In order to give meaning to such series, one often uses assumptions[10–12] of the scaling-hypothesis type, which, in a certain sense, fulfill the function of a small parameter specifying the manner in which the diagrams are to be summed. It is clear that the necessity for this is a matter of principle, and does not depend on the specific description of fully developed turbulence (e.g., by means of the Hopf equation in variational derivatives for the characteristic functional,[13,14] or, equivalently, by means of the corresponding path integral.[15] It is natural, therefore, that the results which are based entirely on the hydrodynamic equations (e.g., those of Refs. 16–19), while undoubtedly being of great interest, are, in the main, very far from direct contact with experiment. One should specially note Refs. 20 and 21, in which, in particular, the asymptotic nonuniqueness of the solutions of the time-dependent Navier-Stokes equations, which leads to turbulence, is proved; for the averages a closed system of equations is presented and a uniqueness theorem is proved.

However, most of the theoretical papers that aim to explain the experimental data either have a phenomenological character or rest on model assumptions. For example, according to the theory of Kolmogorov and Obukhov[1–4] the spectral density of the energy is

$$E(k)\sim k^{-5/3}g(\eta,k), \tag{2}$$

where $g(\varkappa)\to\text{const}$ as $\varkappa=\eta k\to 0$.

We focus on the result of Ref. 22, in which the "5/3 law" was obtained, together with the constant C_1 appearing in it, on the basis of a certain reduction of the Navier-Stokes equations in the discrete cascade system proposed by Obukhov for the description of cascade processes in fully developed turbulence.[23–25] (We use the following standard notation: ε is the rate of dissipation of energy, ν is the kinematic viscosity, and $\eta=\nu^{3/4}\varepsilon^{-1/4}$ is the Kolmogorov scale. For $r\gg\eta$, according to the Kolmogorov-Obukhov law, the longitudinal structure function is $D_{LL}(r)=C(\varepsilon r)^{2/3}$. For $\eta k\ll 1$ the three-dimensional spectrum, integrated over the angles, is $E(k)=C_1\varepsilon^{2/3}k^{-5/3}$. The one-dimensional longitudinal spectrum is given by $E_{LL}(k)=C_2\varepsilon^{2/3}k^{-5/3}$. The constants C, C_1, and C_2 are connected by the relations $C_1=55C/27\Gamma(1/3)\cong 0.760C$ and $C_2=18C_1/55\cong 0.327C_1$.)

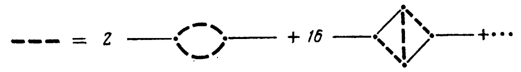
From the papers relating to the investigation of the spectrum for $\eta k\gg 1$ we note the model result of Ref. 26, based on the assumption that the interaction of small-scale vortices is unimportant:

$$E(\eta k)\sim\exp\{-(\eta k)^2\}\quad\text{for}\quad \eta k\gg 1. \tag{3}$$

The most systematic investigation (in the sense of providing

0038-5646/87/121136-06$04.00

support for the starting equations) is the investigation of the short-wavelength asymptotic form carried out in Ref. 27. In this paper it was shown that, for $\eta k \gg 1$, in the right-hand side of (1) one can replace the exact vertices (the heavy points) by bare vertices (points). In addition, since for $\eta k \gg 1$ the turbulent viscosity coincides with the molecular viscosity, one can also replace the exact Green's functions (the heavy solid lines) by bare Green's functions (thin lines), after which we go over to the equation


(4)

In (4) the first term of the right-hand side of (1) has been omitted, since it is assumed that the spectrum of the external force is concentrated in the region $\eta k \ll 1$. But the same difficulties are also encountered in solving the simplified equation (4), since its right-hand side is a series without a small parameter. At the same time, in systems with strong coupling the situation is often such that, in the region of interest, the first term of the formal diagrammatic series, being itself a sum of an infinite number of subdiagrams, gives a largely (to within a few tens per cent) correct answer, to judge both from comparison with the experimental data and from an estimate of the contribution of the next diagram. This situation is encountered, e.g., in quantum chromodynamics,[28] and also in the calculation of critical exponents in the theory of second-order phase transitions.[29] If we argue on the basis of Ref. 27, it is apparent that this situation also obtains in the present case. In fact, in Ref. 27 it is shown that the leading asymptotic term of the solution of the equation


(5)

for $\eta k \to \infty$ gives for the spectrum the expression

$$E(k)=30\nu^2 k(\eta k)^2 e^{-\eta k}, \tag{6}$$

while allowance for the second diagram in the right-hand side of (4) causes the coefficient 30 in (6) to be replaced by 23. The possibility of such a solution was first pointed out in Ref. 30.

In the present paper, on the basis of the above arguments, we too shall adhere to the assumption that the asymptotic form determined from Eq. (5) is close to the true asymptotic form of the spectrum. The principal parameter of the asymptotic form is the exponent in formula (6); in fact, in Ref. 27 this exponent was not determined, since a more detailed analysis[30] points to the possibility of making in formula (6) the replacement $\eta \to \xi\eta$, $e^{-\eta k} \to e^{-\xi\eta k}$, where ξ is an arbitrary number. According to the experimental data, this number can be very significant. Below, on the basis of Eq. (5), we shall give an estimate of ξ and refine the pre-exponential factor in (6). At the same time, using certain integral relations, we succeed in establishing a relationship between the quantity ξ and the value of the dimensionless constant C_1 appearing in the Kolmogorov-Obukhov "5/3 law". This makes it possible to determine uniquely the point where the asymptotic forms of the spectrum for $\eta k \to \infty$ and for $\eta k \to 0$ join. The short-wavelength asymptotic form obtained is in good agreement with experiment for values of ηk greater than $\sim 1/\xi$ ($\xi \cong 10$), and, together with the known power-law asymptotic form for $\eta k \to 0$, gives a smooth approximation of the entire spectrum.

2. ASYMPTOTIC FORM OF THE SPECTRAL DENSITY FOR $\eta k \to \infty$. THE BASIC EQUATION

In accordance with what has been said above, the spectral density decays exponentially in the region $\eta k \gg 1$. The problem that we set ourselves in this section is to estimate the exact value of the exponent for the spectral density and to refine the pre-exponential factor. Both the possibility of solving this problem and the method of solution stem from the following considerations. The exponent in the k-representation is determined by the position of the isolated singularity (in $r = |\mathbf{r}|$ in the coordinate representation) nearest to the real axis. The pre-exponential factor is determined by the character of this singularity. At the same time, the position and character of the singular point can be studied using Eq. (5), written in the r-representation for an arbitrary correlation function[1)] $B_{LL}(r,t)$. This equation, as noted in Ref. 8, coincides with the Chandrasekhar equation, the validity of which in the region $r \ll \eta$ was already noted in Ref. 9.

We turn to the realization of this program. If we introduce the dimensionless variables $\rho = r/\eta$, $\tau = \nu t/\eta^2$, $b(\rho,\tau) = \eta^2\nu^{-2}B_{LL}$, the Chandrasekhar equation can be rewritten as

$$\begin{aligned}&\frac{\partial}{\partial\rho}\left(\frac{\partial^2}{\partial\tau^2}-\left(\frac{\partial^2}{\partial\rho^2}+\frac{4}{\rho}\frac{\partial}{\partial\rho}\right)^2\right)b(\rho,\tau)\\&=b(\rho,\tau)\frac{\partial}{\partial\rho}\left(\frac{\partial^2}{\partial\rho^2}+\frac{4}{\rho}\frac{\partial}{\partial\rho}\right)b(\rho,\tau).\end{aligned} \tag{7}$$

The function $b(\rho,\tau)$ is related, by a Fourier transformation

$$b(\rho,\tau)=\int_0^\infty \tilde{E}_{LL}(\varkappa,\tau)\cos(\varkappa\rho)\,d\varkappa=b(-\rho,\tau), \quad \varkappa=\eta k, \tag{8}$$

to the dimensionless one-dimensional longitudinal spectral density $\tilde{E}_{LL}(\varkappa,\tau)$. For $\tau = 0$ it coincides with the purely spatial spectral density $\tilde{E}_{LL}(\varkappa)$, which is what is usually measured in experiments. From the inversion formula

$$\tilde{E}_{LL}(\varkappa,\tau)=\pi^{-1}\int_{-\infty}^{\infty} b(\rho,\tau)e^{i\varkappa\rho}\,d\rho \tag{9}$$

it follows that the asymptotic form of $\tilde{E}_{LL}$ for $\varkappa \to \infty$ is determined by that isolated singularity of the function $b(\rho,\tau)$ which is nearest to the real axis in the region of complex values of ρ. We shall determine from (7) the possible form of this singularity. If we assume that in the neighbourhood of the singular point $\rho = i\rho_0(\tau)$ (it will be shown below that $\rho_0^* = \rho_0$) $b(\rho,\tau)$ has the form $b(\rho,\tau) \sim (\rho - i\rho_0(\tau))^{-\lambda}$, then, taking into account that the left-hand side contains fifth-order derivatives with respect to ρ while the right-hand side contains third-order derivatives with respect to ρ, we obtain $\lambda + 5 = 2\lambda + 3$, whence $\lambda = 2$. Thus, near the pole $\rho = i\rho_0$ we can write an expansion of the form

$$\begin{aligned}b(\rho,\tau)=&\frac{a_2(\tau)}{[\rho-i\rho_0(\tau)]^2}+\frac{ia_1(\tau)}{\rho-i\rho_0(\tau)}\\&+b_0(\tau)+ib_1(\tau)[\rho-i\rho_0(\tau)]\\&+b_2(\tau)[\rho-i\rho_0(\tau)]^2+ib_3[\rho-i\rho_0(\tau)]^3\\&+b_4(\tau)[\rho-i\rho_0(\tau)]^4+\ldots.\end{aligned} \tag{10}$$

Substituting (10) into (7), in both sides of the equality we obtain poles of up to seventh order. Comparison of the coefficients of the poles, from seventh-order to third-order, gives the relations

$$a_2(\tau)=-30,\quad a_1(\tau)=-\frac{120}{13\rho_0(\tau)}\qquad b_0=-\dot{\rho}_0^2+\frac{1596}{169\rho_0^2},$$

$$b_1(\tau)=\frac{16\,812}{2197\rho_0^3}+\frac{\dot{\rho}_0^2}{\rho_0}+\frac{\ddot{\rho}_0}{4},$$

$$b_2(\tau)=\frac{14}{39}\frac{\dot{\rho}_0^2}{\rho_0^2}+\frac{19}{78}\frac{\ddot{\rho}_0}{\rho_0}-\frac{193\,944}{28\,561\rho_0^4},\ldots. \tag{11}$$

We substitute the resulting expansion (10) into (9) and displace the contour of integration for $\varkappa>0$ upward. Then the residue at the point $\rho=i\rho_0$ gives the leading term of the asymptotic expansion:

$$\tilde{E}_{LL}(\varkappa,\tau)=60\left[\varkappa+\frac{4}{13\rho_0(\tau)}\right]e^{-\varkappa\rho_0(\tau)}+\ldots. \tag{12}$$

The regular part of the expansion (10) gives an exponentially small contribution in comparison with (8), since it is essentially a series expansion, at the point $\rho=i\rho_0$, of the contributions of the singularities lying above the point $i\rho_0$. Since, by definition, $\tilde{E}^*_{LL}=\tilde{E}_{LL}$, it follows from (12) that $\rho_0^*=\rho_0$. We note that the function $\tilde{E}(\varkappa)$, which gives the three-dimensional isotropic spectral density, is connected[8] with $\tilde{E}_{LL}$ by the relation

$$\tilde{E}(\varkappa)=\frac{\varkappa^3}{2}\frac{d}{d\varkappa}\left[\frac{1}{\varkappa}\frac{d}{d\varkappa}\tilde{E}_{LL}(\varkappa)\right]. \tag{13}$$

With the notation

$$\rho_0(0)=\xi,$$

$$\tilde{E}_{LL}(\varkappa)=60\left(\varkappa+\frac{4}{13\xi}\right)e^{-\xi\varkappa}+\ldots=\psi(\varkappa)+\ldots \tag{14}$$

this leads to the asymptotic formula

$$\tilde{E}(\varkappa)=30\varkappa\left[(\xi\varkappa)^2-\frac{9}{13}(\varkappa\xi+1)\right]e^{-\xi\varkappa}+\ldots. \tag{15}$$

The formula (6) can be obtained from this with $\xi=1$, if we neglect the second term in the pre-exponential factor.

3. ESTIMATION OF THE EXPONENT OF THE EXPONENTIAL DECAY OF THE SPECTRUM FROM THE NORMALIZATION CONDITION

To estimate ξ we shall make use of the well-known[8] integral relation

$$\int_0^\infty \varkappa^2\tilde{E}_{LL}(\varkappa)\,d\varkappa=\frac{1}{15}, \tag{16}$$

which expresses the definition of the quantity ε in terms of $\tilde{E}_{LL}$. In the region $\varkappa\ll1$, according to the Kolmogorov-Obukhov law, we have for $\tilde{E}_{LL}(\varkappa)$ the expression

$$\tilde{E}_{LL}(\varkappa)=C_2\varkappa^{-5/3}\equiv\varphi(\varkappa). \tag{17}$$

On the basis of the experimental data analyzed in Ref. 7, and from the theoretical estimate of Ref. 22, the constant C_2 is of the order of 0.42–0.45. In the region $\varkappa\gg1$, for $\tilde{E}_{LL}$ the asymptotic estimate (14) is valid. We shall divide the region of integration in (16) into the segments $(0,\varkappa_0)$ and $(\varkappa_0,\infty)$. On the first of these we substitute into (16) the function $\varphi(\varkappa)$ defined by formula (17), and on the second we substitute the function $\psi(\varkappa)$ from (14). Then the relation (16) takes the form

$$C_2\int_0^{\varkappa_0}\varkappa^{1/3}\,d\varkappa+60\int_{\varkappa_0}^\infty\left(\varkappa^3+\frac{4\varkappa^2}{13\xi}\right)e^{-\xi\varkappa}\,d\varkappa=\frac{1}{15}, \tag{18}$$

or, if we perform the integration and denote $\xi\varkappa_0=\alpha$,

$$\frac{3}{4}C_2\left(\frac{\alpha}{\xi}\right)^{4/3}+\frac{60e^{-\alpha}}{\xi^4}\left(\alpha^3+\frac{43}{13}\alpha^2+\frac{86}{13}\alpha+\frac{86}{13}\right)=\frac{1}{15}. \tag{19}$$

If in (19) we now fix a definite value of $\varkappa_0$, we obtain a transcendental equation for ξ, which is easily solved numerically. To estimate $\varkappa_0$, and hence ξ, in the first approximation we make use of the fact that for the values of C_2 of interest, in the region in which Eq. (19) has a solution, the functions $\varphi(\varkappa)$ and $\psi(\varkappa)$ do not intersect. Therefore, it is natural to choose as $\varkappa_0$ that value of $\varkappa$ for which $\varphi(\varkappa)/\psi(\varkappa)=\min$, i.e., for which on a logarithmic scale (see the figure below) the distance between the asymptotic expansions (14) and (17) is a minimum. This leads to the condition

$$\alpha=\xi\varkappa_0=\frac{46}{39}+\left[\left(\frac{46}{39}\right)^2+\frac{20}{39}\right]^{1/2}\approx2.559\ldots, \tag{20}$$

and when this is taken into account (19) can be rewritten in the form

$$2.626C_2/\xi^{4/3}+287.6/\xi^4=1/15. \tag{21}$$

The solution of Eq. (21) turns out to be stable against variations of C_2. Thus, $\xi=11.10$ for $C_2=0.45$, and $\xi=10.85$ for $C_2=0.42$ (see Sec. 5, in which this result is compared with experimental data).

4. REFINEMENT OF THE DEPENDENCE $\xi=\xi(C_2)$ WITH ELIMINATION OF THE ARBITRARINESS IN THE CHOICE OF THE POINT WHERE THE ASYMPTOTIC FORMS JOIN

Above, we obtained an estimate of ξ based on an entirely natural but nevertheless arbitrary assumption about the position of the joining point $\varkappa_0$. In this section we get rid of this arbitrariness and obtain a unique (in the framework of the model under consideration) dependence $\xi=\xi(C_2)$. We shall start from the relation

$$\oint b(\rho,\tau)\frac{\partial}{\partial\rho}\left(\frac{\partial^2b(\rho,\tau)}{\partial\rho^2}+\frac{4}{\rho}\frac{\partial b(\rho,\tau)}{\partial\rho}\right)d\rho$$
$$=-\oint\frac{\partial b(\rho,\tau)}{\partial\rho}\left(\frac{\partial^2b(\rho,\tau)}{\partial\rho^2}+\frac{4}{\rho}\frac{\partial b(\rho,\tau)}{\partial\rho}\right)d\rho=0, \tag{22}$$

where the integration is performed over a closed contour in the complex ρ-plane around the point $\rho=i\rho_0(\tau)$. The relation (22) is an exact consequence of Eq. (7): Since the function $b(\rho,\tau)$ after complete passage around the pole takes its original value, the integral of the total derivatives of the functions $b(\rho,\tau)$ over the closed contour is equal to zero. We substitute into (22) the expansion (10). Here we must take into account terms of up to fourth order, inclusive, in $\rho-i\rho_0(\tau)$. One can then obtain the following relation, equivalent to the vanishing of the residue of the integrand of (22) at the point $i\rho_0(\tau)$:

$$5b_1\rho_0^3(\tau)-10b_2\rho_0^4(\tau)-13b_3\rho_0^5(\tau)=2250/13. \tag{23}$$

We note two important circumstances. First, the relation (23) does not contain derivatives with respect to τ; i.e., τ appears in it as a parameter. Second, the quantity b_0

(which for the Kolmogorov-Obukhov spectrum is found to be infinite) does not appear in (23). This can be seen immediately from the second equality in (22), since it contains only derivatives of b with respect to ρ, in which the quantity b_0 does not appear. In addition, the quantity b_4 also drops out of (23).

We set $\tau = 0$ in (23). Then $\rho_0(0) = \xi$, and we obtain the relation

$$5b_1\xi^3-10b_2\xi^4-13b_3\xi^5=2250/13. \tag{24}$$

The relation (24) turns out to be equivalent to a certain integral relation for the spectral density $E_{LL}(\varkappa)$. In order to obtain this, we shall consider the function

$$\begin{aligned}(\rho-i\xi)^2b(\rho)=&-30-(120i/13\xi)(\rho-i\xi)+b_0(\rho-i\xi)^2\\&+ib_1(\rho-i\xi)^3+b_2(\rho-i\xi)^4+ib_3(\rho-i\xi)^5+\dots\end{aligned} \tag{25}$$

This function is analytic in a neighborhood of the point $\rho = i\xi$. Differentiating (25) with respect to ρ three, four, and five times, and, after this, setting $\rho = i\xi$, we obtain the formula

$$b_1=\frac{1}{i\cdot 3!}\left\{\left(\frac{d}{d\rho}\right)^3[(\rho-i\xi)^2b(\rho)]\right\}_{\rho=i\xi} \tag{26}$$

and analogous formulas for b_2 and b_3. At the pole $\mathrm{Im}\rho = \xi$ the function $b(\rho)$ can be represented by a Fourier integral:

$$b(\rho)=\int_0^\infty \tilde{E}_{LL}(\varkappa)\cos(\varkappa\rho)\,d\varkappa. \tag{27}$$

Since the relation

$$\rho b(\rho)=-\int_0^\infty \tilde{E}_{LL}'(\varkappa)\sin(\varkappa\rho)\,d\varkappa,$$

$$\rho^2 b(\rho)=-\int_0^\infty \tilde{E}_{LL}''(\varkappa)\cos(\varkappa\rho)\,d\varkappa$$

hold, we have

$$\begin{aligned}(\rho-i\xi)^2b(\rho)=\int_0^\infty&(2i\xi\tilde{E}_{LL}'(\varkappa)\sin(\varkappa\rho)\\&-(\xi^2\tilde{E}_{LL}(\varkappa)+\tilde{E}_{LL}''(\varkappa))\cos(\varkappa\rho))\,d\varkappa.\end{aligned}$$

We separate out here the factors $e^{i\varkappa\rho}$ and $e^{-i\varkappa\rho}$. Then

$$(\rho-i\xi)^2b(\rho)={}^1/_2\int_0^\infty[G^{(+)}(\varkappa)e^{i\varkappa\rho}-G^{(-)}(\varkappa)e^{-i\varkappa\rho}]\,d\varkappa, \tag{28}$$

where we have introduced the notation

$$G^{(\pm)}(\varkappa)=2\xi\tilde{E}_{LL}'(\varkappa)\mp[\tilde{E}_{LL}''(\varkappa)+\xi^2\tilde{E}_{LL}(\varkappa)]. \tag{29}$$

We note an important property of the function $G^{(-)}(\varkappa)$: If $\tilde{E}_{LL}(\varkappa)$ has the form $\tilde{E}_{LL}(\varkappa) = (a + b\varkappa)\exp(-\varkappa\xi)$, then, according to (29), $G^{(-)}(\varkappa)\equiv 0$. Therefore, the leading term of the asymptotic expansion of $\tilde{E}_{LL}(\varkappa)$ makes $G^{(-)}(\varkappa)$ vanish, while the next terms, which are of order $O(e^{-\varkappa\xi})$, upon substitution into (28) not only do not lead to a divergence of the second term at the point $\rho = i\xi$ but give an exponentially small contribution. Therefore, the formula (28) in fact determines the explicit form of the analytic continuation of the function $b(\rho)$ onto the singular point $\rho = i\xi$.

We substitute (28) into (26), and then substitute the coefficients b_1, b_2, and b_3 thus obtained into (24). This leads to the relation

$$\begin{aligned}\int_0^\infty&(\varkappa\xi)^3[e^{-\xi\varkappa}G^{(+)}(\varkappa)(100+50\xi\varkappa+13\xi^2\varkappa^2)\\&+e^{\xi\varkappa}G^{(-)}(\varkappa)(100-50\xi\varkappa+13\xi^2\varkappa^2)]\,d\varkappa=-240\cdot 2250/13.\end{aligned} \tag{30}$$

As in the preceding section, we divide the region of integration in (30) into the segments $(0, \varkappa_0)$ and $(\varkappa_0, \infty)$, and use on these segments the approximations $\varphi(\varkappa)$ and $\psi(\varkappa)$, respectively (here the integral of the second term from $\varkappa_0$ to ∞ vanishes). Then Eq. (30) takes the form

$$26C_2\xi^{8/3}A(\alpha)+9\cdot 240B(\alpha)=9\cdot 240\cdot 2250, \tag{31}$$

where we have introduced the notation

$$\begin{aligned}A(\alpha)=\int_0^\alpha\varkappa^{1/3}\Big[&(5000+840\varkappa^2)\operatorname{ch}\varkappa\\&-(4000+2920\varkappa^2+117\varkappa^4)\frac{\operatorname{sh}\varkappa}{\varkappa}\Big]\,d\varkappa,\end{aligned} \tag{32}$$

$$B(\alpha)=\int_\alpha^\infty e^{-2\varkappa}(169\varkappa^6+533\varkappa^5+850\varkappa^4-900\varkappa^3)\,d\varkappa. \tag{33}$$

Solving Eq. (31) for $C_2\xi^{8/3}$, we obtain

$$C_2\xi^{8/3}=(1080/13)[2250-B(\alpha)]/A(\alpha)\equiv f(\alpha). \tag{34}$$

The equation (34) is the second integral relation [in addition to (19)] connecting ξ, α, and C_2; unlike (19), it is valid only in the framework of the Chandrasekhar equation. Together with (19) it makes it possible to determine uniquely the point where the asymptotic forms join, and to find the dependence $\xi = \xi(C_2)$. Substituting $C_2 = f(\alpha)\xi^{8/3}$ from (34) into (19), we obtain an equation containing only ξ and α. From this equation we easily find

$$\begin{aligned}\xi(\alpha)=\Big\{15\Big[&\frac{3}{4}\alpha^{4/3}f(\alpha)\\&+60e^{-\alpha}\left(\alpha^3+\frac{43}{13}\alpha^2+\frac{86}{13}\alpha+\frac{86}{13}\right)\Big]\Big\}^{1/4}.\end{aligned} \tag{35}$$

Finally, substituting (35) into (34), we have

$$C_2(\alpha)=f(\alpha)(\xi(\alpha))^{-8/3}. \tag{36}$$

The pair of equations (35), (36) specifies in parametric form the dependence $\xi = \xi(C_2)$.

The quantities $A(\alpha)$, $B(\alpha)$, and $f(\alpha)$ appearing in (35) and (36) have been calculated numerically. It is found that $A(\alpha) > 0$ for $\alpha < \alpha_0 \cong 2.6$, whereas $B(\alpha) < 2250$ for $\alpha > \alpha_1 \cong 0.95$. Thus, a positive solution for $C_2\xi^{8/3}$ exists in the interval $0.95 < \alpha < 2.6$. In the interval $2.48 < \alpha < 2.53$ the quantity C_2 determined by the formulas (35), (36) varies from 0.36 (for $\alpha = 2.48$) to 0.49 (for $\alpha = 2.53$), which encompasses the entire range of values of C_2 obtained both from the experiments of Refs. 5–7 and from the theoretical estimates of Ref. 22. In this range of α the dependence $\xi = \xi(C_2)$ specified by Eqs. (35), (36) can be represented by the approximating formula[2)]

$$\xi=10.753+8.324(C_2-0.4164)+4.515(C_2-0.4164)^2, \quad (37)$$

which gives five significant figures coinciding with those obtained from (35), (36).

For $C_2 = 0.42$ we have $\xi = 10.78$, while for $C_2 = 0.45$ we have $\xi = 11.04$. We see that they differ only very slightly from the values 10.85 and 11.10 given in the preceding section. We note also that the value $\alpha \cong 2.6$ adopted in the preceding section lies above the point at which $A(\alpha) = 0$. Until we take into account the restrictions stemming from (22), small variations of α are unimportant. But when the consequences of the relation (22) are taken into account the results become sharply dependent on the choice of α. In this connection we draw attention to the fact that the experimentally admissible values of C_2 correspond to a very narrow range of α—from 2.48 to 2.53. This is the reason why, if we wish to make the results of the calculation of the dependence $\xi = \xi(C_2)$ consistent with (22), it is necessary to make an extremely accurate determination of the position of the point where the asymptotic forms join; the value $\alpha \cong 2.6$ adopted in the preceding section is found to be incompatible with (22).

5. COMPARISON WITH EXPERIMENT

We shall compare the expressions obtained with the results of experiment. In Ref. 7 a composite graph of the dependence $\tilde{E}_{LL} = \tilde{f}(\varkappa)$ is plotted on the basis of three independent experiments carried out in water and in air (the graph is reproduced in Ref. 8). The constant C_2, determined from the power-law segment of this function, is equal to 0.42, so that the corresponding value of ξ is 10.8. In the figure we give both the experimental points and the functions (17) (curve 1) and (14) (curve 2). We emphasize that in plotting curve 2 we have not used any adjustable parameter. If, ignoring the relation (22), we introduce the point $\varkappa_0$ at which the two asymptotic forms join, this has only an insignificant effect on the value of ξ. The magnitude of these discrepancies characterizes the accuracy of our estimate of the quantity ξ. We note also that fully satisfactory agreement of the experimental and calculated data is also observed for the quantity $\varkappa^2 E_{LL}$, which is more sensitive to the form of the spectrum in the region $\varkappa \gg 1$.

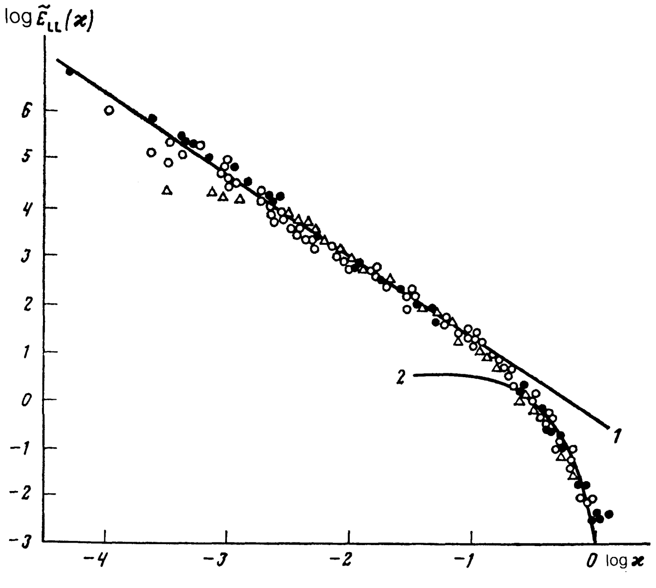


FIG. 1. Comparison of the results of the calculation and the results of measurements of the longitudinal turbulence spectrum. ●, △, and ○ are experimental values from Refs. 5, 6, and 7, respectively; 1) the power-law asymptotic form for $\varkappa \ll 1$, and 2) the asymptotic form (14) for the region $\varkappa \gg 1$, for the value $\xi = 10.85$.

6. CONCLUSION

We shall summarize the paper. Analysis of the equations of the diagram technique leads to the conclusion that in the region $\eta k \gg 1$ the Chandrasekhar equation (7) is valid for the space-time correlation function. Investigation of the singular point of this equation nearest to the real axis makes it possible to establish the general asymptotic form of the turbulence spectrum; this form contains one free parameter ξ, characterizing the exponential decay of the spectrum for $\eta k \gg 1$. For a first estimate of this parameter we use the exact integral relation (16), which determines the dependence of ξ on the constant C_2 in the Kolmogorov-Obukhov "5/3 law" and on the joining point $\varkappa_0$ of the asymptotic forms of the spectrum for $\eta k \to 0$ and $\eta k \to \infty$. An important point is that the estimate of ξ depends only weakly on the choice of the point $\varkappa_0$. This first estimate already leads to good agreement with experiment. The arbitrariness in the choice of the point $\varkappa_0$ is removed by using the second integral relation (34), which, unlike (19), is valid only in the framework of the adopted model, describable by the Chandrasekhar equation. This makes it possible to determine the joining point $\varkappa_0$ uniquely, and thereby to determine the dependence $\xi = \xi(C_2)$. Thus, the constant C_2 plays the role of a "boundary condition" in the determination of the form of the spectrum in the dissipation interval.

It might appear that, having determined the joining point $\varkappa_0$ in a natural manner (as was done in Sec. 3), one could use these two integral relations to determine both ξ and C_2 simultaneously. However, this is impossible because of the instability of the dependence $C_2 = C_2(\varkappa_0)$. Evidence for this is the fact that, near the value $\varkappa_0 \cong 2.6/\psi$ chosen *a priori* in Sec. 3, the function $A(\alpha)$ appearing in the second integral relation vanishes. At the same time, the inverse procedure, including the determination of $\varkappa_0$, turns out to be stable.

One further circumstance should be noted. Into the second integral equation for $\eta k \gg 1$ we have substituted the asymptotic solution (14), which is in fact a consequence of the Chandrasekhar equation. However, the power function used in this integral relation for $\eta k \ll 1$ is not a consequence of the Chandrasekhar equation. Therefore, in substituting the power function into (34) we are, generally speaking, making a certain error, which is difficult to estimate.

Evidently, in evaluating the reliability of the result obtained above we must appeal first of all to the agreement of the calculated and experimental results.

The authors express their deep gratitude to V. V. Tatarskiĭ, who carried out the necessary numerical calculations and independently checked on the computer the cumbersome analytical transformations.

[1]Here r is the distance between the points at which the velocities are taken, and t is the time interval between the observation times.

[2)] In the interval $2.48 < \alpha < 2.53$ the results of a numerical calculation of the function can be represented with a relative error less than $2 \cdot 10^{-4}$ by means of the approximation formula $A(\alpha) = 260.81 - 2824 (\alpha - 2.5055) - 3909.7 (\alpha - 2.5055)^2$. For $B(\alpha)$ we have the exact expression $B(\alpha) = e^{-2\alpha} (169\alpha^6/2 + 520\alpha^5 + 1725\alpha^4 + 3000\alpha^3 + 4500\alpha^2 + 4500\alpha + 2250)$.

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941) [C. R. Acad. Sci. URSS 30, 301 (1941)].
[2] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **32**, 19 (1941) [C. R. Acad. Sci. URSS 32, 16 (1941)].
[3] A. M. Obukhov, Dokl. Akad. Nauk SSSR 32, 22 (1941) [C. R. Acad. Sci. URSS **32**, 19 (1941)].
[4] A. M. Obukhov, Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz. **5**, 453 (1941).
[5] H. L. Grant, R. W. Stewart, and A. Moilliet, J. Fluid Mech. **12**, 241 (1962).
[6] S. Pond, R. W. Stewart, and R. W. Burling, J. Atmos. Sci. **20**, 319 (1963).
[7] V. A. Sandborn and R. D. Marshall, Local isotropy in wind tunnel turbulence, Technical Report, Fluid Dynamics and Diffusion Laboratory, Colorado State University (1965).
[8] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, MIT Press, Cambridge, Mass. (1971), Part II [Russ. original, Nauka, Moscow (1964)].
[9] H. W. Wyld, Jr., Ann. Phys. (N.Y.) **14**, 143 (1961).
[10] A. Z. Patashinskiĭ and V. L. Pokovskiĭ, Zh. Eksp. Teor. Fiz. **46**, 994 (1964) [Sov. Phys. JETP **19**, 677 (1964)].
[11] A. M. Polyakov, Zh. Eksp. Teor. Fiz. **55**, 1026 (1968) [Sov. Phys. JETP **28**, 533 (1969)].
[12] G. A. Kuz'min and A. Z. Patashinskiĭ, Zh. Eksp. Teor. Fiz. **62**, 1175 (1972) [Sov. Phys. JETP 35, 620 (1972)].
[13] E. Hopf, J. Ration. Mech. Anal. **1**, 87 (1952).
[14] R. M. Lewis and R. H. Kraichnan, Commun. Pure Appl. Math. **15**, 397 (1952).
[15] V. I. Tatarskii, Zh. Eksp. Teor. Fiz. **42**, 1386 (1962) [Sov. Phys. JETP **15**, 961 (1962)].
[16] A. A. Konstantinov, Teor. Mat. Fiz. **42**, 79 (1980) [Theor. Math. Phys. (USSR) **42**, 52 (1980)].
[17] S. S. Moiseev, R. Z. Sagdeev, A. V. Tur, G. A. Khomenko, and V. V. Yanovskii, Zh. Eksp. Teor. Fiz. **85**, 1979 (1983) [Sov. Phys. JETP **58**, 1149 (1983)].
[18] A. V. Tur and V. V. Yanovskiĭ, Dokl. Akad. Nauk SSSR **292**, 1368 (1987) [Sov. Phys. Dokl. **32**, 147 (1987)].
[19] S. F. Edwards, J. Fluid Mech. **18**, 239 (1964).
[20] V. P. Maslov, Teor. Mat. Fiz. **69**, 361 (1986) [Theor. Math. Phys. (USSR) **69**, (1986)].
[21] V. P. Maslov, Usp. Mat. Nauk **41**, 19 (1986).
[22] E. B. Gledzer, Zh. Eksp. Teor. Fiz. **91**, 818 (1986) [Sov. Phys. JETP **64**, 483 (1986)].
[23] A. M. Obukhov, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **17**, 1235 (1981).
[24] A. M. Obukhov, Gerlands Beitr. Geophys. **82**, 282 (1973).
[25] A. M. Obukhov, Usp. Mat. Nauk **38**, 101 (1983).
[26] E. A. Novikov, Dokl. Akad. Nauk SSSR **139**, 331 (1961) [Sov. Phys. Dokl. **6**, 571 (1961)].
[27] G. A. Kuz'min and A. Z. Patashinskii, Zh. Eksp. Teor. Fiz. **76**, 2075 (1979) [Sov. Phys. JETP **49**, 1050 (1979)].
[28] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979).
[29] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974) [Russ. transl., Mir, Moscow (1975)].
[30] R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).

Translated by P. J. Shepherd