Travelling stimulated Brillouin scattering in a nonuniform medium with thermal phonons

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A theory is derived for stimulated Brillouin scattering in a "travelling" regime in a medium with prescribed variations in the permittivity. Spontaneous hypersonic vibrations are taken into account. The field of the Stokes wave is calculated rigorously, as are the Green's function and gain for the Stokes pulse. Modulation of the frequency of the Stokes emission should result from periodic steady-state variations in the refractive index in the medium. The amplitude of the pump wave along the axis of the waveguide should also be modulated because of periodic variation in the cross section of the waveguide. Numerical calculations on travelling stimulated Brillouin scattering are reported.

INTRODUCTION

Stimulated Brillouin scattering (SBS) is known to involve the coherent scattering of a pump wave E_L into a Stokes wave E_S by hypersonic vibrations of the medium, q. Our purpose in the present paper is to derive a theory for SBS[1] in the so-called travelling regime,[2] in which the length L of the scattering medium is greater than the spatial length of the pump pulse, $l=\tau_L c/n_0$: $L\gtrsim l$, where τ_L is the duration of the pump pulse, c is the velocity of light, and n_0 is the refractive index of the medium. In this case the pump intensity can be assumed to remain constant. In this regime the pump pulse in a sense travels through the medium at a velocity c/n_0, and the stimulated scattering of the pump wave into the Stokes wave occurs in different layers of the medium. We will examine the process of stimulated Brillouin backscattering. Noting that the pump wave and the Stokes wave are propagating in opposite directions in the case of stimulated Brillouin backscattering, we see that the pulse of Stokes radiation is actually moving opposite the pump pulse at a velocity $c/2n_0$ in this process. We assume that there are steady-state variations $\delta n(z)$ in the refractive index satisfying $|\delta n(z)|\ll n_0$ along the direction of propagation of the interacting waves in the medium. These variations might result from, for example, acoustic waves in the medium.

1. BASIC EQUATIONS DESCRIBING THE TRAVELLING REGIME OF STIMULATED BRILLOUIN SCATTERING

We choose the direction of the x axis in such a way that the pump wave $E_L(z,t)\exp(-i\omega_L t-ik_L z)$ is propagating along the $-z$ direction, and the Stokes wave $E_S(z,t)\exp(-i\omega_S t+ik_S z)$ along the $+z$ direction. The equations describing this process then take the form[1]

$$\frac{n_0}{c}\frac{\partial E_S}{\partial t}+\frac{\partial E_S}{\partial z}-ik_S\delta n(z)E_S=-ik_2(q^*+\delta q^*)E_L, \qquad (1)$$

$$\frac{\partial q^*}{\partial t}+v\frac{\partial q^*}{\partial z}+\Gamma q^*=-ik_1E_SE_L^*, \qquad (2)$$

$$\frac{\partial(\delta q^*)}{\partial t}+v\frac{\partial(\delta q^*)}{\partial z}+\Gamma\delta q^*=f_T(z,t), \qquad (3)$$

$$\frac{n_0}{c}\frac{\partial E_L}{\partial t}-\frac{\partial E_L}{\partial z}+ik_L\delta n(z)E_L=0. \qquad (4)$$

Here $E_L(z,t)$ and $E_S(z,t)$ are the "slow" complex amplitudes of the pump and Stokes waves; q and δq, respectively, are the complex amplitudes of the stimulated and spontaneous hypersonic vibrations of the medium; f_T is a source of spontaneous thermal phonons; $\Gamma=\alpha_0 v$, where α_0 and v are the attenuation factor and velocity of hypersound in the medium; $k_1=k_S Y/4\rho v^2$, $k_2=2G\Gamma\rho v^2/k_S Y$, where $Y=\rho\partial\varepsilon/\partial\rho$, ρ is the density, and the coefficient G is a measure of the nonlinearity of the active medium; and $k_L\approx k_S=k$ is the wave number for the pump wave and for the Stokes wave. Equations (2) and (3) incorporate the spatial derivatives of the hypersound, $v\partial q^*/\partial z$; i.e., they incorporate the fact that the hypersonic vibrations are not spatially localized. It is important to recognize that incorporating the spatial derivatives of the hypersound in these equations brings out several interesting effects, namely a change in the frequency spectrum of the field E_S at the exit from the medium, as we will be discussing below.

It can be seen from (4) that we are assuming that the inverse effect of the Stokes wave on the pump field is negligible, as are the diffraction loss and the absorption of light. The pump-wave amplitude E_L in Eqs. (1)–(3) is thus a given function which satisfies Eq. (4). We note, however, that the amplitude of the pump wave may also vary along the z direction because of a change in the cross section of the volume of the active medium. For generality we incorporate the possibility of such a modulation at the outset by introducing an additional factor of $1+r(z)$ in the expression for the amplitude E_L in the solution of Eq. (4). Correspondingly, we introduce a factor of $[1+r(z)]^2$ in the expression for the intensity of the pump wave. The function $r(z)$ characterizes the change in the cross section of the waveguide along the z axis.

We consider the propagation through an active medium which occupies the spatial volume $z\leqslant 0, |z|\leqslant L$, of a temporally bounded pump pulse. We choose origins for the z and t scales in such a way that the front of the pump pulse intersects the entrance plane of the medium ($z=0$) at the time $t=0$ (Fig. 1). It then follows from Eq. (4) that the locus of points of the front of the pump pulse in the zt plane is the straight line $z+ct/n_0=0$ (Fig. 1). Each point z_0, t_0 on this straight line is a source of an induced Stokes wave which is propagating along the family of characteristics $z-ct/n_0=z_0-ct_0/n_0$. The $z=0$ plane is the entrance plane for

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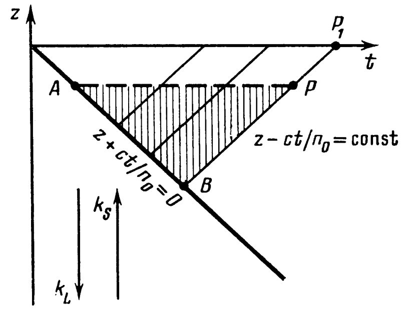

FIG. 1. Characteristic lines for the pump wave and the Stokes wave in the zt plane. The $z = 0$ plane is the entrance plane for the pump wave and the exit plane for the Stokes wave.

the pump wave and the exit surface for the Stokes wave. The boundary condition on the Stokes-wave amplitude E_S is $E_S = 0$ on the line $z + ct/n_0 = 0$. Taking account of the propagation direction and the finite value of the velocities of the pump and Stokes waves, and also imposing a temporal causality principle, we see that the strength of the Stokes wave at an arbitrary point P is determined by the strengths of the pump wave and the hypersonic vibrations of the medium in triangle ABP in Fig. 1.

2. SOLUTION OF THE EQUATION FOR THE STOKES WAVE; DETERMINATION OF THE GREEN'S FUNCTION

To solve the system (1)–(3) we write the amplitudes of the pump and Stokes waves in the form

$$E_L = A_L \exp\left(ik \int_z^0 \delta n(z'')\,dz''\right),$$

$$E_S = A_S \exp\left(-ik \int_z^0 \delta n(z'')\,dz''\right), \tag{5}$$

and we write q^* and δq^* in the form

$$q^* = ue^{-\Gamma t}, \quad \delta q^* = \delta u e^{-\Gamma t}. \tag{6}$$

Furthermore, to eliminate spatial derivatives from Eqs. (2) and (3) for the hypersonic vibrations, we introduce the coordinate transformation

$$z' = (z - vt)/(1 - vn_0/c), \quad t' = t. \tag{7}$$

In terms of the variables z' and t', the equations for the amplitudes A_S, A_L, u and δu then take the form

$$\left(\frac{n_0}{c}\frac{\partial A_S}{\partial t'} + \frac{\partial A_S}{\partial z'}\right)e^{\Gamma t'} = -ik_2(u + \delta u)\,\alpha(z', t'), \tag{8}$$

$$\frac{\partial u}{\partial t'} = ik_1 e^{\Gamma t'} A_S(z', t')\,\alpha^*(z', t'), \tag{9}$$

$$\frac{\partial(\delta u)}{\partial t'} = f_T(z', t')\,e^{\Gamma t'}, \tag{10}$$

$$\frac{\partial A_L}{\partial t'} - \frac{c}{n_0}\frac{\partial A_L}{\partial z'}\left(1 + \frac{v}{c}n_0\right)\Big/\left(1 - \frac{v}{c}n_0\right) = 0, \tag{11}$$

where

$$\alpha(z', t') = A_L(z', t')\exp\left[2ik\int_{z(z',t')}^{0} \delta n(z'')\,dz''\right].$$

The solution of the equation for the amplitude $A_L(z', t')$ is

$$A_L = f(p) = f\left[z' + \frac{ct'}{n_0}\left(1 + \frac{v}{c}n_0\right)\Big/\left(1 - \frac{v}{c}n_0\right)\right];$$

i.e., the pump field is propagating in the $z't$ plane along characteristics

$$p = z' + \frac{ct'}{n_0}\left(1 + \frac{v}{c}n_0\right)\Big/\left(1 - \frac{v}{c}n_0\right) = \text{const}.$$

The form of the function f is determined by the particular shape of the pump pulse. As we mentioned above, we are allowing for a variation in the cross section of the waveguide by multiplying the expression found for A_L from (11) by a factor of $1 + r(z)$. We thus write

$$A_L = f(p)[1 + r(z)].$$

A corresponding factor of $1 + r(z)$ should undoubtedly be incorporated in the final expression for the Stokes-wave amplitude A_S. However, in the expression for A_S this factor, which directly describes the change in the intensity of the Stokes wave due to the change in the cross section of the medium, does not play an important role for the effects in which we are interested here, while the associated modulation of the pump wave gives rise (as we will see below) to substantial modulation of the frequency spectrum of the Stokes radiation because of the nonlinearities and the exponential gain.

We turn now to the solution of system (8)–(11). Differentating (8) with respect to t, using Eqs. (9) and (10), and also using the expression for A_L, we find the following second-order partial differential equation for $A_S(z', t')$:

$$\frac{n_0}{c}\frac{\partial^2 A_S}{\partial t'^2} + \frac{\partial^2 A_S}{\partial z'\,\partial t'} + \tilde{\Gamma}\left(\frac{n_0}{c}\frac{\partial A_S}{\partial t'} + \frac{\partial A_S}{\partial z'}\right) = k_1 k_2 I_L(z', t')\,A_S(z', t') - ik_2 f_T \alpha(z', t'), \tag{12}$$

$$\tilde{\Gamma} = \Gamma - \frac{c}{n_0 A_L}\frac{\partial A_L}{\partial z'}\left(1 + \frac{v}{c}n_0\right)\Big/\left(1 - \frac{v}{c}n_0\right) + 2ikv\delta n(z(z', t')), \quad I_L = |E_L|^2. \tag{13}$$

We assume that the space-time dependence of the pump pulse f, of duration τ_L, is

$$f(p) = E_{L0}e^{\gamma p} = E_{L0}\exp\left\{\gamma\left[z' + \frac{c}{n_0}t'\left(1 + \frac{v}{c}n_0\right)\Big/\left(1 - \frac{v}{c}n_0\right)\right]\right\} = E_{L0}\exp\left[\gamma\left(z + \frac{c}{n_0}t\right)\Big/\left(1 - \frac{v}{c}n_0\right)\right]$$

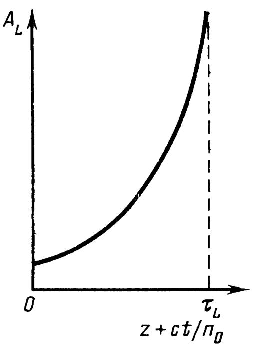

FIG. 2. Space-time dependence of the amplitude of a pump pulse of duration τ_L.

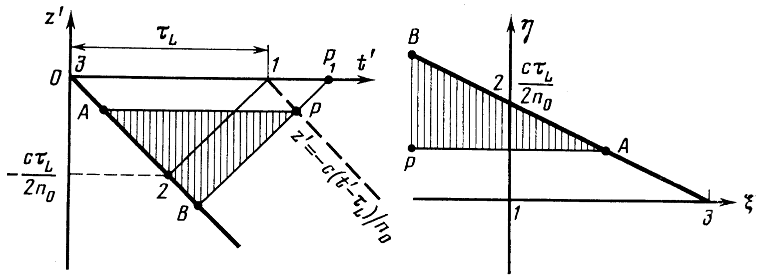

FIG. 3. Mapping of characteristic points in the $z't'$ plane into the $\xi\eta$ plane.

(Fig. 2). We then write

$$\frac{1}{A_L}\frac{\partial A_L}{\partial z'}=\gamma+\frac{dr}{dz}\frac{\partial z}{\partial z'}\frac{1}{1+r(z)}=\gamma+\left(1-\frac{v}{c}n_0\right)\frac{r_z'}{1+r(z)}$$

and the quantity $\tilde{\Gamma}$ takes the form

$$\tilde{\Gamma}=\Gamma-\frac{c\gamma}{n_0}\left(1+\frac{v}{c}n_0\right)\Big/\left(1-\frac{v}{c}n_0\right)-\frac{c}{n_0}\left(1+\frac{v}{c}n_0\right)\frac{r_z'}{1+r(z)}+2ikv\delta n(z(z',t')). \tag{14}$$

The particular case $\gamma=0$ corresponds to a square pump pulse.

To put Eq. (12) in canonical form, we introduce the variables ξ and η:

$$\xi=z'-c(t'-\tau_L)/n_0,\quad \eta=-z'. \tag{15}$$

Figure 3 shows a mapping of characteristic points from the $t'z'$ plane into the $\xi\eta$ plane, as specified by Eqs. (15). In terms of the variables ξ, η the expression for A_L takes the form

$$A_L=I_{L\max}^{1/2}\exp\left\{-\gamma\left[2\eta+\xi\left(1+\frac{v}{c}n_0\right)\right]\Big/\left(1-\frac{v}{c}n_0\right)\right\}\cdot[1+r(z)], \tag{16}$$

$$I_{L\max}=|E_{L0}|^2\exp\left[\frac{2\gamma\tau_L(c/n_0+v)}{1-vn_0/c}\right].$$

We write $A_S(\xi,\eta)$ in the form

$$A_S(\xi,\eta)=W(\xi,\eta)\exp\left[\frac{n_0}{c}\int_0^{\xi}\Gamma(\xi',\eta)d\xi'\right]. \tag{17}$$

Making use of the small parameters $vn_0/c\ll1, r_z'/k\ll1$, and $|r(z)|\ll1$, we write the equation for W in the following form, where we are using (12) and (17):

$$\frac{\partial^2W}{\partial\tilde{\eta}\,\partial\tilde{\xi}}-\frac{G\Gamma}{2c}n_0I_{L\max}W=-\frac{ik_2}{c}n_0f_T(\xi(\tilde{\xi}),\eta(\tilde{\eta}))\frac{I_{L\max}^{1/2}e^{\mu}}{1+r(z(\tilde{\xi},\tilde{\eta}))}, \tag{18}$$

$$\mu=-\frac{n_0}{c}\Gamma\xi(\tilde{\xi})+2\gamma\xi(\tilde{\xi})\left(1+\frac{v}{c}n_0\right)\Big/\left(1-\frac{v}{c}n_0\right)+2\gamma\eta(\tilde{\eta})\Big/\left(1-\frac{v}{c}n_0\right)+\left(1+\frac{v}{c}n_0\right)\int_0^{\xi(\tilde{\xi})}\frac{r_z'd\xi_1}{1+r(z)}+2ik\int_{v\tau_L-\eta(\tilde{\eta})}^{0}\delta n(z'')\,dz''.$$

Here, we have introduced the variables

$$\tilde{\xi}=\int_0^{\xi}\exp\left[-2\gamma\xi'\left(1+\frac{v}{c}n_0\right)\Big/\left(1-\frac{v}{c}n_0\right)\right]d\xi',$$

$$\tilde{\eta}=\int_0^{\eta}\exp\left[-4\gamma\eta'\Big/\left(1-\frac{v}{c}n_0\right)\right][1+r(-\eta')]^2d\eta'. \tag{19}$$

In the expression for $\tilde{\eta}$, we ignore the additional small parameter $\sim v/c$ in the argument of the function $r(z)$ in the integral, since the function $r(z)$ is itself small. In the expression for η, $r(z)$ is then a function of $\tilde{\eta}$ alone, since we have $z(v=0)=-\eta$. We will also be using this approximation below.

The solution of Eq. (18) is expressed in terms of the Riemann function $R(\tilde{\xi},\tilde{\eta})$ (Ref. 3), found by solving the Goursat problem of the equation

$$\frac{\partial^2R}{\partial\tilde{\xi}\,\partial\tilde{\eta}}-AR=0,\quad A=\frac{G\Gamma n_0}{2c}I_{L\max}=\text{const} \tag{20}$$

with the boundary conditions $R(\tilde{\xi},0)=R(0,\tilde{\eta})=1$. This solution is

$$R(\tilde{\xi},\tilde{\eta})=I_0([2GI_{L\max}(\Gamma n_0/c)\tilde{\xi}\tilde{\eta}]^{1/2}),$$

where I_0 is the modified Bessel function.

The solution $W(P)$ of inhomogeneous equation (18) at an arbitrary point P is expressed in terms of the solution of the corresponding homogeneous equation, $R(\tilde{\xi},\tilde{\eta})$, as follows[3]:

$$W(P)=\frac{1}{2}\int_{AB}I_0\left(\left[2GI_{L\max}\frac{\Gamma n_0}{c}\tilde{\xi}(\xi)\tilde{\eta}(\eta)\right]^{1/2}\right)\cdot\left(\frac{\partial W}{\partial\xi}d\xi-\frac{\partial W}{\partial\eta}d\eta\right)+\frac{ik_2n_0}{c}I_{L\max}^{1/2}\int_{\Delta PAB}I_0\left(\left[2GI_{L\max}\frac{\Gamma n_0}{c}\tilde{\xi}\tilde{\eta}\right]^{1/2}\right)$$
$$\cdot f_T(\xi,\eta)\exp\left(-\frac{\Gamma n_0}{c}\xi-\frac{2\gamma\eta}{1-vn_0/c}\right)[1+r(z(\xi,\eta))]\cdot\exp\left[\int_0^{\xi}\frac{r_z'd\xi_1}{1+r}+2ik\int_{v\tau_L-\eta}^{0}\delta n(z'')dz''\right]d\xi\,d\eta. \tag{21}$$

The first term on the right side of expression (21) is an integral along the straight line AB in Fig. 3; the second term is an integral over the area bounded by triangle PAB. This second term describes the contribution of spontaneous thermal phonons ($\sim f_T$) to the Stokes wave.

Substituting the expression for Γ into (17), and integrating over ξ, we find

$$A_S(P)=W(P)e^{\nu(P)}, \tag{22}$$

$$\nu(P)=\frac{n_0}{c}\Gamma\xi^{(P)}-\gamma\xi^{(P)}\left(1+\frac{v}{c}n_0\right)\Big/\left(1-\frac{v}{c}n_0\right)$$

$$-2ik\int_{v\tau_L-\eta^{(P)}}^{z(P)}\delta n(z'')\,dz''-\frac{r_z'\xi^{(P)}}{1+r(z^{(P)})},\qquad(22a)$$

where, as follows from (15),

$$z^{(P)}=v\tau_L-\eta^{(P)}-n_0v\xi^{(P)}/c.$$

The superscript P means that the corresponding coordinate refers to the point P: $P(z^{(P)},t^{(P)})$ or $(P\xi^{(P)},\eta^{(P)})$. In order to determine the Stokes wave in the exit plane of the medium, $E_S(z=0)$ we need to determine the amplitude $A_S(P)$ at points P lying on the characteristic $z=-c(t-\tau_L)n_0$ which corresponds to the trailing edge of the pump pulse (Fig. 3). It is not difficult to see that for such points P we have $E_S^{ex}(z=0)=A_S(P)$ [see (5)]. Using (22) along with Eqs. (8)–(10) and the boundary condition $A_S|_{AB}=0$, and carrying out several manipulations, we find

$$A_S(P)=e^{\nu(P)}I_{L\,\max}^{1/2}ik_2\Big\{\frac{1}{2}\int_{AB}I_0\Big(\Big[2GI_{L\,\max}\frac{\Gamma n_0}{c}\xi\tilde{\eta}\Big]^{1/2}\Big)$$
$$\cdot\delta q^*(\xi,\eta)\,(1+r(z(\xi,\eta)))]$$
$$\cdot\exp\Big[-\frac{n_0\Gamma}{c}\xi-2\gamma\eta\Big/\Big(1-\frac{v}{c}n_0\Big)+\Delta_r(\xi,\eta)$$
$$+\Delta_n(\xi,\eta)\Big]d\eta+\frac{n_0}{c}\int_{\Delta PAB}f_T(\xi,\eta)I_0\Big(\Big[2GI_{L\,\max}\frac{\Gamma n_0}{c}\tilde{\xi}\tilde{\eta}\Big]^{1/2}\Big)$$
$$\cdot\exp\Big[-\frac{n_0\Gamma\xi}{c}-2\gamma\eta\Big/\Big(1-\frac{v}{c}n_0\Big)+\Delta_r+\Delta_n\Big]$$
$$\cdot[1+r(z)]\,d\xi\,d\eta\,,\qquad(23)$$

where

$$\Delta_n(\xi,\eta)=2ik\int_{v\tau_L-\eta}^{0}\delta n(z'')\,dz'',\quad \Delta_r(\xi,\eta)=\frac{r_z'}{1+r}\xi,$$

and $\tilde{\xi}(\xi)$ and $\tilde{\eta}(\eta)$ are given by (19). Under our assumptions, expression (23) is an exact equation for determining the complex amplitude of the Stokes wave at an arbitrary point in the active medium when we allow for thermal hypersonic vibrations $\sim\delta q^*(z,t)f_T(z,t)$ and possible perturbations of the refractive index $\delta n(z)$ in the medium and also a modulation of the intensity of the pump wave along the z axis due to a variation in the cross section of the waveguide. From (23) we easily find that the function

$$H_1(\xi,\eta)=I_0\Big(\Big[2GI_{L\,\max}\frac{\Gamma n_0}{c}\tilde{\xi}(\xi)\tilde{\eta}(\eta)\Big]^{1/2}\Big)[1+r(z)]$$
$$\cdot\exp\Big\{-\frac{n_0\Gamma\xi}{c}-2\gamma\eta\Big/\Big(1-\frac{v}{c}n_0\Big)+\Delta_r+\Delta_n\Big\}\qquad(24)$$

is a three-dimensional Green's function for the region bounded by triangle *PAB*, while it is a surface Green's function for line *AB*. Working in a similar way, we can calculate the amplitude and the Green's function for hypersonic vibrations of the medium from Eqs. (8)–(10).

3. CALCULATION OF THE GREEN'S FUNCTION AND GAIN FOR TRAVELLING STIMULATED BRILLOUIN SCATTERING OF A SQUARE PUMP PULSE IN A HOMOGENEOUS MEDIUM

Let us examine in more detail the particular case of a square pump pulse (i.e., one with $\gamma=0$) of duration τ_L and intensity I_L in a homogeneous medium $[\delta n(z)=0]$ whose cross section remains constant along $z[r(z)=0]$. In this case, taking the limit $\gamma\to0$ in (23), we find

$$A_S(P)=\exp\Big(\frac{\Gamma n_0}{c}\xi^{(P)}\Big)I_L^{1/2}ik_2\Big\{\frac{1}{2}\int_{AB}I_0\Big(\Big[2GI_L\frac{\Gamma n_0}{c}\xi\eta\Big]^{1/2}\Big)$$
$$\cdot\delta q^*(\xi,\eta)\exp\Big(-\frac{\Gamma n_0}{c}\xi\Big)d\eta+\frac{n_0}{c}\int_{\Delta PAB}f_T(\xi,\eta)$$
$$\cdot\exp\Big(-\frac{\Gamma n_0}{c}\xi\Big)I_0\Big(\Big[2GI_L\frac{\Gamma n_0}{c}\xi\eta\Big]^{1/2}\Big)d\xi\,d\eta\Big\}.\qquad(25)$$

From (25) we find an expression for the function $H_1(\xi,\eta)$ for the case of a square pump pulse in a homogeneous medium:

$$H_1(\xi,\eta)=\exp\Big(-\frac{\Gamma n_0}{c}\xi\Big)I_0\Big(\Big[2GI_L\frac{\Gamma n_0}{c}\xi\eta\Big]^{1/2}\Big).\qquad(26)$$

The exact expression (25) for travelling SBS differs from the corresponding expressions derived for the cases of time-varying SBS[4] and time-varying stimulated Raman scattering[5] under the assumptions used in those previous studies.

To determine the Stokes-pulse gain for travelling SBS in the case of a square pump pulse (i.e., with $\gamma=0$) in a homogeneous medium $[\delta n(z)\equiv0]$ we need to calculate the eigenvalues for the exponential solution of the homogeneous equation for the Stokes-wave amplitude E_S [i.e., $E_S(z',t')=A_S(z',t')$ with $\delta n(z)=0$], found from Eq. (12):

$$\frac{n_0}{c}\frac{\partial^2E_S}{\partial t'^2}+\frac{\partial^2E_S}{\partial z'\,\partial t'}+\Gamma\Big(\frac{n_0}{c}\frac{\partial E_S}{\partial t'}+\frac{\partial E_S}{\partial z'}\Big)=k_1k_2I_LE_S(z',t').\qquad(27)$$

Equation (27) incorporates the fact that we have $\tilde{\Gamma}=\Gamma$ for a square pump pulse in a homogeneous medium. Writing E_S in the form

$$E_S(z',t')=E_{S0}\exp[\lambda(z'+ct'/n_0)],$$

we find the following quadratic equation for the eigenvalues λ from (27):

$$\lambda^2+\Gamma n_0\lambda/c-k_1k_2I_Ln_0/2c=0.\qquad(28)$$

The solutions of this equation are

$$\lambda_{1,2}=\frac{\Gamma n_0}{2c}\Big[-1\pm\Big(1+\frac{2k_1k_2I_Lc}{\Gamma^2n_0}\Big)^{1/2}\Big].\qquad(29)$$

The eigenvalue λ_2, which corresponds to the minus sign in front of the square root, is negative, and it does not contribute to the amplification of the Stokes wave. Since the amplification of the Stokes wave in the case of a pump pulse of duration τ_L occurs over a distance τ_Lc/n_0, we find the following expression for the overall gain g for the amplitude of the Stokes field:

$$g=\lambda_1\frac{c}{n_0}\tau_L=\frac{\Gamma\tau_L}{2}\Big[-1+\Big(1+\frac{GI_Lc}{\Gamma n_0}\Big)^{1/2}\Big].\qquad(30)$$

Here we have used $k_1k_2=G\Gamma/2$. At large values of the nonlinearity coefficient of the medium, G, or for high pump intensities I_L, such that the relation $GI_Lc/\Gamma n_0\gg1$, holds, we find from (30)

$$g\approx(\Gamma GI_Lc/4n_0)^{1/2}\tau_L.$$

For the case of a small amplification ($GI_Lc/\Gamma n_0\ll1$), on the other hand, the gain is

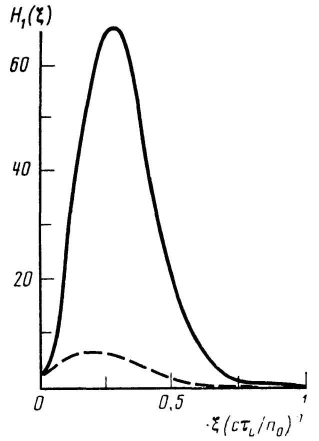

FIG. 4. The surface Green's function for a pump pulse of duration τ_L with $\Gamma\tau_L = 10$. Solid line—$GI_L\tau_L c/n_0 = 40$; dashed line—$GI_L\tau_L c/n_0 = 20$.

$$g \approx {}^1\!/_2 GI_L c\tau_L/2n_0.$$

This expression is equal to the steady-state gain over a distance $c\tau_L/2n_0$, which is equal to half the spatial length of the pump pulse (half because of the opposite propagation directions of the pump and Stokes waves).

Figure 4 shows plots of the surface Green's function $H_1(\xi,\eta)$ for a square pump pulse of duration τ_L for various pump-wave intensities I_L in a homogeneous medium $[\delta n(z) \equiv 0]$. The maximum of this function lies within the segment AB.

4. MODULATION OF THE FREQUENCY SPECTRUM OF THE STOKES WAVE DUE TO VARIATIONS IN THE REFRACTIVE INDEX OF THE MEDIUM

We return to the most general case, of a medium with variations in its refractive index, $\delta n(z)$, and with a possible modulation of the cross section of the waveguide, described by the function $r(z)$. A detailed analysis shows that each of these factors leads to modulation of the frequency spectrum of the Stokes wave at the exit from the medium, as can be seen with the help of (23) for the amplitude of the Stokes wave.

We begin by considering the effect of variations $\delta n(z)$ in the medium [i.e., we set $r(z) \equiv 0$] on the frequency spectrum of the Stokes radiation for the case of travelling stimulated Brillouin backscattering. The exponential phase factors

$$\exp \Delta_n = \exp\left[2ik \int_{v\tau_L-\eta}^{0} \delta n(z'')\,dz''\right]$$

in the integrands do not affect the spectrum $E_S(P)$. The reason is that since the functions $\delta q^*(\xi,\eta)$ and $f_T(\xi,\eta)$ are of a random nature these exponential factors are equivalent even without them to a change in the random phase of the quantities δq^* and f_T. Consequently, the frequency modulation of the Stokes wave by the variations $\delta n(z)$ is described by the exponential factor

$$\exp\left[-2ik \int_{v\tau_L-\eta^{(P)}}^{z^{(P)}} \delta n(z'')\,dz''\right],$$

which appears in front of the expression in braces in (23) [see expression 22(a) for $\nu(P)$]. Significantly, only by taking account of the finite magnitude of the sound velocity ($v \neq 0$) or, in other words, the fact that the hypersonic vibrations are not spatially localized [the term $\sim v\partial/\partial z$ in Eqs. (9) and (10)] are we able to bring out the effect of a change in the frequency spectrum of the Stokes wave due to $\delta n(z)$. In the opposite case, i.e., in the approximation $v = 0$, we have $z^{(P)} = -\eta^{(P)}$, and the exponential coefficient in front of the expression in braces in (23) and thus the modulation of the frequency spectrum of the Stokes wave due to $\delta n(z)$ disappear.

Let us assume for definiteness that the modulation of the refractive index in the medium is sinusoidal:

$$\delta n(z) = \delta n_0 \sin Qz.$$

Under this assumption, calculations carried out with to first order in the small quantity v/c lead to the following time dependence of the coefficient

$$\exp\left[-2ik \int_{v\tau_L-\eta^{(P)}}^{z^{(P)}} \delta n(z'')\,dz''\right]$$

at points P lying the characteristic of the trailing edge of the pump pulse $z^{(P)} = -c(t^{(P)} - \tau_L)/n_0$:

$$\exp\left(-2ik \int_{v\tau_L-\eta^{(P)}}^{z^{(P)}} \delta n(z'')\,dz''\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_m\left(\frac{2\delta n_0 k}{Q}\right) J_n\left(\frac{2\delta n_0 k}{Q}\right) \times i^{m-n} \exp\left\{i(t-\tau_L)Q\left[(m+n)\frac{c}{2n_0} + nv\right]\right\}. \quad (31)$$

Here Q is the wave number of the sinusoidal modulation of the refractive index, $\delta n(z)$; v is the sound velocity; k is the magnitude of the wave vector of the Stokes wave; J_m is the Bessel function; and t is the value of the time at point P_1 in the exit plane of the medium ($z = 0$) which corresponds to the point P on the characteristic of the trailing edge of the pump pulse.

A new result follows from expression (31): The Stokes radiation leaving a medium with sinusoidal variations in the permittivity in the case of travelling stimulated Brillouin backscattering is the sum of sinusoidal oscillations

$$\exp\{i(t-\tau_L)Q[(m+n)c/2n_0 + nv]\},$$

and the factor $\exp[i(t-\tau_L)Qnv]$ describes broadening of the spectrum of each harmonic with frequency $NQc/2n_0$ ($m + n = N = \text{const}$). Under the conditions $LQ/2\pi \lesssim 10^3$ and $\delta n_0 \lesssim 0.03$ cm/L (cm), which would ordinarily hold in practice, we find the following approximate expression for the Fourier time transform of the Stokes wave from (23) and (31):

$$F_S(\omega) \approx F_S^{(0)}(\omega) + {}^1\!/_2 kvT\delta n_0 e^{i\varphi} F_S^{(0)}(\omega - cQ/2n_0), \quad T = 2Ln_0/c. \quad (32)$$

Here $F_S^{(0)}$ is the amplitude of the Fourier transform of the Stokes wave in a homogeneous medium [i.e., for the case $\delta n(z) \equiv 0$], and $\varphi = \text{const}$. We thus see that an additional term at the frequency $\omega_0 = Qc/2n_0$, with the spectral width of the Stokes signal in a homogeneous medium and with an amplitude proportional to $(1/2)kvT\sigma\delta n_0$, appears in a medium with periodic variations δn of spatial frequency Q in the temporal spectrum of the Stokes wave. This change in the frequency of the optical signal does not occur in linear optics as light propagates through media with periodic variations in the refractive index.

If the length of the active medium is $L\sim 10^4$ cm (optical fibers), we would have threshold values $\delta n_0 \sim 10^{-6}$, and these values could be reduced to 10^{-7} through appropriate processing of a packet of signals. The results of Fourier analysis of a Stokes wave found through numerical simulation of travelling SBS confirm the conclusions of this theory.

5. EFFECT OF MODULATION OF THE CROSS SECTION OF THE ACTIVE MEDIUM ON THE FREQUENCY SPECTRUM OF THE STOKES WAVE

How is the frequency spectrum of the Stokes radiation affected by modulation of the intensity of the pump field along the z axis caused by variation of the cross-sectional area of the waveguide [i.e., $r(z)\neq 0$]? We see from (23) that a modulation of this sort gives rise to exponential factors in $A_S(P)$ which in this case have a purely real argument instead of an imaginary argument [as results from $\delta n(z)$]. These factors in the integral in (23) are no longer canceled out by the random nature of the phases of δq^* and f_T; they lead to an amplitude modulation of the Stokes wave, which converts into a phase modulation. A detailed analysis shows that the modulation of the frequency of the Stokes wave in this case results from specifically the factors

$$\exp\Delta_r = \exp[\xi r_z'/(1+r)]$$

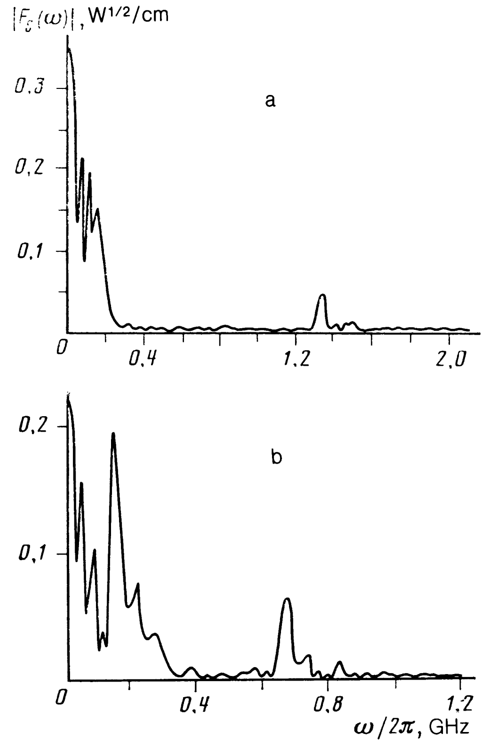

FIG. 5. Stokes-wave spectrum $|F_S(\omega)|$ found by numerical calculations for the following values of the parameters of the pump and the variations of the properties of the medium: $I_L = 3$ MW/cm^2, $\tau_L = 100$ ns, $\delta n_0 \approx 10^{-5}$. a—The spatial frequency of the periodic variations in the medium is $Q = 0.144$ cm^{-1}; b—$Q = 0.072$ cm^{-1}.

in the integrand. This modulation can be determined in the following way. If the cross section is modulated in accordance with $r(z) = r_0 \sin\beta z$ $(r_0 \ll 1)$, the amplitude $F_l^{(r)}(\omega)$ of the Fourier transform of the Stokes wave E_S with frequency $\omega_S + \omega_l$, which deviates from the fundamental Stokes frequency ω_S by $\omega_l = l\beta c/n_0$ $(l = 0,1,2,3...)$, is given by the function

$$F_l^r(\omega) = \text{const}\, I_l(r_0 c\tau_L\beta/n_0),$$

where I_l is the modified Bessel function.

An important point is that the modulation of the frequency spectrum of the Stokes wave due to the modulation of the pump intensity which results from a periodic variation in the cross section of the waveguide also occurs if we assume that hypersonic vibrations of the medium are spatially localized, i.e., when we ignore the sound velocity, $v = 0$. In contrast with this situation, as we mentioned earlier, the change in the frequency of the Stokes signal due to variations $\delta n(z)$ in the course of travelling stimulated Brillouin backscattering can be manifested only when we allow the circumstance that the hypersound is not spatially localized $(v\neq 0)$.

The modulation of the frequency spectrum of the Stokes wave due to variations $\delta n(z)$ and $r(z)$ predicted in this paper might be utilized in practice in problems requiring the remote detection of corresponding variations in the medium through frequency analysis of the Stokes radiation in the case of travelling stimulated Brillouin backscattering.

6. NUMERICAL SIMULATION OF TRAVELLING SBS

The results of this theoretical analysis of stimulated Brillouin backscattering in the travelling regime have been confirmed by numerical simulation.[1] Specifically, an algorithm was developed for constructing a mathematical model of the process of travelling stimulated Brillouin backscattering, and a corresponding computer program was written. The numerical simulation was carried out for sinusoidal variations:

$$n(z) = n_0 + \delta n_0 \sin Qz.$$

Figure 5 shows the data of a Fourier analysis of the Stokes wave calculated by this program. We see that the results of the numerical simulation of travelling stimulated Brillouin scattering and also the results of actual experiments and numerical simulations reported in Ref. 2 confirm the theoretical prediction of a change in the spectrum of the Stokes signal in a medium with variations in its refractive index. On this basis it can be concluded that travelling SBS could be utilized for remote sounding of variations in a medium and in several other promising applications.

CONCLUSION

In summay, this theoretical study of travelling stimulated Brillouin backscattering has resulted in a rigorous calculation of the field of the Stokes wave in the volume of, and at the exit from, the active medium. The analysis has incorporated the effect on the stimulated Brillouin scattering of hypersonic thermal vibrations of the medium and possible perturbations of the refractive index of the medium, $\delta n(z)$. Exact expressions have been derived for the amplitude of the Stokes wave, the volume and surface Green's functions, and the gain for travelling stimulated Brillouin scattering.

The results of this analysis predict frequency modulation of the Stokes radiaion resulting from periodic variations in the refractive index in the medium. Numerical calculations have been carried out on travelling stimulated Brillouin scattering. The results of these calculations confirm the conclusions of the analytic theory.

Travelling stimulated Brillouin scattering is clearly of interest for possible applications in the remote study of variations in a scattering medium.

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