Tunnel effect at the boundary of state stability: optical cavity and highly excited hydrogen atom in a magnetic field

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Using as examples two non-one-dimensional quasiclassical systems with nonseparable variables, viz., an open optical cavity and a highly excited hydrogen atom in a strong magnetic field, it is shown that certain structural transitions that occur in them are accompanied by tunneling processes. States localized in the vicinity of a stable classical trajectory that coincides with the symmetry axis of the problem are considered. A structural transition occurs at the instant when the trajectory loses stability. In some cases it turns out that the stability loss is accompanied by a tunneling decay of the states. In other cases, after passing through the stability boundary, instanton-type states are produced or bifurcation with doubling of the trajectory period takes place. In the case of a homogeneous optical cavity near the stability boundary, it is possible to determine analytically the width and level splitting for the quasistationary and instanton solutions, respectively. For a hydrogen atom in a magnetic field, states are considered which are elongated along the field, and also states located near the symmetry plane perpendicular to the field. In the former case it turns out that, near the stability boundary, families of closed stable or unstable trajectories are detached, and analytic equations are obtained for them. The appearance of unstable trajectories corresponds to formation of an effective potential barrier. It is shown that recently obtained numerically closed trajectories, which explain the experimentally observed resonances [A. Holle *et al.* Phys. Rev. Lett. **56**, 2594 (1986)], belong to a family of stable trajectories. In the latter case, closed stable trajectories are detached and have U-shaped projections on the symmetry plane. The stability boundary as well as the parameters of the solutions were found numerically for the general case and analytically for high energies. Unstable states localized near the zero-energy symmetry plane are considered and their lifetimes are calculated.

INTRODUCTION

An adiabatic change of the parameters of the Hamiltonian can cause the stationary state of a system to go over into a state with a different structure. The parameter values corresponding to the transition determine the stability boundary of the state. Many forms of structural transitions are accompanied by narrowing, way down to zero, of some initially exceedingly broad (practically infinite) effective potential barrier. Processes of this kind were considered in the theory of phase transitions at low temperatures,[1–3] and particularly in the theory of macroscopic tunneling.[4–6] They are, of course not a feature of many-body theory only, but are typical of many quantum-mechanical and wave systems with one or more degrees of freedom.[7,8] The one-dimensional problem is in a certain sense trivial, since the corresponding potential barriers are determined in elementary fashion by the form of the potential, and the tunneling probability can be found either in the WKB approximation, or by solving the one-dimensional Schrödinger equation by any other method.

For non-one-dimensional problems, the parameters of the effective potential barriers produced on the stability boundary are patently not obvious, and the boundary itself is as a rule unknown. Similar interesting problems were solved recently. These include an investigation of an asymmetric top[9] as well as the problem of highly excited states of a hydrogen atom in magnetic and electric fields.[10–12] What was used in fact in the cited papers was a quasiclassical perturbation theory for degenerate systems, developed in general form in Ref. 13. In This theory the Hamiltonian takes the form $H = H_\phi + V$, with the variables in H_0 separable and the increment V regarded as small in the sense of classical mechanics.

In contrast of Ref. 13, the states investigated in the present paper are in general not the products of perturbation of a system with exactly seperable variables, but comprise distinctive stability islands in a stochastic "sea."[14] The small parameters that we shall introduce will determine not the deviation of the Hamiltonian from an exactly integrable one, but localization of the solution in the vicinity of a manifold having a dimensionality smaller than that of the initial system. This approach to solving the Schrödinger and a few other equations, stemming from the Leontovich and Fock parabolic-equation method,[15] was intensively developed in diffraction theory and in mathematical physics (Refs. 16 and 17).[1)] We know, in particular, how to find the asymptotic wave functions and the quantization rules for states localized near stable classical trajectories. From the physical point of view, interest attaches also to effects produced when seemingly heretofore not investigated trajectories pass through the stability boundary.

To study the effects due to passage through bifurcation point of this type it is necessary to retain in the expansion of the Hamiltonian, and also in the Ansatz of the asymptotic solution in the zeroth approximation, not only the terms quadratic in the coordinates perpendicular to the trajectories (in the general case—to the manifold), as is usually

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done,[15-20] but also the terms of next order. Near the stability boundary certain coefficients of the quadratic terms in the solution tend to zero, thus indicating delocalization, but the next terms can determine a new characteristic dimension of the localized state.

The theory is developed in the present paper for the case when an initially stable trajectory coincides with the symmetry axis of the problem. The Hamiltonian is in this case an even function of the coordinates perpendicular to the symmetry axis.

The phenomenon considered below can be illustrated with a one-dimensional anharmonic oscillator with potential $U(x) = bx^2 + cx^4$ as the example. Depending an the ratio of the parameters b and c, the state can belong either to a discrete spectrum ($c > 0$) or to a continuous one ($c < 0$). If c is small enough and b is positive, the anharmonic increment cx^4 is inessential for the low-lying states. With decrease of b and at $c < 0$, the potential barriers become narrower and account must be taken of the level width, which determines the decay probability of the state. A similar effective potential for the motion perpendicular to the symmetry axis appears in the problems considered below. The parameters b and c are determined in this case in terms of Schrödinger-equation solutions localized in the vicinity of the symmetry axis. The stability limit of the state corresponds to the condition $b = 0$.

The theory developed permits a full investigation of the effects produced in the vicinity of a bifurcation of a localized state. We consider below tunnel phenomena that accompany the stability-loss process in an open optical cavity, and also similar phenomena for a highly excited hydrogen atom in a strong magnetic field. The latter problem, of great interest to theoreticians and experimenters (Refs. 10–13 and 20–23), is sometimes called the principal unsolved problem of elementary quantum mechanics.

1. EXPOSITION

1.1. Construction of asymptotic solution of the Schrödinger equation

Although we consider henceforth both quantum-mechanical and optical applications, we shall use the quantum terminology. Let a wave function Ψ satisfy the Schrödinger equation:

$$\tfrac{1}{2}\hbar^2\Delta\Psi+[E-V(\mathbf{r})]\Psi=0. \qquad (1.1)$$

We assume the potential $V(\mathbf{r})$ to be axisymmetric. The substitution $\Psi=\rho^{-1/2}\exp(im\varphi)u$ reduces Eq. (1.1) to the two-dimensional Schrödinger equation:

$$\tfrac{1}{2}\hbar^2(u_{zz}+u_{\rho\rho})+[E-V(z,\rho)-\hbar^2(m^2-\tfrac{1}{4})/2\rho^2]u=0. \qquad (1.2)$$

We shall consider below both the three-dimensional and the two-dimensional problems. In the latter case, we start from Eq. (1.2) with $m=\frac{1}{2}$.

Consider the solutions of (1.2) which are localized near the z axis, i.e., we assume that the characteristic dimension ρ_0 of the region in which the wave function is concentrated along the ρ axis is substantially smaller than the corresponding dimension z_0 relative to the z axis. To this end, we introduce the small parameter ε using the equation $\rho=\varepsilon^{1/2}x$ and assume that $x_0=\rho_0\varepsilon^{-1/2}$ and z_0 are of the same order of magnitude. Equation (1.2) then takes the form

$$\tfrac{1}{2}\hbar^2(u_{zz}+u_{xx}/\varepsilon)+[E-V(z,\varepsilon^{1/2}x)-\hbar^2(m^2-\tfrac{1}{4})/2\varepsilon x^2]u=0. \qquad (1.3)$$

We assume also that the potential V is quasiclassical or, formally, that the parameter $\hbar$ is small. We choose a system of units in which all the problem parameters are dimensionless, and assume for the time being that they are all of the order of unity, with the exception of the small parameters $\hbar$ and ε. We seek an asymptotic solution of Eq. (1.3) in the form

$$u=\Phi(\nu)\exp\{(i/\hbar)[\tau_0(z)+\hbar\tau_{01}(z)+\varepsilon\tau_1(z)x^2+\varepsilon^2\tau_2(z)x^4]\},\quad \nu=x/\sigma(z), \qquad (1.4)$$

neglecting in the exponential the terms of order $\hbar$, $\varepsilon^3/\hbar$, and higher. Substituting (1.4) in (1.3), we obtain

$$\begin{aligned}&\hbar^2\Phi_{\nu\nu}(1+O(\varepsilon))+\hbar\varepsilon\Phi_\nu[2i\nu\sigma(2\sigma\tau_1-\sigma_z\tau_{0z})+O(\hbar)+O(\varepsilon)]\\&+\varepsilon\Phi\{\sigma^2[2(\varepsilon-V^0(z))-\tau_{0z}^2]\\&-\hbar^2(m^2-\tfrac{1}{4})/\varepsilon\nu^2+\hbar\sigma^2(i\tau_{0zz}+2i\tau_1-2\tau_{0z}\tau_{01z})\\&-\varepsilon\nu^2\sigma^4(V_{\rho\rho}^0+2\tau_{0z}\tau_{1z}+4\tau_1^2)\\&-\varepsilon^2\nu^4\sigma^6(\tfrac{1}{12}V_{\rho\rho\rho\rho}^0+\tau_{1z}^2+2\tau_{0z}\tau_{2z}+16\tau_1\tau_2)\\&+O(\hbar^2)+O(\varepsilon\hbar)+O(\varepsilon^3)\}=0,\end{aligned} \qquad (1.5)$$

where $V^0=V(z,0)$, $V_{\rho\rho}^0=V_{\rho\rho}(z,\rho)|_{\rho=0}$ etc. At first glance the substitution (2.4) does not differ in any manner from the substitution used in the parabolic-equation method.[16] Our approach, which leads to a generalization of this method to the case of passage through the stability boundary, consists in the following. In the parabolic-equation method it is assumed that the sum of the terms of zeroth order in ν in the curly brackets of (1.5) is of the order of $O(\varepsilon)$, i.e., the same as of the expression quadratic in ν that follows it. It is natural in this case to discard the term proportional to ν^4, since it is of higher order $O(\varepsilon^2)$. As the state stability limit is approached, however, the coefficient of ν^2 tends to zero (the wave function spreads out as a function of ρ), so that the terms proportional to ν^4 also become important. We assume therefore hereafter that all the terms in the curly brackets in (1.5) are of the same order $O(\varepsilon^2)$, and stipulate satisfaction of the following additional conditions that determine the arbitrary functions $\sigma(z)$, $\tau_j(z)$, and $\tau_{01}(z)$:

$$2\sigma\tau_1-\sigma_z\tau_{0z}=0, \qquad (1.6)$$

$$i\tau_{0zz}+2i\tau_1-2\tau_{0z}\tau_{01z}=0, \qquad (1.7)$$

$$\sigma^2\{2[E-V^0(z)]-\tau_{0z}^2\}=\varepsilon^2 a, \qquad (1.8)$$

$$\sigma^4(V_{\rho\rho}^0+2\tau_{0z}\tau_{1z}+4\tau_1^2)=\varepsilon b, \qquad (1.9)$$

$$\sigma^6(\tfrac{1}{12}V_{\rho\rho\rho\rho}^0+\tau_{1z}^2+2\tau_{0z}\tau_{2z}+16\tau_1\tau_2)=c, \qquad (1.10)$$

where a, b, and c are for the time being arbitrary constants. Taking Eqs. (1.6)–(1.10) into account, we discard from (1.5) the terms of order $\varepsilon\hbar^2$, $\varepsilon^2\hbar$, ε^4 and higher. Equation (1.5) takes then the form

$$\frac{\hbar^2}{\varepsilon^3}\Phi_{\nu\nu}+[a-U_m(\nu)]\Phi=0,\quad U_m(\nu)=b\nu^2+c\nu^4+\frac{\hbar^2(m^2-\frac{1}{4})}{\varepsilon^3\nu^2}. \qquad (1.11)$$

The terms indicated can be neglected if the inequalities $\varepsilon\hbar^2$, $\varepsilon\hbar$, $\varepsilon^4 \ll \hbar^2$, ε^3 are pairwise satisfied. It is easily seen that all are equivalent to the inequalities $\varepsilon^2 \ll \hbar \ll \varepsilon$. Putting, for example, $\varepsilon=\hbar^q$, we get $\frac{1}{2}<q<1$.

All the known functions in (1.6)–(1.10) can be expressed in terms of the function $\sigma_0(z)$, which obeys the nonlinear ordinary differential equation

$$p^2\sigma_{0zz}+pp_z\sigma_{0z}+V_{\rho\rho}^0\sigma_0=\sigma_0^{-3},\quad p(z)=2^{1/2}[E-V^0(z)]^{1/2}. \tag{1.12}$$

Then

$$\sigma=(\varepsilon b)^{1/4}\sigma_0,\qquad \tau_0=d+\int_{z_1}^{z} p\,dz-\frac{\varepsilon^{3/2}a}{2b^{1/2}}\int_{z_1}^{z}\frac{dz}{p\sigma_0^2}, \tag{1.13}$$

$$\tau_{01}={}^1/_2 i\ln(p\sigma_0),\quad \tau_1=p\sigma_{0z}/2\sigma_0, \tag{1.14}$$

$$\tau_2=\sigma_0^{-4}\left\{f-\frac{1}{2}\int_{z_1}^{z}\left[\sigma_0^4\left(\frac{p\sigma_{0z}}{2\sigma_0}\right)_z^2-\frac{\sigma_{0z}^2}{\sigma_0^2}-\frac{c}{(\varepsilon b)^{3/2}\sigma_0^2}+\frac{\sigma_0^4}{12}V^0_{\rho\rho\rho\rho}\right]\frac{dz}{p}\right\}, \tag{1.15}$$

where d and f are arbitrary constants. In turn, all the solutions of (1.12) are expressed in terms of the solution of a homogeneous equation that is linearized with respect to ρ by the equation for the classical trajectories adjacent to the z axis:

$$p^2y_{zz}+pp_zy_z+V_{\rho\rho}^0y=0, \tag{1.16}$$

using the equation

$$\sigma_0(z)=[a_{11}y_1^2(z)+2a_{12}y_1(z)y_2(z)+a_{22}y_2^2(z)]^{1/2},\quad a_{11}a_{22}-a_{12}^2=1, \tag{1.17}$$

where a_{ij} are constants, and y_1 and y_2 are solutions of (1.16) and meet the condition

$$w(p^{1/2}y_1,\ p^{1/2}y_2)=(p^{1/2}y_1)_zp^{1/2}y_2-p^{1/2}y_1(p^{1/2}y_2)_z=1. \tag{1.18}$$

The Wronskian (1.18) does not depend on z.

We seek next the stationary or quasistationary states in the classically accessible z in the form of the real part of (1.4)

$$u=\Phi(\nu)(p\sigma_0)^{-1/2}\cos[\hbar^{-1}(\tau_0+\varepsilon\tau_1x^2+\varepsilon^2\tau_2x^4)]. \tag{1.19}$$

It follows from (1.13)–(1.17) that the wave function (1.19) contains, besides the normalization constant, also the seven arbitrary parameters E, f, d, a_{11}, a_{12} and

$$\tilde{a}=\varepsilon^{3/2}b^{-1/2}a,\quad \tilde{c}=(\varepsilon b)^{-3/2}c. \tag{1.20}$$

These constants can be determined after formulating the boundary-value problem.

1.2 Rules for bypassing the zeros of $\sigma_0(z)$

We obtained the asymptotic solution by assuming that the function $\sigma(z)$ is of the order of unity and therefore, as follows from (1.13), $\sigma_0(z)$ is large. The zeros of $\sigma(z)$, however, which correspond to foci, can lie asymptotically close to or on the real z axis. The approximation employed may therefore not be valid here. To establish the rules for bypassing these (asymptotically small) regions, it is necessary to add to the original Ansatz one more arbitrary function $\lambda(z)$, putting $\nu = x/\sigma + \varepsilon^{3/2}b^{1/2}\lambda(x/\sigma)^3$. The last equation in the system (1.6)–(1.10) is then replaced by two equations:

$$2p(\sigma_0^4\tau_2)_z-8\lambda/\sigma_0^2=\tilde{c}/\sigma_0^2-\sigma_0^4(p\sigma_{0z}/2\sigma_0)_z^2+\sigma_{0z}^2/\sigma_0^2-{}^1/_{12}\sigma_0^4V^0_{\rho\rho\rho\rho},$$
$$2p\lambda_z+8\sigma_0^2\tau_2=\sigma_0\sigma_{0z}(p\sigma_{0z}/\sigma_0)_z. \tag{1.21}$$

For most σ_0 this system splits, λ becomes small, and the first equation goes over into (1.10). In the general case the system (1.21) can also be integrated.[16] To this end it suffices to note that the corresponding homogeneous system has the solutions

$$\tau_2=\sigma_0^{-4}\exp\left(\pm 4i\int^{z}\frac{dz}{p\sigma_0^2}\right),\quad \lambda=\pm i\exp\left(\pm 4i\int^{z}\frac{dz}{p\sigma_0^2}\right).$$

Let $z^\times$ be the root of the equation $y_1(z)+(a_{12}/a_{11})y_2(z)=0$, let a_{ij} be real, and let $a_{11}\to\infty$. The function $\sigma_0^2(z)$ can be represented near $z^\times$ in the form

$$\sigma_0^2(z)=\xi_1(z-z^\times)^2+\xi_2,\quad \xi_1=a_{11}\left(y_{1z}+\frac{a_{12}}{a_{11}}y_{2z}\right)^2\bigg|_{z=z^\times}\to\infty,$$
$$\xi_2=\frac{y_2^2(z^\times)}{a_{11}}\to 0.$$

The zeros of $\sigma_0(z)$ are in this case complex. Using the explicit solutions of the system (1.21), we can now show that on passing through the vicinity of $z^\times$ only the term proportional to $\tilde{c}$ in expression (1.15) for τ_2 will acquire a jumplike increase

$$\tilde{c}\int_{z_1}^{z}\frac{dz}{p\sigma_0^2}\bigg|_{z<z^\times}\to\tilde{c}\int_{z_1}^{z}\frac{dz}{p\sigma_0^2}\bigg|_{z>z^\times}-\pi\tilde{c}.$$

This means that the integral of the indicated term in (1.15) must be calculated by bypassing the complex zero $\sigma_0(z)$ as shown in Fig. 1. The bypass rule for the case when $z^\times$ coincides with a turning point of the momentum $p(z)$ is obtained in the same manner.

1.3 Structure of wave function and rule for quantization with respect to the transverse coordinate

The possible situations that set the structure of the solution of the one-dimensional Schrödinger equation (1.11), as well as of the wave function (1.19), are shown in Fig. 2. A distinction must be made here between the two-dimensional problem and the three-dimensional one. For three dimensions, depending on the relations between the parameters $\tilde{a}, \tilde{c}$ and $\hbar^2(m-\frac{1}{4})$, there can be realized the case A 1 corresponding to the discrete spectrum of Eq. (1.11), and to solutions localized over x or the case B 1 when the states localized near the z axis are quasi-stationary in view of the possibility of tunneling through the potential barrier along the coordinate ν. Finally, in the case C 1 there are no localized states near

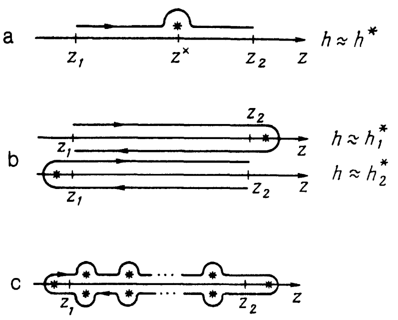

FIG. 1. Integration contour C in the complex z plane (the asterisks mark the zeros of the function $\sigma(z)$: a—for an optical cavity at $h\approx h^*$; b—for an optical cavity at $h\approx h_1^*$ and $h\approx h_2^*$; c—characteristic form of C for a hydrogen atom in a magnetic field (there is only one zero of $\sigma(z)$ for states localized near a symmetry plane—either near z_1 or near z_2). The locations of the zeros of $\sigma(z)$ correspond to location of the symmetry axis in the stability region (to real coefficients a_{ij}).

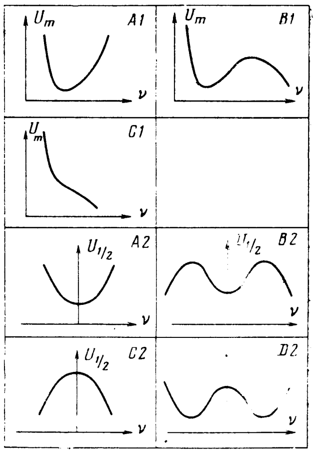

FIG. 2. Form of effective potential $U_m(\nu)$ in the three- and two-dimensional cases.

the z axis in the approximation considered. For the two-dimensional problem, $m = 1/2$ and there is no centrifugal term, so that besides the cases $A\,2$, $B\,2$, $C\,2$, which are similar to the corresponding cases of the three-dimensional problem, there is also a possible case $D\,2$ when a two-well potential, for which instanton solutions can be determined, appears in Eq. (1.11). The sign of b is reversed when the parameters of the physical system are changed. The case $A\,1$ does not change qualitatively in this case, and transitions $B\,1 \to C\,1$. $B\,1 \to C\,2$, and $A\,2 \to D\,2$ take place.

To ensure localization of the wave function in the vicinity of the z axis, we require that the solution (1.19) be exponentially small at large ρ (or ν). This condition can be met in the cases $A\,1$, $B\,1$, $A\,2$, $B\,2$, and $D\,2$ (Fig. 2) by choosing the corresponding solution of Eq. (1.11); in the cases $B\,1$, $B\,2$, $D\,2$ the barriers are assumed to be wide enough to make the corresponding quasistationary state long-lived and the level splitting exponentially small[2)] The condition that $\Phi(\nu)$ be exponentially small at large ν in case A leads to a quantization rule that takes in the semiclassical approximation the form

$$\frac{1}{\hbar}\oint[\tilde{a}-U_m(\tilde{\nu})]^{1/2}\,d\tilde{\nu}=2\pi\left(k+\frac{1}{2}\right),$$

$$k,\ |m|\gg 1 \text{ and integers,} \qquad (1.22)$$

$$\tilde{U}_m(\tilde{\nu})=\tilde{\nu}^2+\tilde{c}\tilde{\nu}^4+m^2\hbar^2/\tilde{\nu}^2,\qquad \tilde{\nu}=\varepsilon^{3/4}b^{1/4}\nu,$$

where the integration contour encloses a cut joining the turning points in the potential well in the complex ν (or $\tilde{\nu}$) plane. At $k \sim |m| \sim 1$ and $\tilde{c}\hbar \ll 1$ the term $\tilde{c}\nu^4$ in the well can be neglected and the quantization rule becomes particularly simple:

$$\tilde{a}=2\hbar(2k+|m|+1) \text{ in case } A1,$$

$$\tilde{a}=2\hbar(k+1/2) \text{ in case } A2, \qquad (1.23)$$

$$\tilde{a}+(4\tilde{c})^{-1}=2^{1/2}\hbar(k+1/2) \text{ in case } D2. \qquad (1.24)$$

In case B, Eqs. (1.22) and (1.23) determine the real parts of the transverse quasilevels. In case $D\,2$, Eqs. (1.22) and (1.24) take no account of the exponentially small level splitting.

1.4 Equations for the level width and splitting

Confining ourselves to semiclassical potential barriers and to states with small quantum numbers k and m, we assume that the condition $\hbar\tilde{c} \ll 1$ is met. We also assume satisfaction of conditions $\tilde{a}\tilde{c} \ll 1$ and $\hbar|m| \lesssim \tilde{a}$ in case B and of the condition $|4\tilde{a}\tilde{c}+1| \ll 1$ in case $D\,2$. We assume throughout that the motion along the z coordinate is between the turning points z_1 and z_2.

For semiclassical potential barriers, the level width Γ is known to be the product of $\hbar$ by the total flux of the wave function normalized to unity in the vicinity of a finite classically allowed region.[24] If there are no zeros of $\sigma(z)$ near the interval (z_1, z_2), after continuing in standard fashion the wave function of the bound state through the barrier and calculating the flux integral, we have in case $B\,1$

$$\Gamma=\frac{\hbar}{k!\,(|m|+k)!}\left(\frac{4}{\hbar|\tilde{c}|}\right)^{2k+|m|+1}\times\int_{z_1}^{z_2}\frac{dz}{p\sigma_0^2}\left(\int_{z_1}^{z_2}\frac{dz}{p}\right)^{-1}\exp\left(-\frac{2}{3\hbar|\tilde{c}|}\right), \qquad (1.25)$$

and in case $B\,2$

$$\Gamma=\frac{2^k\hbar}{\pi^{1/2}k!}\left(\frac{4}{\hbar|\tilde{c}|}\right)^{k+1/2}\int_{z_1}^{z_2}\frac{dz}{p\sigma_0^2}\left(\int_{z_1}^{z_2}\frac{dz}{p}\right)^{-1}\exp\left(-\frac{2}{3\hbar|\tilde{c}|}\right), \qquad (1.26)$$

if zeros of $\sigma(z)$ are present near the interval (z_1, z_2), they lie near the intersection points $z_l^\times$ of the detached closed trajectories with the z axis, and are arranged as shown in Fig. 1. The integrand $(p\sigma_0^2)^{-1}$ in (1.25) or (1.26) has a sharp peak near the points $z_l^\times$ (see Sec. 1.2), and the corresponding integral converges in an asymptotically small vicinity of these points. The reason for this peak is that the classical trajectories in the effective potential well near the stability limit are tangent to the internal caustics $\rho = \pm(\varepsilon a/b)^{1/2}\sigma(z)$ predominantly near the points $z_l^\times$ and have there a higher density. The tunneling probability, in turn, is proportional both to the density and to the number of tangencies.

At values of z in the close vicintiy of $z_l^\times$, the obtained asymptotic relation remains in force only when $\nu \ll (-b/c)^{1/2}$. If we assume that we can use in these vicinities an Ansatz in which, in contrast to (1.4), both the argument of the exponential and the argument of the function $\Phi(\nu)$ are represented as series in ν, and leave the standard equation (1.21) unchanged, we can show that the results (1.25) and (1.26) remain in force also in the general case. This statement is actually equivalent to the hypothesis[26] of existence of a caustic that limits the potential barrier from the outside and coincides with one of the level lines $\mathrm{Im}\,\tau(\rho,z) = \mathrm{const}$ ($\tau(\rho,z)$ is the complex classical action below the barrier).

The level splitting in case $D\,2$ can be calculated by an asymptotic method,[24] introducing a symmetric $(\Psi_1 + \Psi_2)$ and antisymmetric $(\Psi_1 - \Psi_2)$ combination of wave functions Ψ_1 and Ψ_2 localized respectively near each of the detached trajectories. If one of the points $z_l^\times$ coincides with the turning point z_j, loss of stability is accompanied by a detachment not of two closed trajectories symmetric about the z axis, but one closed trajectory with double the period. No instanton states are produced in this case. If, however, the points $z_l^\times$ lie outside the segment (z_1, z_2), the level splitting is

$$\Delta E=\frac{\hbar}{\pi^{1/2}k!}\,(2^{3/2}\hbar\,|\tilde c\,|)^{-k-1/2}\left|\int_{z_1}^{z_2}\frac{s\,dz}{p\sigma_0^{\,2}}\left(\int_{z_1}^{z_2}\frac{s\,dz}{p}\right)^{-1}\right|$$

$$\times\exp\left(-\frac{2^{1/2}}{3\hbar\,|\tilde c\,|}\right),\qquad(1.27)$$

where we have introduced the function $s(z) = (-1)^l$ at $z_l^{\times} < z < z_{l+1}^{\times}$, in which the intersection points $z_l^{\times}$ are numbered from unity of L and we use the notation $z_0^{\times} = z_1$, $z_{L+1}^{\times} = z_2$. The integral of $s(p\sigma_0^2)^{-1}$ in (1.27) must be taken near the singularities in the sense of the principal value.

2. TUNNEL EFFECTS IN AN OPEN OPTICAL CAVITY

The diffraction losses in an open optical cavity can be caused not only by edge effects due to the finite dimensions of the reflecting surfaces,[25] but also to tunneling processes which are generally speaking not connected with these dimensions. This circumstance was pointed out in Ref. 26. Specific results on this subject were obtained for slightly bent surfaces of the cavity.[27] Similar phenomena take place also near its stability boundary.[28]

We investigate in the present section tunnel phenomena in the case of a uniformly filled cavity. Depending on the relations obtained below between the parameters of the reflecting surfaces, and on the distance between them, we show that in the approximation considered there can exist near the stability limit states that are quasistationary, stationary, or, in the two-dimensional case, instantons. Analytic expressions are obtained for the corresponding values of the level widths and splitting.

2.1 Boundary-value problem and quantization rule

We consider the solutions of Eq. (1.1) in a narrow tube near the z axis between surface S_1 and S_2. We assume for simplicity that the boundary conditions satisfied on S_1 and S_2 are

$$u|_{S_1}=u|_{S_2}=0.\qquad(2.1)$$

Equations (2.1) are equivalent to $\Psi|_{S_1} = \Psi|_{S_2} = 0$. The conditions (2.1) can be met by stipulating that in the function (1.19) the value of $\tau_0 + \varepsilon\tau_1\chi^z + \varepsilon^2\tau_2\chi^4$ be constant on S_1 and S_2 accurate to $O(\varepsilon^2)$. Let the equation of the boundary S_j be at small ρ of the form

$$z^{(j)}(\rho)=z_j+\alpha_j\rho^2+\beta_j\rho^4+\ldots=z_j+\varepsilon\alpha_jx^2+\varepsilon^2\beta_jx^4+\ldots.\qquad(2.2)$$

This requirement leads then to four additional conditions for τ_j ($j = 1, 2$):

$$\alpha_jp(z_j)+\tau_1(z_j)=0,\qquad(2.3)$$

$$\beta_jp(z_j)+{}^1/_2\alpha_j^2p_z(z_j)+\alpha_j\tau_{1z}(z_j)+\tau_2(z_j)=0.\qquad(2.4)$$

The boundary condition for u will be met if we stipulate also satisfaction of the quantization rule

$$\frac{1}{\hbar}\int_{z_1}^{z_2}p\,dz-\frac{\tilde a}{2\hbar}\int_{z_1}^{z_2}\frac{dz}{p\sigma_0^{\,2}}=\pi n,\quad n\gg1\ \text{ and integer}\qquad(2.5)$$

and put $d = \pi\hbar/2$. The conditions (2.3)–(2.5) together with (1.22) determine completely the remaining free parameters of the wave function (1.19), with two parameters of the function $\sigma_0(z)$ determined from two equations (2.3) independently of the remaining ones. After obtaining them we have from two equations (2.4), with allowance for the results of Sec. 1.2,

$$\tilde c=\left(\int_{z_1}^{z_2}\frac{dz}{p\sigma_0^{\,2}}\right)^{-1}\Bigg\{2(\varkappa_1-\varkappa_2)$$

$$+\int_{z_1}^{z_2}\left[\sigma_0^{\,4}\left(\left(\frac{p\sigma_{0z}}{2\sigma_0}\right)_z^2+\frac{V_{\rho\rho\rho\rho}}{12}\right)-\frac{\sigma_{0z}^{\,2}}{\sigma_0^{\,2}}\right]\frac{dz}{p}\Bigg\},$$

$$f=-\varkappa_1,\qquad(2.6)$$

$$\varkappa_j=\sigma_0^{\,4}(z_j)\,[\beta_jp(z_j)+{}^1/_2\alpha_j^2p_z(z_j)+\alpha_j(p\sigma_{0z}/2\sigma_0)_z|_{z=z_j}],$$

where the integral in the first factor is taken with the complex zeros of $\sigma_0(z)$ bypassed as shown in Fig. 1a.

2.2 Homogeneous open optical cavity

In the homogeneous case we have $p(z) = p_0 = \text{const}$, $V_{\rho\rho}^0 = V_{\rho\rho\rho\rho}^0 = 0$. Equation (1.16) is easily solved, and the condition (2.3) on the boundaries S_1 and S_2 of the cavity determine all the free parameters of the function $\sigma_0(z)$. As a result we have[16]

$$\sigma_0(z)=p_0^{-1/2}(a_{11}+2a_{12}z+a_{22}z^2)^{1/2},$$

$$a_{11}=-a_{12}/2\alpha_1$$

$$=[h(1+2\alpha_2h)]^{1/2}[2(1-2\alpha_1h)(\alpha_1-\alpha_2+2\alpha_1\alpha_2h)]^{-1/2},\qquad(2.7)$$

$$a_{22}=2^{1/2}(\alpha_1-\alpha_2+4\alpha_1\alpha_2h)$$

$$\times[h(1-2\alpha_1h)(1+2\alpha_2h)(\alpha_1-\alpha_2+2\alpha_1\alpha_2h)]^{-1/2},$$

$$h=z_2-z_1=z_2.$$

The origin is here on the symmetry axis and coincides with $z = z_1$.

We determine now the critical distances between the mirrors:

$$h^*=(\alpha_2-\alpha_1)/2\alpha_1\alpha_2,\quad h_1^*=1/2\alpha_1,\quad h_2^*=-1/2\alpha_2.\qquad(2.8)$$

Let initially the Surfaces S_1 and S_2 be identically concave and let $\alpha_2 \geqslant \alpha_1 > 0$. It follows from (2.7) and (2.8) that $h_1^* > h_2^*$ and the symmetry axis is stable only in the interval $h^* < h < h_1^*$. Expressions (2.7) for a_{ij} can be simplified near the critical value $h = h^*$. The simplifications, which will be treated in greater detail at the end of Sec. 3.1, can be carried out also in Eq. (2.6). Recognizing, finally, that in this case ther are no points of intersection of the detached trajectories with the z axis, we get at $|h - h^*| \ll 1$

$$\tilde c=\frac{(\alpha_1\alpha_2)^{1/2}}{4p_0(\alpha_2-\alpha_1)}\left[\alpha_1-\alpha_2+\frac{\beta_1\alpha_2}{\alpha_1^{\,3}}-\frac{\beta_2\alpha_1}{\alpha_2^{\,3}}\right]\left(\frac{h}{h^*}-1\right)^{-3/2},$$

$$\int_{z_1}^{z_2}\frac{dz}{p_0\sigma_0^{\,2}}\left(\int_{z_1}^{z_2}\frac{dz}{p_0}\right)^{-1}=2p_0\left[\alpha_1\alpha_2\left(\frac{h}{h^*}-1\right)\right]^{1/4}.\qquad(2.9)$$

It follows from this equation that if

$$\alpha_1-\alpha_2+\beta_1\alpha_2/\alpha_1^{\,3}-\beta_2\alpha_1/\alpha_2^{\,3}<0\qquad(2.10)$$

a stable state approaching the stability limit becomes quasistationary with an intrinsic-energy width determined by expressions (2.9), (1.26), and (2.9), (1.25) for the two- and three-dimensional cases, respectively. If the inverse of inequality (2.10) is satisfied, the states in the stability region near the boundary are stationary in the approximation considered, but on the other side of the boundary they take in the two-dimensional case the form of instantons, with the level splitting determined by Eqs. (2.9) and (1.27).

At $h = h_1^*$ the point of intersection of the detached trajectories coincides with the turning point z_2 and bifurcation with doubling of the period takes place, such that a V-shaped closed trajectory is detached from the symmetry axis. It can be shown in this case in analogy with Sec. 1.2 that the first integral in (2.6) should be evaluated along the contour shown in Fig. 1b and divided by two. As a result we get for $|h - \hbar_1^*|$

$$\tilde{c} = \frac{(\alpha_1+\alpha_2)^{1/2}}{2p_0\alpha_1^{5/2}}[\alpha_1^3-\beta_1]\left(1-\frac{h}{h_1^*}\right)^{-3/2},$$

$$\int_{z_1}^{z_2}\frac{dz}{p_0\sigma_0^2}\left(\int_{z_1}^{z_2}\frac{dz}{p_0}\right)^{-1} = \pi p_0\alpha_1. \qquad (2.11)$$

It follows hence that $\alpha_1^3 > \beta_1$ the passage through the stability limit is accompanied by detachment of the V-like stable closed trajectory, in the vicinity of which are localized V-like states. If, however, $\alpha_1^3 < \beta_1$, then at $0 < h < h_1^* \ll h_1^*$ the states near the symmetry axis are quasistationary with a level width determined from Eqs. (2.11), (1.25), and (1.26).

Let now the surfaces S_1 and S_2 be inwardly concave, i.e., $\alpha_1 > 0$, $\alpha_2 < 0$. We assume for the sake of argument at $\alpha_1 \leqslant -\alpha_2$. It follows from (2.7) that instability takes place in this case only at $h^* > h > h_1^*$ and at $h < h_2^*$. If $\alpha_1 = -\alpha_2$, the entire interval $0 < h < h_2^*$ is stable. Finally, if $\alpha_1 = 0$, the stability region coincides with the interval $0 < h < h_2^*$. Let us consider the situation when the distance between the mirrors is close to h^*. The intersection point $z^{\times} = (2\alpha_1)^{-1}$ of the detached closed trajectories lies then inside the interval (z_1, z_2). Taking this circumstance into account, we chose for (2.6) the integration contour shown in Fig. 1a. As a result we have

$$\tilde{c} = \frac{(-\alpha_1\alpha_2)^{1/2}}{4p_0(\alpha_1-\alpha_2)}\left[\alpha_1-\alpha_2+\frac{\beta_1\alpha_2}{\alpha_1^3}-\frac{\beta_2\alpha_1}{\alpha_2^3}\right]\left(1-\frac{h}{h^*}\right)^{-3/2},$$

$$\int_{z_1}^{z_2}\frac{s\,dz}{p_0\sigma_0^2}\left(\int_{z_1}^{z_2}\frac{s\,dz}{p_0}\right)^{-1} = -2p_0\left[\alpha_1\alpha_2\left(\frac{h}{h^*}-1\right)\right]^{1/2}, \qquad (2.12)$$

$$\int_{z_1}^{z_2}\frac{dz}{p_0\sigma_0^2}\left(\int_{z_1}^{z_2}\frac{dz}{p_0}\right)^{-1} = \frac{\pi p_0}{h^*}.$$

This case is similar to the case $h \approx h^*$ for $\alpha_2 \geqslant \alpha_1 > 0$. In particular, the expression for c differs from the corresponding value in (2.9) only in sign. Equations (2.12) and (1.25)–(1.27) determine the width and splitting of the levels for the quasistationary and instanton solutions, respectively.

We present finally the expressions for the parameters of Eqs. (1.25)–(1.27) near the critical distances h_1^* and h_2^*. At $|h - h_1^*| \ll h$ we have

$$\tilde{c} = \frac{(-\alpha_1-\alpha_2)^{1/2}}{2p_0\alpha_1^{5/2}}[\beta_1-\alpha_1^3]\left(\frac{h}{h^*}-1\right)^{-3/2},$$

$$\int_{z_1}^{z_2}\frac{dz}{p_0\sigma_0^2}\left(\int_{z_1}^{z_2}\frac{dz}{p_0}\right)^{-1} = \pi p_0\alpha_1,$$

and at $|h - h_2^*| \ll h$

$$\tilde{c} = \frac{(-\alpha_1-\alpha_2)^{1/2}}{2p_0(-\alpha_2)^{5/2}}[\beta_2-\alpha_2^3]\left(1-\frac{h}{h_2^*}\right)^{-3/2},$$

$$\int_{z_1}^{z_2}\frac{dz}{p_0\sigma_0^2}\left(\int_{z_1}^{z_2}\frac{dz}{p_0}\right)^{-1} = -\pi p_0\alpha_2.$$

These cases are perfectly analogous to the case $h \approx h_1^*$ at $\alpha_2 \geqslant \alpha_1 > 0$.

3. HIGHLY EXCITED HYDROGEN-ATOM STATES LOCALIZED IN A STRONG MAGNETIC FIELD NEAR A SYMMETRY AXIS AND A SYMMETRY PLANE

The classical motion of a particle in a combination of a Coulomb field with a magnetic field of comparable strength is generally speaking stochastic.[22] Experiments (including numerical ones), however, with highly excited atoms in a strong magnetic field point to a definite regularity of the spectrum. This is attribuited in Refs. 21 and 22 to the presence of closed trajectories along which the heuristic quantization rule agrees with the experimental data. The question is, how correct is such a quantization rule? If the considered closed trajectory is stable, the answer is known.[17] It is also known that the quasiclassical wave function of a system has in the stochastization region an increased density near unstable closed trajectories.[30] If such states have a sufficiently long lifetime, they should also be manifested in the observed spectrum.

From among the closed trajectories of the considered problem, a special role is played by trajectories on the symmetry axis z, and also on the symmetry axis ρ for the two-dimensional equation (1.2), since the most pronounced are transitions into states that are elongated along the z axis in the case of longitudinal polarization of the radiation, and states located near the symmetry plane in the case of transverse polarization.[20,31] The trajectories lying on the symmetry axis z are stable only in certain bands.[20] A stability region for trajectories lying on the ρ axis (or in the symmetry plane $z = 0$) is also known.[32] In addition, an additional burst of radiation can occur on the stability region.[20,31]

We show in the present section that stable states localized near the z axis are separated from the other states by a potential barrier whose parameters have been found in analytic form. The z axis loses stability after its calescence on the ρz plane with two closed trajectories. We obtain below analytic expressions for families of such trajectories (some of which were obtained numerically in Refs. 21, and 29). Elementary equations for the coalescence points[20] are in good agreement with the numerical calculation.[29]

We show next the states located near the symmetry plane, outside the stability boundary, become U-shaped. In this case a trajectory that is stable on the ρ axis loses stability after passing through the boundary, and a stable U-shaped trajectory not lying on the ρ axis is detached from it. The quantization rules and the parameters of the solutions are obtained in analytic form at high energy E, and numerically in the general case. We show also that at $E = 0$ the symmetry plane is unstable and we calculate the lifetime of the states localized near a symmetry plane with $E = 0$.

3.1. Semiclassical quantization rule

If the potential $V(r)$ is an analytic function of the coordinates in all of space, and the motion along z is restricted between two turning points z_1 and z_2, we require, to determine the remaining free parameters, that the continuations of the wave function (1.19) beyond the turning point z_1 and z_2 into the region classically forbidden in z decrease exponentially. The momentum $p(z)$ has in this case a square-root instability at the point z_1 and z_2. We can therefore use Zwann's method[33] and stipulate that the asymptotic form (1.19) be transformed into itself after bypassing, along the closed countour C, the turning points z_1 and z_2 in the complex z plane with a cut $[z_1, z_2]$. The contour C should in this case enclose the zeros of the function $\sigma^2(z)$ (Fig. 1c). For the method to be valid it suffices if the momentum $p(z)$ in the vicinity of the turning point z_j is of the form $p(z) \sim \eta_j(z)(z-z_j)^{1/2}$, where $\eta j(z)$ is an arbitrary meromorphic function.[33] In particular, putting $\eta_j = (z-z_j)^{-1}$ we find that Zwann's method is valid if the potential has at the point z_j a Coulomb singularity. This statement was verified for the three-dimensional case in Ref. 20 by directly matching the solutions in the vicinity of a three-dimensional Coulomb singularity.

The "transition-into-itself" condition for the asymptotic form of the solution (1.19) can be represented as an aggregate of conditions for the functions σ_0, τ_0, τ_1 and τ_2. These conditions for σ_0, τ_0, and τ_1 were discussed earlier in Ref. 34 for two simple turning points and in Ref. 20 for the case when one of them is a Coulomb point.

In the vicinity of a simple turning point z_j, those solutions of (1.16) which satisfy the condition (1.18) are of the form

$$y_1^{(j)}=1+O(z-z_j),\quad y_2^{(j)}=\int\limits_{z_j}^{z}\frac{dz}{p}+O(|z-z_j|^{3/2}),\tag{3.1}$$

while in the vicinity of a Coulomb point, where $V^0(z) = -\alpha/|z-z_j|$, they can be expressed as

$$y_1^{(j)}(z)=(z-z_j)+O((z-z_j)^2),$$
$$y_2^{(j)}(z)=p(z)(z-z_j)/\alpha+O(|z-z_j|^{3/2}).\tag{3.2}$$

Let y_1 and y_2 be solutions of (1.16) which go over as $z \to z_1$ into the solutions $y_1^{(1)}$ and y_2^2, respectively. In the vicinity of the turning point z_2 the functions y_1 and y_2 go over then into linear combinations of the solutions $y_1^{(2)}$ and $y_2^{(2)}$ and it is possible to determine the constant matrix $\mathbf{T}$ from the equation

$$\mathbf{y}\underset{z\to z_2}{\approx}\mathbf{T}\mathbf{y}^{(2)},\quad \mathbf{y}=\begin{pmatrix}y_1\\ y_2\end{pmatrix},\quad \mathbf{y}^{(2)}=\begin{pmatrix}y_1^{(2)}\\ y_2^{(2)}\end{pmatrix},\quad \mathbf{T}=\|t_{ij}\|.\tag{3.3}$$

This matrix transforms the solutions $y_k^{(2)}$ of Eq. (11.6) in the vicinity of z_1 into analogous solutions y_k^2 in the vicintiy of z_2. It turns out that the condition for transformation into itself for the solution $\sigma_0(z)$ determines completely the arbitrary constants in (1.17):[3)]

$$a_{11}=a_{22}^{-1}=(-t_{22}t_{21}/t_{11}t_{12})^{1/2},\quad a_{12}=0.\tag{3.4}$$

It follows from (1.14) that the condition of transformation into itself coincides for τ_1 with (3.4). For the function τ_0 to meet this condition, it is necessary to satisfy the quantization rule

$$\frac{1}{\hbar}\oint\limits_c p\,dz-\frac{\tilde a}{2\hbar}\oint\limits_c\frac{dz}{p\sigma_0^2}=2\pi\left(n+\frac{1}{2}\right),\quad n\gg1 \text{ and integer}\tag{3.5}$$

and the equality $d=\pi\hbar/4$. Equation (3.5) can be transformed into

$$\frac{1}{\hbar}\oint\limits_c p\,dz-\frac{\tilde a}{\hbar}\operatorname{sgn}\left(\frac{t_{21}}{t_{11}}\right)\left\{\operatorname{arctg}\left[\left(-\frac{t_{12}t_{21}}{t_{11}t_{22}}\right)^{1/2}\right]+\pi N\right\}$$
$$=2\pi\left(n+\frac{1}{2}\right),\tag{3.6}$$

where N is the number of zeros of the solution $y_1(z)$ on the open interval (z_1, z_2), and $\operatorname{sgn}(t_{21}/t_{11})$ was unfortunately omitted in Refs. 20 and 21. The region of the symmetry-axis stability corresponds to a positive radicand in (3.6), and its limit corresponds to a zero or infinite radicand.

Finally, the indicated condition, when applied to the function τ_2, yields

$$\tilde c=\left(\oint\limits_c\frac{dz}{p\sigma_0^2}\right)^{-1}\oint\limits_c\left\{\sigma_0^4\left[\left(\frac{p\sigma_{0z}}{2\sigma_0}\right)_z^2+\frac{V^0_{\rho\rho\rho\rho}}{12}\right]-\frac{\sigma_{0z}^2}{\sigma_0^2}\right\}\frac{dz}{p},\quad f=0.\tag{3.7}$$

It can be shown that near the stability limit the expression for $\tilde c$ simplifies to

$$\tilde c=\frac{1}{4}\left(\oint\limits_c\frac{dz}{p\sigma_0^2}\right)^{-1}\oint\limits_c\left[(\sigma_0^2V_{\rho\rho}^0+\sigma_{0z}^2p^2)^2+\frac{1}{3}\sigma_0^4p^2V^0_{\rho\rho\rho\rho}\right]\frac{dz}{p^3},\tag{3.8}$$

in which we must put $\sigma_0 = a_{11}{}^{1/2}y_1$ or $\sigma_0 = a_{22}{}^{1/2}y_2$, depending on whether the stability limit corresponds to a zero of a_{22} or to a zero of a_{11}. A similar simplifying assumption of (26) was made in the calculations of Sec. 2.

3.2 Highly excited states localized near the *z* axis

The potential in the Schrödinger equation (1.1) for a hydrogen atom in a magnetic field B can be represented in the form

$$V(\mathbf{r})=-\alpha/r+\gamma^2\rho^2/8,\quad \gamma=B/B_0,\quad B_0=2.35\cdot10^5\ \text{T}.\tag{3.9}$$

We expand (3.9) near the z axis accurate to terms of fourth order in ρ:

$$V(\mathbf{r})\approx V^0(z)+{}^1/_2V_{\rho\rho}{}^0(z)\rho^2+{}^1/_{24}V^0_{\rho\rho\rho\rho}(z)\rho^4,$$
$$V^0(z)=-\alpha/z,\quad V_{\rho\rho}{}^0(z)=\alpha/z^3+\gamma^2/4,\quad V^0_{\rho\rho\rho\rho}(z)=-9\alpha/z^5.\tag{3.10}$$

In this case the momentum $p(z)$ has one Coulomb turning point and one simple one. We introduce the dimensionless coordinate $\zeta = -Ez/a(E<0)$ and the dimensionless parameter $\omega = \gamma^2a^2/|2E|^3$. Equation (1.6) takes then the form

$$2\zeta^2(1-\zeta)y_{\zeta\zeta}-\zeta y_\zeta+(1-8\omega\zeta^3)y=0.\tag{3.11}$$

The solutions of this equation for weak ($\omega \ll 1$) and strong ($\omega \gg 1$) magnetic fields were investigated in Ref. 20. We consider only the case of a strong field. In this case the stability region of the trajectory lying on the z axis constitutes a set of bands defined by the inequalities[20]

$$\omega_q^-<\omega<\omega_q^+,\quad (\omega_q^\pm)^{1/2}=q+{}^1/_2\pm{}^1/_6\tag{3.12}$$

or by the equivalent inequalities (see Fig. 3a)

$$\gamma_{nq}^-<\gamma<\gamma_{nq}^+,\quad \gamma_{nq}^\pm=(\alpha^2/n^3)(q+{}^1/_2\pm{}^1/_6),\tag{3.13}$$

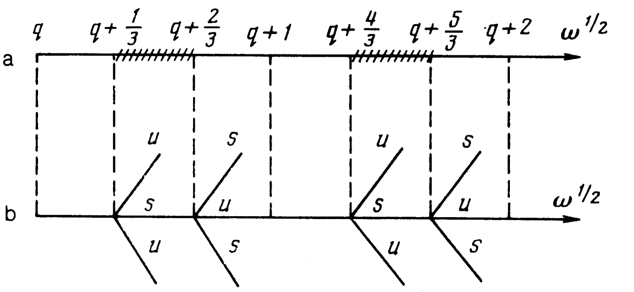

FIG. 3. a) Stability bands for hydrogen atoms in a strong magnetic field (hatched). b) Scheme of detachment of closed stable (s) and unstable (u) trajectories at the stability boundary.

where $q \gg 1$ is an integer, and the quantum number n together with the integers k and $m \ll n$ and with the number q of the band determine the energy levels:

$$E_{mnk} = -\frac{\alpha^2}{2\hbar^2 n^2} + \frac{(-1)^{q+1}(2k+|m|+1)\alpha^2}{\pi\hbar^2 n^3} \times \operatorname{arctg}\left\{\left[\operatorname{tg}\left(\frac{\pi\hbar^3 n^3\gamma}{2\alpha^2}+\frac{\pi}{3}\right) \times \operatorname{tg}\left(\frac{\pi\hbar^3 n^3\gamma}{2\alpha^2}-\frac{\pi}{3}\right)\right]^{1/2}\right\}. \quad (3.14)$$

This expression was obtained in Ref. 20 by using uniform asymptotics of the solutions of Eq. (3.11) at $\omega \gg 1$, in the form[4)]

$$y_1 = C_1 u^{1/2} J_{1/6}(u), \quad y_2 = C_2 u^{1/2} J_{-1/6}(u),$$
$$C_1 = \Gamma(1/6)\alpha/2^{3/2}|E|(3\omega)^{1/3}, \quad C_2 = 3^{1/3}\Gamma(5/6)/|E|^{1/2}\omega^{1/6}, \quad (3.15)$$
$$u = \omega^{1/2}\int_0^{\zeta}\frac{\zeta^{1/2}}{(1-\zeta)^{1/2}}d\zeta = \omega^{1/2}[\arcsin(\zeta^{1/2}) - \zeta(1-\zeta)^{1/2}],$$

where $J_\alpha(u)$ is a Bessel function and $\Gamma(x)$ a gamma function. Estimates show that when the second integral of (3.8) is calculated in the leading order ω the region $\omega\zeta^3 \lesssim 1$ makes a small contribution, and only the region $\omega^{-1/3} \ll \zeta < 1$ in which the WKB asymptotics can be used to solve (3.15) is significant:

$$y_1 \approx C_1(2/\pi)^{1/2}\cos(u-\pi/3), \quad y_2 \approx C_2(2/\pi)^{1/2}\cos(u-\pi/6). \quad (3.16)$$

A surprising property of the hydrogen atom in a strong magnetic field is that the solutions (3.6) are valid also in the vicinity of the turning point $\zeta = 1$.

Using the explicit value obtained for T in Ref. 20 and substituting (3.16) in (3.18) we obtain for $|\omega^{1/2} - (\omega_q^+)^{1/2}| \ll 1$

$$\tilde{c} = -\frac{3^{1/4}(q+2/3)^2|2E|^{1/2}}{2^{5/2}\pi^{3/2}\alpha(q+2/3-\omega^{1/2})^{3/2}}\oint\frac{d\zeta\,\zeta^{3/2}}{(1-\zeta)^{3/2}} = \frac{3^{5/4}(q+2/3)^2}{2^{5/2}\pi^{1/2}\hbar n(q+2/3-\omega^{1/2})^{3/2}}, \quad (3.17)$$

and for $|\omega^{1/2} - (\omega_q^-)^{1/2}| \ll 1$

$$\tilde{c} = \frac{3^{1/4}(q+1/3)^2|2E|^{1/2}}{2^{5/2}\pi^{3/2}\alpha(\omega^{1/2}-q-1/3)^{3/2}}\oint\frac{d\zeta\,\zeta^{3/2}}{(1-\zeta)^{3/2}} = -\frac{3^{5/4}(q+1/3)^2}{2^{5/2}\pi^{1/2}\hbar n(\omega^{1/2}-q-1/3)^{3/2}}. \quad (3.18)$$

In the case of (3.17), $c > 0$ and splitting of the closed stable trajectories takes place. In the case (3.18) the negative value of c indicates that the stable states near the symmetry axis are separated from the other states of the discrete spectrum by a potential barrier. The width of the "decay" is determined here by (3.18) and by (1.25), with the following substitutions in the latter:

$$\int_{z_1}^{z_2}\frac{dz}{p} = \frac{\pi}{\gamma}\left(q+\frac{1}{3}\right), \quad \int_{z_1}^{z_2}\frac{dz}{p\sigma^2} = \frac{\pi q}{2}. \quad (3.19)$$

The dynamic features of an analogous one-dimensional system were investigated in Ref. 35.

Two closed stable (unstable) trajectories, having mirror symmetry relative to the z axis and corresponding to the bottom of the well (to the top of the barrier), $\nu_0^\pm = \pm(-b/2c)^{1/2}$ in Fig. 2, are merged at the stability limit with an unstable (stable) trajectory lying on the axis, and are transformed into one stable (unstable) trajectory lying on the axis (Fig. 3b). The trjectory corresponding to ν_0^+ at $\omega^{1/2} - (\omega_q^+)^{1/2} \ll 1$ is given by

$$\rho = \rho_1^{(q)}(z) = |a_{11}/2\tilde{c}|^{1/2}y_1(z),$$

or, after substituting the explicit values of a_{11} and $\tilde{c}$,

$$\rho = \rho_1^{(q)}(z) = \frac{\pi^{1/2}\alpha(\omega^{1/2}-q-2/3)^{1/2}}{3^{1/2}(q+2/3)^{3/2}|E|}u^{1/2}J_{1/6}(u). \quad (3.20)$$

Analogously at $\omega^{1/2} - (\omega_q^-)^{1/2} \ll 1$ point ν_0^+ corresponds to a closed trajectory $\rho = \rho_2^{(q)}(z) = |a_{22}/2\tilde{c}|^{1/2}y_2(z)$, or in explicit form

$$\rho = \rho_2^{(q)}(z) = \frac{\pi^{1/2}\alpha(\omega^{1/2}-q-1/3)^{1/2}}{3^{1/2}(q+1/3)^{3/2}|E|}u^{1/2}J_{-1/6}(u). \quad (3.21)$$

The family of closed trajectories (3.20), recently obtained numerically,[21,29] explains the structure of the resonances observed in the experiments.[21] According to (4.12), the energy at which the separation of the closed trajectories takes place has the following dependence on the index q:

$$E_q^\pm = -1/2[\alpha\gamma/(q+1/2\pm1/6)]^{2/3}.$$

The value of E_q^+ differs from the result of Ref. 29 by 8% at $q = 1$, by 4% at $q = 2$, and does not differ from Ref. 29 at $q \geqslant 3$ within the limits of the accuracy of the plot of that paper. The period of the trajectories can be easily found:

$$T_q^\pm = 2\pi(q+1/2\pm1/6)/\gamma.$$

This formula gives the $T_q^+ = 2/3$ an asymptotic value of the "quantum defect" equal to 2/3, or a value of 1/3 for the numbering used in Ref. 21. The quantum defect was obtained in Ref. 21 numerically at $E = 0$ and was found to be close to 0.3.

3.3. Highly excited states localized near a symmetry plane

To explain the experiments with Rydberg atoms in a strong magnetic field, a two-dimensional model was proposed in which the motion of the electron is investigated only in a symmetry plane perpendicular to the field (see Ref. 36). The semi-empirical quantization rule obtained contains only two quantum numbers responsible for the motion in the symmetry plane, and ignores the motion in the transverse direction. A correct quantization rule should include an integer quantum number in the limit of the purely Coulomb problem, and a half-integer number in the purely oscillator problem. A planar model cannot meet this requirement.[36] Nonetheless, good agreement was observed in Refs. 37 and 22 between the two-dimensional model and an exact quantum calculation (which covers the regions of the regular and irregular spectra).

Some results of the present subsection were reported to a conference.[32] The problem reduces to an investigation of states localized on the ρz plane near a segment bounded by the turning points of the effective potential $V_e = V + \hbar^2 m^2/2\rho^2$ at $z=0$. (In the semi-classical approximation we left out the term $\hbar^2/8\rho^2$,) we expand V_e up to terms of fourth order in z:

$$V_e(\mathbf{r}) = V^0(\rho) + {}^1/_2 V_{zz}^0(\rho) z^2 + {}^1/_{24} V_{zzzz}^0(\rho) z^4,$$
$$V^0(\rho) = -\frac{\alpha}{\rho} + \frac{\gamma^2\rho^2}{8} + \frac{\hbar^2 m^2}{2\rho^2},\qquad (3.22)$$
$$V_{zz}^0(\rho) = \frac{\alpha}{\rho^3},\quad V_{zzzz}^0(\rho) = -\frac{9\alpha}{\rho^5}.$$

Note that in the case $\gamma = 0$ the exact solutions of Eq. (1.16) (in which z and ρ must now change places) are $y_1 = \rho^2 - m^2/a$ and $y_2 = -\alpha p\rho(\alpha^2 + 2Em^2\hbar^2)^{-1}$. It can be easily found that now the arctangent is zero and $N = 1$ in (3.6), and an integer quantum number appears in the right-hand side of (3.6) with allowance for (1.22) (case A 2). As a result we arrive at a quantization rule for the Coulomb field. In the opposite case, at $\alpha = 0$, the solutions (3.1) are exact and the arctangent and N are both zero. We then obtain a quantization rule for an oscillator with half-integer quantum number.

To solve the problem in the case when the Coulomb and magnetic fields are, generlly speaking, comparable in magnitude, we introduce dimensionless coordinate $\chi = \gamma^{2/3}\alpha^{-1/3}\rho/2$ and dimensionless parameters $\eta = 2E(a\gamma)^{-2/3}$ and $\mu = \hbar m\gamma^{1/3}\alpha^{-2/3}/2$. Equation (1.6) can then be reduced to

$$2\chi(\eta\chi^2+\chi-\chi^4-\mu^2)y_{\chi\chi} - (\chi+2\chi^4-2\mu^2)y_\chi + y = 0. \qquad (3.23)$$

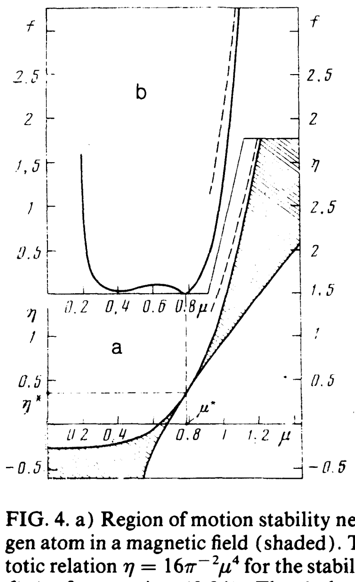

FIG. 4. a) Region of motion stability near a symmetry plane for a hydrogen atom in a magnetic field (shaded). The dashed line shows the asymptotic relation $\eta = 16\pi^{-2}\mu^4$ for the stability region. b) Plot of the function $f(\mu)$ of expression (3.24). The dashed line is the asymptotic relation $f = 2^{15/2}\pi^{-4}\mu^8$.

The motion stability region near the ρ axis was obtained numerically.[5)] It is shown in Fig. 4a. In the present subsection we confine ourselves to the case of large m, when the turning points ρ_1 and ρ_2 of the momentum $p = 2^{1/2}(E - V^0)^{1/2}$ are simple. For small m, stable states far from the stability boundary were investigated in Ref. 31.

The lower boundary of the stability region in Fig. 4a corresponds to coalescence of the turning points ρ_1 and ρ_2. It is obtained from the condition that the momentum $p(\rho)$ and the derivative of the potential $V^0(\rho)$ be simultaneously equal to zero. The upper boundary corresponds to loss of the stability of the trajectory on the ρ axis. The point of tangency of the boundaries is the point of the classical resonance. It corresponds to $\mu = \mu^* = 3^{1/6}6^{1/2}4$, $\eta = \eta^* = 3^{-1/3}/2$. To the left of μ^* the stability boundary corresponds to a zero coefficient t_{23}, and to the right—to a zero coefficient t_{11}. In a small vicinity of the point (μ^*, η^*) the period of motion along the z axis is double the period of the motion along the ρ axis. The theory considered is no longer valid here and the motion can be investigated by the methods of classical and semiclassical perturbation theory for degenerate systems.[38,13]

If the dependence of η on μ for the upper boundary is written in the form $\eta_0(\mu)$, it is convenient to represent the expression (3.8) for $\tilde{c}$ near this boundary as follows:

$$\tilde{c} = \gamma^{1/3}\alpha^{-2/3} f(\mu)[\eta_0(\mu) - \eta]^{-3/2}. \qquad (3.24)$$

The dimensionless function $f(\mu)$ was calculated with a computer. To this end, the second integral in (3.8) was transformed into an integral on the real axis, containing no singularities at the points ρ_1 and ρ_2, by adding and subtracting a known very simple integral with the same singularities. The first integral in (3.8) can be simply expressed in terms of the elements of the matrix $\mathbf{T}$. The calculation result is shown in Fig. 4b. It can be seen that c is positive. It turns out here that passage through the stability boundary is accompanied by the appearance of a U-shaped closed stable trajectory that does not lie on the ρ axis and has double the period [see also Eq. (3.29)]. We get thus U-shaped states with exponentially small wave-function density on the ρ axis everywhere except in the vicinities of the point ρ_1 or ρ_2.

Near the point (μ^*, η^*) (Fig. 4a) and also at large μ and η, the solution of Eq. (3.23), and also the value of c, can be obtained in the analytic form. We consider the case $\eta \sim \mu^4 \gg 1$, in which Eq. (3.23) has the asymptotic solutions:

$$y_1(\chi) = 1 - {}^1/_2\eta^{1/2}\mu^{-2}\arcsin(\eta^{-1/2}\chi),$$
$$y_2(\chi) = \gamma^{-1}\arccos(1 - 2\eta^{-1}\chi^2 + 2\mu^2\eta^{-2}). \qquad (3.25)$$

This yields for the matrix $\mathbf{T}$

$$\mathbf{T} = \begin{pmatrix} 1 - {}^1/_4\pi\eta^{1/2}\mu^{-2} & -{}^1/_4\gamma\eta^{1/2}\mu^{-2} \\ \pi\gamma^{-1} & 1 \end{pmatrix}. \qquad (3.26)$$

It follows from (3.26) that the asymptotic formula for the stability boundary is of the form $\eta_0 = (\mu) \approx 16\pi^{-2}\mu^4$. This dependence agrees well with the result of the numerical calculation already at $\mu > 1$ (see Fig. 4a). The motion along the ρ axis is stable below the boundary. Substituting (3.26) and (1.23) (case A 2) in (3.6), we obtain for the corresponding energy levels

$$E_{kmn}=\frac{\gamma\hbar}{2}\left\{n+\frac{1}{2}+\frac{2k+1}{2\pi}\operatorname{arctg}\left[\left(\frac{\hbar^{4/3}m^2\gamma^{1/2}}{\pi n^{1/2}\alpha}-1\right)^{-1/2}\right]\right\}. \tag{3.27}$$

The main contribution to the closed-contour integral in (38) is made in this case by the vicinity of the point $\chi_1=\mu\eta^{-1/2}$. As a result we get

$$\tau=\frac{2^{1/2}.6\gamma^{1/3}\mu^4}{\alpha^{8/3}(\eta_0-\eta)^{3/2}}\int_{\mu\eta^{-1/2}}^{\infty}\frac{d\chi}{\chi^4(\chi^2-\mu^2\eta^{-1})^{1/2}}$$
$$=\frac{2^{23/2}\mu^{12}}{\hbar n\pi^6(\eta_0-\eta)^{3/2}}. \tag{3.28}$$

Assuming η close enough to $\eta_0(\mu)$, we obtain from (3.24) and (3.28) $f(\mu)\approx 2^{15/2}\pi^{-4}\mu^8$. This function is compared in Fig. 4b with the numerically obtained one. It is easy also to find an asymptotic equation for the detached closed stable trajectories:

$$z(\rho)=\frac{2\pi\alpha^{7/3}[\eta-\eta_0(\mu)]^{1/2}}{\gamma^{5/3}\hbar^3m^3}\arccos\left(\frac{\pi\alpha\rho}{2\hbar^2m^2}\right),$$
$$0<\eta-\eta_0(\mu)\ll\eta_0(\mu),\quad m\gg 1. \tag{3.29}$$

The states investigated experimentally and theoretically in greatest detail within the framework of the two-dimensional model are those with near-zero energy and small m. It is seen from Fig. 4a that these states are unstable. Experiment, nonetheless, reveals clearly resonances with periodic structure corresponding to quantization within the framework of the two-dimensional model in the symmetry plane (see Ref. 6 and the literature cited therein).

Let us calculate the lifetimes of the unstable states at $E=0$ and $\hbar m=0$. Equation (3.23) can be solved in this case exactly (it reduces to Eq. (2.410 of Ref. 39). In this case, one turning point is of the Coulomb type and the other is simple. Without presenting the detailed calculations, we write down the final expression for the matrix **T**:

$$\mathbf{T}=\begin{pmatrix}2\pi^{1/2}\Gamma(7/6)[\Gamma(5/6)]^{-2}\alpha^{1/3}\gamma^{-2/3} & 3\pi^{1/2}\Gamma(7/6)[\Gamma(1/3)]^{-2}\alpha^{-1/6}\gamma^{1/3}\\ -2\pi^{1/2}\Gamma(5/6)[\Gamma(2/3)]^{-2}\alpha^{1/6}\gamma^{-1/3}, & -3\pi^{1/2}\Gamma(5/6)[\Gamma(1/6)]^{-2}\alpha^{-1/3}\gamma^{2/3}\end{pmatrix}. \tag{3.30}$$

Substituting (3.30) in (3.6), we obtain for the width of the level with quantum number k

$$\Gamma=2\,\mathrm{Im}\,E=\frac{2k+1}{\pi}\,\mathrm{Im}[\operatorname{arctg}(3^{1/2}i)]\Delta E=0.43(2k+1)\Delta E, \tag{3.31}$$

where the distance between the levels ΔE is connected with the period T of the electron motion by the relation $T\Delta E=2\pi\hbar$, and the lifetime of the state is $\tau\approx\hbar/\Gamma$. It follows from (3.31), in particular, that a state with $k=0$ has a lifetime three times longer than one with $k=1$ and its level width is smaller by two or three times than the level spacing. We can therefore assume, by analogy with the arguments of Refs. 30 and 40, that these resonances are experimentally observable. However, the detachment of the U-shaped stable trajectories considered above takes place for small m at a very low energy E (see Fig. 4a). As $E=0$ is approached, their half-life will probably not differ greatly from that of the unstable trajectory with $E=0$ on the ρ axis and U-shaped states can make the main contribution to the observed spectrum.

CONCLUSION

1. The theory developed allow us to investigate in explicit form the tunnel effects produced on passage of certain series of classical resonances in a semiclassical system with nonseparable variables. The phenomenon itself has a rather general character and is well known for systems with separable variables (e.g., the effective centrifugal barrier for the spherically symmetric polarization potential αr^{-4}). In the two-dimensional case, for an arbitrary closed classical trajectory, the term quadratic in ν in the exponential of the Ansatz (1.4) and in Eq. (1.11) will be followed primarily by a term proportional to ν^3. An interesting problem is the generalization of the theory to include the vicinity of the stability boundary of closed trjectories in N-dimensional space.

2. The experimental data and theoretical calculations have shown that stable closed trajectories in the stochastic region can act as centers for experimentally observed highly excited states. The oscillator strengths of transitions into such states outside the stability limits can be expressed, in analogy with Ref. 20, in analytic form in terms of the matrix **T**, and were numerically determined near the boundary.

As the stability boundary $\omega=\omega_q^+$ is approached (Sec. 3.2) the intensity of the radiation should increase. The structure predicted for the resonances in Ref. 10[6)] accords with the resonances observed experimentally in Ref. 21. The maximum values of the oscillator strengths can be estimated by noting that Eqs. (42) and (43) of Ref. 20 do not hold at $\tilde{a}\tilde{c}\sim 1$. After obtaining a_{22} from this estimate for the condition $q^{+2/3}-\omega^{1/2}\ll 1$ and substituting in the indicated equations, we get (in atomic units)

$$\max|\langle\Psi_0|z|\Psi_{0nk}^-\rangle|^2\sim n^{-11/3}(k+1)^{-1/3}q^{-1/3},$$
$$\max|\langle\Psi_0|x|\Psi_{\pm 1nk}\rangle|^2\sim n^{-13/3}(k+1)^{1/3}q^{-2/3}.$$

Comparing these expressions with the corresponding equations from Ref. 20 far from the stability limits, we find that at $q\sim k\sim 1$, near the boundary, transitions from the ground state into the considered highly excited states have intensities approximately $n^{1/3}$ times larger if $m=0$ and $n^{2/3}$ times larger if $m=\pm 1$.

3. Investigation of size effects in an open cavity can be useful not only for optical applictions but also in the theory of electrophysical structures of extremely small size. Such structures can be reproduced with the aid of a scanning tunnel microscope.[41] The central link in future transistors based on the resonant-tunneling effect is an open cavity of nanometer size. It is of interest to investigate the current-voltage characteristic of such a system at an applied voltage close to the cavity stability limit.

4. It is customarily assumed that in non-one-dimensional systems without a definite symmetry the tunneling through the potential barrier is effected as a rule in the vicinity of the most probable path.[18,19] It was shown above that the inverse situation is also possible, with the probability of penetrating through the potential barrier is uniformly distributed over the entire length of a stable trajectory. The point is that we have calculated the tunnel effects by using an expansion in ρ and assuming the barriers to be narrow enough. The most probable tunneling path may be formed with increase of distance from the stability limit.

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[1)] This method was used in the theory of multidimensional tunnel processes (see Refs. 18 and 19) and also in investigations of highly excited localized states of atoms in Ref. 10 and in other papers.

[2)] These restrictions do not affect in principle the validity of the theory, and are connected with the possibility of solving Eq. (1.11) in the classically forbidden region by the WKB method.

[3)] A somewhat more cumbersome expression of the coefficient a_{ij} in terms of the elements of the matrix **T** and the parameters α_k and β_k can be easily obtained also for the problem considered in Sec. 2.

[4)] Owing to an error by the author, Eq. (24) of Ref. 20 gives for Eq. (20) of the paper an asymptotic solution that is valid at $\omega\zeta^3 \lesssim 1$ and not over the entire range $0<\zeta<1$ as stated there. In addition, the sign $(-1)^{q+1}$ was left out of the quantization rule (3.14), and correced by the author when P. A. Braun pointed out the incorrect behavior of the term in Fig. 1 of the cited paper.

[5)] In Ref. 32 the stability region is illustrated with allowance for the paramagnetic energy component $\gamma m/2$, which is left out in the present article.

[6)] An elementary consequence of Eq. (25) of that references is that the signs + and − for b in Eq. (46) should be interchanged.

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