# Radiation from relativistic particles colliding in a medium in the presence of an external field

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We develop for the consideration of bremsstrahlung an approach which takes account of the medium and an external field. Kinetic equations are derived which enable one to allow for the effects of scattering and the external field. Making use of these, we analyze bremsstrahlung in a screened Coulomb potential and from a nucleus. The corresponding cross sections are calculated to power-law accuracy. To logarithmic accuracy, we study effects involved in the collision of $e^+e^-$ beams in linear colliders.

## 1. INTRODUCTION

At very high energies, the production length involved in certain fundamental processes of quantum electrodynamics is very large, and the processes come to depend on the medium in which they are played out, as well as on external fields. Landau and Pomeranchuk[1] were the first to draw attention to the fact that if the production length for bremsstrahlung becomes comparable to the distance over which multiple scattering becomes important, the bremsstrahlung will be suppressed. Migdal[2,3] developed a quantitative theory of this phenomenon. Two of the present authors have shown[4] that external fields can also modify the bremsstrahlung process.

One of the interesting applications of these effects occurs with radiation derived from the collision of $e^+e^-$ beams in so-called linear colliders (at particle energies of hundreds of GeV or more). In the present paper, we develop a theoretical approach to the consideration of bremsstrahlung, taking both the medium and an external field into account. Previous (valid) results are derived as simple limiting cases.

Quantum effects in an external field $F_{\mu\nu}$ are characterized by the parameter[1)] $\chi$ (where $\chi^2 = (e^2/m^6)|(F_{\mu\nu}p^\nu)^2|$, and $p^\mu$ is the particle four-momentum); for the sake of definiteness, in a transverse electric (magnetic) field,

$$\chi=\frac{E\gamma}{E_0}\quad\left(\chi=\frac{H\gamma}{H_0}\right), \tag{1.1}$$

where $\gamma=\varepsilon/m$ is the Lorentz factor, and $E_0=m^2/e = (m^2c^3/e\hbar) = 1.32\cdot 10^6$ V/cm, so that with $\chi \gtrsim 1$ we are already well into the quantum domain. The situation $\chi > 1$ will prevail in linear colliders, where particles move in the field of the oncoming beam. In that case, the external field is produced by the very medium traversed by the particles, and is therefore inseparable from that medium.

We consider the effect of an external field on the bremsstrahlung process. This effect is associated with a reduction in the production length of a photon (either bremmstrahlung or virtual) due to the relatively large change in particle velocity over this length, and the corresponding increase in the vertex angle of the radiation cone. If a photon of frequency $\Omega$ is emitted by an electron (positron) of energy $\varepsilon$ at an angle $\vartheta$ to its velocity, the production length of such a photon will be given by[2)]

$$l_\Omega\sim(\varepsilon-\Omega)/\varepsilon\Omega\vartheta^2=1/\varepsilon u_\Omega\vartheta^2, \tag{1.2}$$

where $u_\Omega \equiv \Omega/(\varepsilon-\Omega)$. The characteristic radiation angles in weak fields are $\vartheta\sim 1/\gamma$, and we can neglect the influence of the external field if

$$\dot{v}l_\Omega=eHl_\Omega/\varepsilon\ll 1/\gamma. \tag{1.3}$$

Substituting (1.2) into (1.3), we have the criterion for a field to be weak,

$$\gamma\frac{eH}{\varepsilon}\frac{\gamma^2}{\varepsilon u_\Omega}=\frac{\chi}{u_\Omega}\ll 1. \tag{1.4}$$

In strong fields, where $\chi/u \gg 1$, with characteristic radiation angles $\vartheta \gg 1/\gamma$, the effective radiation angle $\vartheta_{\text{eff}}$ is determined by a self-consistency argument: the deviation angle of the particle in the field over one photon production length must not exceed $\vartheta_{\text{eff}}$, i.e.,

$$\dot{v}l_\Omega(\vartheta_{\text{eff}})\sim\vartheta_{\text{eff}}\qquad \vartheta_{\text{eff}}\sim\frac{1}{\gamma}\left(\frac{\chi}{u_\Omega}\right)^{1/3}=\left(\frac{eH}{\varepsilon^2u_\Omega}\right)^{1/3},$$
$$l_\Omega(\vartheta_{\text{eff}})\sim\frac{\gamma}{mu_\Omega}\left(\frac{u_\Omega}{\chi}\right)^{2/3}=\left(\frac{\varepsilon}{u_\Omega e^2H^2}\right)^{1/3}. \tag{1.5}$$

It can be seen from (1.5) that when $\chi/u \gg 1$, neither the characteristic radiation angle nor the photon production length depends on the mass of the radiating particle. A parameter characterizing the effect of an external field on the radiation process was derived in Ref. 4.

It follows from (1.5) that, for $\chi/u_\Omega \gg 1$, the characteristic photon production length is reduced by a factor $(\chi/u_\Omega)^{2/3}$, and the emission angles are increased by a factor $(\chi/u_\Omega)^{1/3}$. Since the relevant parameter $x/u_\Omega$ depends on the frequency $u$, this effect is manifested earlier for soft photons. This is the reason why the field has a significant effect on virtual photon production in $e^+e^-$ collisions ($\Omega\approx q_0' = \omega m^2/4\varepsilon(\varepsilon-\omega)$, where $\omega$ is the frequency of the real photon) even for moderate fields and at relatively low particle energies, when the parameter $\chi(\varepsilon-\omega)/\omega\equiv\chi/u$ is small, and $\chi(\varepsilon-q_0')\approx 4\gamma^2\chi/u$ is large. In that event, the diagram for emission of a bremsstrahlung photon (radiation vertex) does not change directly, but a significant change takes place in the virtual photon spectrum at momentum transfers $|q|\lesssim q_{\min}(4\gamma^2\chi/u)^{1/3}$, increasing the lower bound on the effective momentum transfer and resulting in a corresponding decrease in the cross section. The radiation cross section was derived to logarithmic accuracy under these conditions in Ref. 4 using the equivalent photon method. The relativistic problem was solved in Ref. 6.

It follows from (1.4), and was demonstrated in Ref. 4,

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that even for $\chi/u \gtrsim 1$ there is a change in the radiation vertex corresponding to absorption of a virtual photon and emission of a real one. Since for $\chi/u \gg 1$ the production length $l_\omega$ of a real photon falls off as $(u/\chi)^{2/3}$ (see (1.5)), the bremsstrahlung cross section falls off in just the same way. There have indeed been previous attempts to calculate this cross section. Thus, the bremsstrahlung cross section for a nucleus in an external field was calculated to logarithmic accurcy in Ref. 7. However, an incorrect application of the standard equivalent-photon method in Ref. 7 for this configuration resulted in an erroneous dependence of the argument of the logarithm on the parameter $\chi/u$. The cross section for a screened Coulomb potential was recently calculated[8] to the same accuracy. The logarithmic function obtained there is not in agreement with Ref. 7.

In the present paper, we carry out a relativistic (semiclassical) calculation of the bremsstrahlung cross section for particle collisions in the presence of a field. Section 2 contains a derivation and analysis of kinetic equations which are much more broadly applicable than necessary for the subsequent calculation. In Sections 3 and 4 we analyze radiation in a screened Coulomb potential and a nuclear field, respectively. Section 5 discusses the features of bremsstrahlung in $e^+e^-$ beam collisions to logarithmic accuracy. In our calculations, we neglect polarization of the medium, which is important only for the emission of very soft photons, and which, moreover, is diminished by a factor $(\chi/u)^{2/3}$ when $\chi/u \gg 1$.

## 2. BASIC EQUATIONS

We start out with the quasiclassical theory of radiation, as developed in Ref. 5. In this theory, the emission probability (Eq. (9.27) of Ref. 5) is

$$dw = \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\int dt_1 \int dt_2\, R^*(t_2)R(t_1) \times \exp\left\{-\frac{i\varepsilon}{\varepsilon-\omega}[kx(t_2)-kx(t_1)]\right\}, \tag{2.1}$$

where $\alpha = e^2 = 1/137$, $k = (\omega,\mathbf{k})$ is the four-momentum of the photon, $k^2 = 0$, $x(t) = (t,\mathbf{r}(t))$, $t$ is the time, and $\mathbf{r}(t)$ is the particle location on a classical trajectory. For relativistic spinors, we have

$$R(t)=\varphi_f^+(A+i\sigma\mathbf{B})\varphi_i,$$
$$A=\frac{1}{2}\left(1+\frac{\varepsilon}{\varepsilon-\omega}\right)\mathbf{ev}\approx\frac{1}{2}\left(1+\frac{\varepsilon}{\varepsilon-\omega}\right)\mathbf{e}\boldsymbol{\vartheta},$$
$$B=\frac{\omega}{2(\varepsilon-\omega)}[\mathbf{eb}],\quad \mathbf{b}=\mathbf{n}-\mathbf{v}+\mathbf{n}/\gamma\approx-\boldsymbol{\vartheta}+\mathbf{n}/\gamma, \tag{2.2}$$

where the angle $\boldsymbol{\vartheta} = v^{-1}(\mathbf{v}=\mathbf{n}(\mathbf{n}\cdot\mathbf{v}))\approx\mathbf{v}_\perp$, and $\mathbf{v}_\perp$ is the component of particle velocity perpendicular to the vector $\mathbf{n} = \mathbf{k}/\omega$. If we are not interested in the initial and final particle polarizations, then

$$R^*(t_2)R(t_1)\to\frac{1}{2(\varepsilon-\omega)^2}\left[\frac{\omega^2}{\gamma^2}+(\varepsilon^2+(\varepsilon-\omega)^2)\boldsymbol{\vartheta}\boldsymbol{\vartheta}'\right] =\frac{1}{2}\mathcal{L}(\boldsymbol{\vartheta}',\boldsymbol{\vartheta}), \tag{2.3}$$

where we have used the notation $\boldsymbol{\vartheta}' = \boldsymbol{\vartheta}(t_2)$, $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}(t_1)$. If the particle moves along a definite trajectory, then by substituting the classical values of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ into Eqs. (2.1)–(2.3), we obtain the desired probability for this process. When there is scattering, Eq. (2.1) must be averaged over all possible particle trajectories. This operation is performed with the aid of the distribution function, averaged over atomic positions in the scattering medium and satisfying the kinetic equation with the external field taken into account (particle acceleration). The emission probability per unit time is then (see Ref. 5)

$$dW=\left\langle\frac{dw}{dt}\right\rangle=\frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\,\mathrm{Re}\int_0^\infty d\tau\exp\left(-\frac{i\varepsilon}{\varepsilon'}\omega\tau\right) \times\int d^3v\,d^3v'\,d^3r\,d^3r'\,\mathcal{L}(\boldsymbol{\vartheta}',\boldsymbol{\vartheta})F_i(\mathbf{r},\mathbf{v},t)F_f(\mathbf{r}',\mathbf{v}',\tau;\mathbf{r},\mathbf{v}) \times\exp\left\{i\frac{\varepsilon}{\varepsilon'}\mathbf{k}(\mathbf{r}'-\mathbf{r})\right\},$$
$$\varepsilon'=\varepsilon-\omega. \tag{2.4}$$

The distribution function $F$ in (2.4) satisfies the kinetic equation

$$\frac{\partial F(\mathbf{r},\mathbf{v},t)}{\partial t}+\mathbf{v}\frac{\partial F(\mathbf{r},\mathbf{v},t)}{\partial\mathbf{r}}+\mathbf{w}\frac{\partial F(\mathbf{r},\mathbf{v},t)}{\partial\mathbf{v}} =n\int\sigma(\mathbf{v},\mathbf{v}')[F(\mathbf{r},\mathbf{v}',t)-F(\mathbf{r},\mathbf{v},t)]d^3v', \tag{2.5}$$

where $n$ is the number density of atoms in the medium, and $\sigma(\mathbf{v},\mathbf{v}')$ is the scattering cross section. The normalization condition

$$\int d^3r\int d^3v\,F(\mathbf{r},\mathbf{v},t)=1 \tag{2.6}$$

should also be satisfied, as well as the initial condition for $F_f$:

$$F_f(\mathbf{r}',\mathbf{v}',0;\mathbf{r},\mathbf{v})=\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{v}-\mathbf{v}').$$

In Eq. (2.4), we integrate over $d^3r d^3(r'-r)$, taking advantage of the fact that $F_f(\mathbf{r}',\mathbf{v}',\tau;\mathbf{r},\mathbf{v})$ can only depend on the coordinate difference $\mathbf{r}'-\mathbf{r}$:

$$dW=\frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\,\mathrm{Re}\int_0^\infty d\tau\int d^3v\,d^3v'\,\mathcal{L}(\boldsymbol{\vartheta}',\boldsymbol{\vartheta}) \times F_i(\mathbf{v},t)F_k(\mathbf{v}',\tau;\mathbf{v}), \tag{2.7}$$

where

$$F_i(\mathbf{v},t)=\int d^3r\,F_i(\mathbf{r},\mathbf{v},t),$$
$$F_k(\mathbf{v}',\tau;\mathbf{v})=\exp\left(-\frac{i\varepsilon\omega\tau}{\varepsilon'}\right)\int d^3r'\exp\left\{i\frac{\varepsilon}{\varepsilon'}\mathbf{k}(\mathbf{r}'-\mathbf{r})\right\} \times F_f(\mathbf{r}',\mathbf{v}',\tau;\mathbf{r},\mathbf{v}). \tag{2.8}$$

Here

$$F_k(\mathbf{v}',\tau;\mathbf{v})=U(\boldsymbol{\vartheta}',\boldsymbol{\vartheta};\tau)\delta(|\mathbf{v}'|-|\mathbf{v}|),$$
$$U(\boldsymbol{\vartheta}',\boldsymbol{\vartheta};0)=\delta(\boldsymbol{\vartheta}'-\boldsymbol{\vartheta}), \tag{2.9}$$

if it is assumed that in the ultrarelativistic limit $\mathbf{w}\cdot\mathbf{v}\sim O(1/\gamma^3)$, $\mathbf{w}\approx\mathbf{w}_\perp$, and the scattering cross section is

$$\sigma(\mathbf{v},\mathbf{v}')=\delta(|\mathbf{v}'|-|\mathbf{v}|)\sigma(\boldsymbol{\vartheta},\boldsymbol{\vartheta}'). \tag{2.10}$$

Making use of Eq. (2.5) for $F$, we obtain the following equation for $U(\vartheta',\vartheta;\tau)$:

$$\frac{\partial U}{\partial\tau}+i\frac{\omega\varepsilon}{2\varepsilon'}\left(\frac{1}{\gamma^2}+\vartheta'^2\right)U+\mathbf{w}\frac{\partial U}{\partial\vartheta'}$$
$$=n\int d^2\vartheta''\,\sigma(\vartheta',\vartheta'')\left[U(\vartheta'',\vartheta,\tau)-U(\vartheta',\vartheta,\tau)\right]. \quad (2.11)$$

Reference 5 contains a derivation of this equation for $\mathbf{w}=0$, where the use of the quasiclassical method considerably simplifies the treatment of Ref. 3. A similar equation for the $\vartheta$-dependence of $U$ can be obtained by letting $\vartheta'\to\vartheta$ and $\mathbf{w}\to-\mathbf{w}$. If the final state of the charged particle is of no interest, the probability (2.7) must be integrated over $d^3v'$. The resulting emission probability per unit time, normalized to a single particle moving at a speed $v$ is then

$$dW=\frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\,\mathrm{Re}\int_0^\infty d\tau\exp\left(-i\frac{a\tau}{2}\right)$$
$$\times\left[\frac{\omega^2}{\varepsilon'^2\gamma^2}V_0(\vartheta,\tau)+\left(1+\frac{\varepsilon^2}{\varepsilon'^2}\right)\vartheta\mathbf{V}(\vartheta,\tau)\right], \quad (2.12)$$

where $a=\omega m^2/\varepsilon\varepsilon'$, and $V_\mu(\vartheta,\tau)$ satisfies the equation

$$\frac{\partial V_\mu}{\partial\tau}+i\frac{b\vartheta^2}{2}V_\mu-\mathbf{w}\frac{\partial V_\mu}{\partial\vartheta}$$
$$=n\int d^2\vartheta'\,\sigma(\vartheta,\vartheta')\left[V_\mu(\vartheta',\tau)-V_\mu(\vartheta,\tau)\right]. \quad (2.13)$$

Here $b=\omega\varepsilon/\varepsilon'$; the initial conditions for $V_\mu$ are

$$V_0(\vartheta,0)=1,\quad \mathbf{V}(\vartheta,0)=\vartheta. \quad (2.14)$$

**If the scattering cross section $\sigma(\vartheta',\vartheta)$ depends solely on the angle difference $\vartheta-\vartheta'$, then Eq. (2.13) is most conveniently solved by Fourier transforming with respect to the variable $\vartheta$:**

$$\varphi_\mu(\mathbf{x},\tau)=\frac{1}{(2\pi)^2}\int d^2\vartheta e^{i\vartheta\mathbf{x}}V_\mu(\vartheta,\tau),$$
$$V_\mu(\vartheta,\tau)=\int d^2x e^{-i\vartheta\mathbf{x}}\varphi_\mu(\mathbf{x},\tau). \quad (2.15)$$

From Eq. (2.13), with initial conditions (2.14), we have

$$\frac{\partial\varphi_\mu(\mathbf{x},\tau)}{\partial\tau}-\frac{ib}{2}\Delta_x\varphi_\mu(\mathbf{x},\tau)+i\mathbf{w}\mathbf{x}\varphi_\mu(\mathbf{x},\tau)$$
$$=n\int d^2\vartheta\,(e^{i\vartheta\mathbf{x}}-1)\,\sigma(\vartheta)\,\varphi_\mu(\mathbf{x},\tau)$$
$$\equiv(2\pi)^2n\left[\Sigma(\mathbf{x})-\Sigma(0)\right]\varphi_\mu(\mathbf{x},\tau)$$
$$\varphi_0(\mathbf{x},0)=\delta(\mathbf{x}),\quad \boldsymbol{\varphi}(x,0)=-i\nabla\delta(\mathbf{x}). \quad (2.16)$$

If the angular dependence of the radiation distribution is of no interest, Eq. (2.12) must be integrated over the photon emission angles $\vartheta$. Bearing in mind that $d^3k=\omega^2d\omega d^2\vartheta$, and that

$$\frac{1}{(2\pi)^2}\int V_0(\vartheta,\tau)\,d^2\vartheta=\varphi_0(0,\tau),$$
$$\frac{1}{(2\pi)^2}\int\vartheta\mathbf{V}(\vartheta,\tau)\,d^2\vartheta=-i\nabla\boldsymbol{\varphi}(\mathbf{x},\tau)\,|_{\mathbf{x}=0}, \quad (2.17)$$

we obtain the following expression for the spectral distribution of emission probability per unit time:

$$\frac{dW}{d\omega}=\alpha\omega\,\mathrm{Re}\int_0^\infty d\tau\exp\left(-i\frac{a\tau}{2}\right)$$
$$\times\left[\frac{\omega^2}{\gamma^2\varepsilon'^2}\varphi_0(0,\tau)-i\left(1+\frac{\varepsilon^2}{\varepsilon'^2}\right)\nabla\boldsymbol{\varphi}(0,\tau)\right]. \quad (2.18)$$

We now introduce the function

$$\Phi_\mu(\mathbf{x})=\int_0^\infty d\tau e^{-ia\tau/2}\varphi_\mu(\mathbf{x},\tau), \quad (2.19)$$

which satisfies the equation

$$\tfrac{1}{2}i(a-b\Delta)\Phi_\mu(\mathbf{x})+i\mathbf{w}\mathbf{x}\Phi_\mu(\mathbf{x})$$
$$=(2\pi)^2n\left[\Sigma(\mathbf{x})-\Sigma(0)\right]\Phi_\mu(\mathbf{x})+\varphi_\mu(\mathbf{x},0). \quad (2.20)$$

Making use of the solution (2.20), the emission probability spectrum may be put in the form

$$\frac{dW}{d\omega}=\alpha\omega\,\mathrm{Re}\left[\frac{\omega^2}{\gamma^2\varepsilon'^2}\Phi_0(0)-i\left(1+\frac{\varepsilon^2}{\varepsilon'^2}\right)\nabla\boldsymbol{\Phi}(0)\right]. \quad (2.21)$$

An equation like (2.20) was examined in Ref. 9 (p. 420) in the classical limit and with $\mathbf{w}=0$.

## 3. RADIATION IN A SCREENED COULOMB POTENTIAL

The scattering cross section in a screened Coulomb potential takes the form[3)]

$$\sigma(\vartheta)=4Z^2\alpha^2/\varepsilon^2(\vartheta^2+\vartheta_1^2)^2, \quad (3.1)$$

where $\vartheta_1=\varkappa/\varepsilon$, $\varkappa=1/a_s$, $a_s$ is the screening radius, and its Fourier transform may be expressed in terms of the Bessel function $K_1$:

$$\Sigma(x)=\frac{1}{(2\pi)^2}\int e^{i\vartheta x}\frac{4Z^2\alpha^2}{\varepsilon^2(\vartheta^2+\vartheta_1^2)^2}d^2\vartheta$$
$$=\frac{Z^2\alpha^2}{\pi\varepsilon^2}\frac{x}{\vartheta_1}K_1(x\vartheta_1). \quad (3.2)$$

**Bearing in mind that contributions to the cross section come from $x\sim1/\vartheta_{\text{eff}}\ll\vartheta_1^{-1}$, and expanding $K_1(x\vartheta_1)$ as a power series in $x\vartheta_1$, we obtain the following equation for $\varphi_\mu(\mathbf{x},\tau)$ from (2.16), to power-series accuracy:**

$$\frac{\partial\varphi_\mu}{\partial\tau}=i\frac{b}{2}\Delta\varphi_\mu+i\mathbf{w}\mathbf{x}\,\varphi_\mu$$
$$=\frac{2\pi nZ^2\alpha^2}{\varepsilon^2}x^2\left(\ln\frac{x\vartheta_1}{2}+C-\frac{1}{2}\right)\varphi_\mu, \quad (3.3)$$

where $C=0.577\ldots$ is Euler's constant. When the characteristic emission angles $\vartheta_{\text{eff}}\gg\vartheta_2\approx(Z^{1/3}\alpha\gamma)^{-1}$, no further "tuning" of the scattering angle to the emission angle is possible, and Eq. (3.3) simplifies considerably:

$$\frac{\partial\varphi_\mu}{\partial\tau}-i\frac{b}{2}\Delta\varphi_\mu+i\mathbf{w}\mathbf{x}\varphi_\mu=\frac{2\pi nZ^2\alpha^2}{\varepsilon^2}x^2\ln\frac{\vartheta_1}{\vartheta_2}\varphi_\mu. \quad (3.4)$$

Equation (3.3) can be expressed in this form if we wish to obtain the emission probability to logarithmic accuracy. We must then take as $\vartheta_2$ the characteristic emission angle $\vartheta_{\text{eff}}$, which in weak fields and with no multiple scattering is $\sim1/\gamma$. In strong fields ($\chi/u\gg1$), $\vartheta_{\text{eff}}\sim\gamma^{-1}(\chi/u)^{1/3}$, so that for estimates we can use the approximate formula

$$\vartheta_{\text{eff}}\sim\gamma^{-1}\left[1+(\chi/u)^{1/3}\right], \quad (3.5)$$

and the equation for $\varphi_\mu$ then takes the form

$$\frac{\partial\varphi_\mu}{\partial\tau}-i\frac{b}{2}\Delta\varphi_\mu+i\mathbf{w}\mathbf{x}\varphi_\mu=-qx^2\varphi_\mu, \tag{3.6}$$

where

$$q=2\pi n\frac{Z^2\alpha^2}{\varepsilon^2}\ln\frac{\vartheta_{\text{eff}}}{\vartheta_1}. \tag{3.7}$$

Any subsequent estimate of $\vartheta_{\text{eff}}$ requires that a wider range of radiation angles be taken into account, because of multiple scattering. This effect (the Landau-Pomeranchuk effect) can become the dominating influence in a weak field, where $\chi/u \ll 1$. Let us introduce the variable $\zeta = 1+\gamma^2\vartheta^2$ and estimate the change in the square of the particle deviation angle over the photon production length $l \sim \varepsilon(\varepsilon-\omega)/m^2\omega\zeta = \gamma/mu\zeta$ due to the external field and multiple scattering. We obtain as a result the following estimate for $\zeta_{\text{eff}}$:

$$\frac{\chi^2}{u^2\zeta^2}+\frac{\omega_0^2 Z\alpha\gamma}{m^2u\zeta}\ln\frac{\zeta}{\gamma^2\vartheta_1^2}\leqslant\zeta, \tag{3.8}$$

where $\omega_0^2 = 4\pi nZ\alpha/m$.

Equation (3.6) is of the same form as the Schrödinger equation for an oscillator in an external field, and with initial conditions (2.16) its solution takes the form of a Gaussian distribution. We therefore seek a solution of (3.6) in the form (compare Ref. 10)

$$\varphi_0=\exp[\alpha(\tau)+\beta(\tau)x^2+\gamma(\tau)\mathbf{w}\mathbf{x}],$$
$$\varphi=[A(\tau)\mathbf{x}+B(\tau)\mathbf{w}]\varphi_0. \tag{3.9}$$

Substituting this expression into Eq. (3.6) and solving the system of equations thus obtained with initial conditions (2.16), we have

$$\varphi_0=\frac{q}{\pi\nu\,\mathrm{sh}\,\nu\tau}e^{f(\tau)},\quad \nu=(2ibq)^{1/2},$$
$$\boldsymbol{\varphi}=\left(\frac{\nu\mathbf{x}}{b\,\mathrm{sh}\,\nu\tau}+\frac{\mathbf{w}}{\nu}\mathrm{th}\frac{\nu\tau}{2}\right)\varphi_0,$$
$$f(\tau)=-\frac{qx^2}{\nu}\mathrm{cth}\,\nu\tau-\frac{w^2\tau}{4q}+\frac{w^2}{2\nu q}\mathrm{th}\frac{\nu\tau}{2}-\frac{i\mathbf{w}\mathbf{x}}{\nu}\mathrm{th}\frac{\nu\tau}{2}. \tag{3.10}$$

The solution (3.10) and Eq. (2.18) give the following expression for the emission probability spectrum per unit time:

$$\frac{dW}{d\omega}=\frac{\alpha}{\pi}\mathrm{Re}\frac{q\omega}{\nu}\int_0^\infty\frac{d\tau}{\mathrm{sh}\,\nu\tau}\left[\frac{\omega^2}{\gamma^2\varepsilon'^2}-\left(1+\frac{\varepsilon^2}{\varepsilon'^2}\right)\left(\frac{2i\nu}{b\,\mathrm{sh}\,\nu\tau}+\frac{w^2}{\nu^2}\mathrm{th}^2\frac{\nu\tau}{2}\right)\right]\exp\left(-\frac{ia\tau}{2}-\frac{w^2\tau}{4q}+\frac{w^2}{2\nu q}\mathrm{th}\frac{\nu\tau}{2}\right). \tag{3.11}$$

Integrating the term containing $\sinh^{-2}\nu\tau$ by parts, we have, to logarithmic accuracy,

$$\frac{dW}{d\omega}=-\frac{\alpha}{\pi}\mathrm{Im}\,\nu\int_0^\infty d\tau\left[\frac{1}{\gamma^2\,\mathrm{sh}\,\nu\tau}+\frac{1}{2}\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}\right)\times\left(\frac{1}{\gamma^2}+\frac{w^2}{\nu^2}\right)\mathrm{th}\frac{\nu\tau}{2}\right]\times\exp\left(-\frac{ia\tau}{2}-\frac{w^2\tau}{4q}+\frac{w^2}{2\nu q}\mathrm{th}\frac{\nu\tau}{2}\right). \tag{3.12}$$

with $\mathbf{w}=0$, Eq. (3.12) becomes the probability derived by Migdal.[3] The probability (3.12) is also identical with the basic result of Ref. 8, wherein the calculation was performed by functional integration. For weak multiple scattering, where the second term in Eq. (3.8) can be neglected, we can expand (3.12) in powers of $\nu\tau$. The principal term in this expansion gives the emission probability in a constant field in the absence of scattering, and a correction $\propto q$ gives, to logarithmic accuracy, the emission cross section for scattering by a screened Coulomb potential in the presence of an external field:

$$\frac{d\sigma}{d\omega}=\frac{4Z^2\alpha^3}{15m^2\omega}\ln\left[\left(1+\left(\frac{\chi}{u}\right)^{1/3}\right)\frac{1}{\gamma\vartheta_1}\right]\left[\frac{\omega^2}{\varepsilon^2}(x^4\Upsilon-3x^2\Upsilon'-x^3)+\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)(x^4\Upsilon+3x\Upsilon+5x^2\Upsilon'-x^3)\right], \tag{3.13}$$

where

$$\Upsilon(x)=\int_0^\infty d\tau\sin\left(x\tau+\frac{\tau^3}{3}\right),\quad x=\left(\frac{u}{\chi}\right)^{2/3}. \tag{3.14}$$

The cross section (3.13) differs from the result obtained in Ref. 8 under the same assumptions. It must be noted that since the lower bound on the effective momentum transfer increases as $(\chi/u)^{2/3}$ in strong fields, as will be shown in the next section, a situation in which there is total screening in the absence of an external field may not be so when the fields are sufficiently strong.

We now return to Eq. (2.16) and introduce the notation

$$V(\mathbf{x})=(2\pi)^2n[\Sigma(\mathbf{x})-\Sigma(0)]=n\int d^2\boldsymbol{\vartheta}(e^{i\boldsymbol{\vartheta}\mathbf{x}}-1)\sigma(\boldsymbol{\vartheta}). \tag{3.15}$$

We will solve this equation in the Born approximation for scattering by $V(\mathbf{x})$, with no restrictions on the magnitude of the external field. Assuming $V=0$, we obtain in the zeroth approximation

$$\frac{\partial\varphi_\mu^{(0)}}{\partial\tau}-\frac{ib}{2}\Delta\varphi_\mu^{(0)}+i\mathbf{w}\mathbf{x}\varphi_\mu^{(0)}=0. \tag{3.16}$$

We can obtain the solution of this equation from (3.10), letting $q\to 0$:

$$\varphi_0^{(0)}(\mathbf{x},\tau)=\frac{1}{2ib\pi\tau}\exp\left(-\frac{x^2}{2ib\tau}-\frac{i\mathbf{w}\mathbf{x}\tau}{2}-\frac{ibw^2\tau^3}{24}\right),$$
$$\boldsymbol{\varphi}^{(0)}=\frac{\mathbf{y}}{b}\varphi_0^{(0)},\quad \mathbf{y}=\frac{\mathbf{x}}{\tau}+\frac{b\mathbf{w}\tau}{2}. \tag{3.17}$$

Substituting the solution (3.17) into Eq. (2.18), we obtain the well known emission probability in a constant external field. In the first Born approximation, we have

$$\varphi_\mu^{(1)}=\varphi_\mu^{(0)}+\psi_\mu\varphi_0^{(0)}. \tag{3.18}$$

Transforming then to variables $y$, $s=1/\tau$, we find that the function $\psi_\mu(\mathbf{y},s)$ satisfies the equation

$$-\frac{\partial\psi_\mu(\mathbf{y},s)}{\partial s}-\frac{ib}{2}\Delta\psi_\mu(\mathbf{y},s)=\frac{y_\mu}{bs^2}V\left(\frac{\mathbf{y}}{s}-\frac{b\mathbf{w}}{2s^2}\right), \tag{3.19}$$

where we have put $y_0=b$. We seek a solution of (3.19) using the Green's function $G(y,s)$ which satisfies the equation

$$\frac{\partial G}{\partial s}+i\frac{b}{2}\Delta G=-\delta(y)\delta(s), \tag{3.20}$$

which we can solve, for example, via a Fourier transformation:

$$G(\mathbf{y}-\mathbf{y}', s-s')=\frac{\vartheta(s'-s)}{2ib\pi(s'-s)}\exp\left\{-\frac{(\mathbf{y}-\mathbf{y}')^2}{2ib(s'-s)}\right\}. \quad (3.21)$$

Making use of (3.21), we obtain a solution of the equation for $\psi_\mu(\mathbf{y},s)$ in the form

$$\psi_\mu(\mathbf{y},s)\begin{pmatrix}\psi_0(\mathbf{y},s)\\ \boldsymbol{\psi}(\mathbf{y},s)\end{pmatrix}$$

$$=\int G(\mathbf{y}-\mathbf{y}', s-s')\frac{y_\mu'}{bs'^2}V\left(\frac{\mathbf{y}'}{s'}-\frac{b\mathbf{w}}{2s'^2}\right)d^2y'\,ds'$$

$$=\frac{1}{2ib^2\pi}\int_s^\infty\frac{ds'}{s'-s}\int d^2y'\begin{pmatrix}y_0\\ s'\mathbf{y}'\end{pmatrix}V\left(\mathbf{y}'-\frac{b\mathbf{w}}{2s'^2}\right)$$

$$\times\exp\left\{-\frac{(s'\mathbf{y}'-\mathbf{y})^2}{2ib(s'-s)}\right\}. \quad (3.22)$$

Transforming variables to $s'=(\xi+1)s$ and $\mathbf{y}'=\boldsymbol{\eta}+\mathbf{x}/(1+\xi)+b\mathbf{w}\tau^2/2(1+\xi)$, and reverting to $\tau=s^{-1}$, $\mathbf{x}=\mathbf{y}\tau-b\mathbf{w}\tau^2/2$, we gave

$$\psi_0(\mathbf{x},\tau)=\frac{1}{2ib\pi}\int_0^\infty\frac{d\xi}{\xi}\int\exp\left\{-\frac{(1+\xi)^2\eta^2}{2ib\xi\tau}\right\}$$

$$\times V\left(\boldsymbol{\eta}+\frac{\mathbf{x}}{1+\xi}+\frac{b\xi\mathbf{w}\tau^2}{2(1+\xi)^2}\right)d^2\eta,$$

$$\boldsymbol{\psi}(\mathbf{x},\tau)=\frac{1}{2ib^2\pi\tau}\int_0^\infty\frac{d\xi}{\xi}\int d^2\eta\left[\boldsymbol{\eta}(1+\xi)+\mathbf{x}+\frac{b\mathbf{w}\tau^2}{2}\right]$$

$$\times V\left[\boldsymbol{\eta}+\frac{\mathbf{x}}{1+\xi}+\frac{b\xi\mathbf{w}\tau^2}{2(1+\xi)^2}\right]\exp\left\{-\frac{(1+\xi)^2\eta^2}{2ib\xi\tau}\right\}. \quad (3.23)$$

Substituting the solution (3.23) into (3.18) and then into (2.18), we obtain in the first Born approximation

$$\frac{dW^{(1)}}{d\omega}=\frac{\alpha\omega}{4b^2\pi^2}\mathrm{Re}\int_0^\infty\frac{d\tau}{\tau}\exp\left(-\frac{ia\tau}{2}-\frac{ibw^2\tau^3}{24}\right)$$

$$\times\int_0^\infty\frac{d\xi}{\xi}\int d^2\eta\,V\left[\boldsymbol{\eta}+\frac{b\mathbf{w}\xi\tau^2}{2(1+\xi)^2}\right]\exp\left\{\frac{i(1+\xi)^2\eta^2}{2b\xi\tau}\right\}$$

$$\times\left[\left(1+\frac{\varepsilon^2}{\varepsilon'^2}\right)\left(\frac{\eta^2(1+\xi)^2}{b^2\xi\tau^2}+\frac{\mathbf{w}\boldsymbol{\eta}(1+\xi)^2}{2b\xi}+\frac{w^2\tau^2}{4}\right)-\frac{\omega^2}{\gamma^2\varepsilon'^2}\right]. \quad (3.24)$$

It is convenient to transform variables in (3.24), letting $\tau\to2\tau$, $\xi=e^{2z}$, whereupon the emission probability spectrum takes the form

$$\frac{dW^{(1)}}{d\omega}=\frac{\alpha\omega}{b^2\pi^2}\mathrm{Re}\int_0^\infty\frac{d\tau}{\tau}e^{-\rho(\tau)}\int_0^\infty dz\int d^2\eta\exp\left(\frac{i\eta^2\,\mathrm{ch}^2 z}{b\tau}\right)V\Bigg(\boldsymbol{\eta}$$

$$+\frac{\mathbf{w}b\tau^2}{2\,\mathrm{ch}^2 z}\Bigg)\left\{\left(1+\frac{\varepsilon^2}{\varepsilon'^2}\right)\left[\mathrm{ch}^2 z\left(\left(\frac{\eta}{b\tau}\right)^2+\frac{2}{b}\mathbf{w}\boldsymbol{\eta}\right)+w^2\tau^2\right]\right.$$

$$\left.-\frac{\omega^2}{\gamma^2\varepsilon'^2}\right\},$$

$$\rho(\tau)=i\left(a\tau+\frac{1}{3}bw^2\tau^3\right). \quad (3.25)$$

This can be expressed directly in terms of the scattering cross section. In order to do so, we must use the definition (3.15) and carry out the appropriate Gaussian integrals over $\eta$ in (3.25). We then obtain

$$\frac{dW^{(1)}}{d\omega}=\frac{\alpha\omega n}{b\pi}\mathrm{Im}\int_0^\infty d\tau e^{-\rho(\tau)}\int_0^\infty\frac{dz}{\mathrm{ch}^2 z}\int d^2\vartheta\sigma(\boldsymbol{\vartheta})\left[F(0)-F(\boldsymbol{\vartheta})\right], \quad (3.26)$$

where

$$F(\boldsymbol{\vartheta})=\left[\left(1+\frac{\varepsilon^2}{\varepsilon'^2}\right)\left(\frac{i}{b\tau}+\frac{\vartheta^2}{4\,\mathrm{ch}^2 z}-\mathbf{w}\boldsymbol{\vartheta}\tau+w^2\tau^2\right)-\frac{\omega^2}{\gamma^2\varepsilon'^2}\right]$$

$$\times\exp\left\{-\frac{ib\tau}{4\,\mathrm{ch}^2 z}(\vartheta^2-2\mathbf{w}\boldsymbol{\vartheta}\tau)\right\}. \quad (3.27)$$

Equations (3.25)–(3.27) describe the emission, to quasi-classical (relativistic) accuracy, upon particle scattering in an external field when the scattering cross section is of the form $\sigma(\boldsymbol{\vartheta},\boldsymbol{\vartheta}')=\sigma(\boldsymbol{\vartheta}-\boldsymbol{\vartheta}')$. This subsumes scattering from a screened Coulomb potential (see Eq. (3.1)).

In weak fields, we can expand the functions which enter into (3.25)–(3.27) in powers of $w\tau$. Keeping the most significant terms of this expansion and terms $\sim\omega^2\tau^2$, we obtain the following expression for the emission cross section in a weak field ($\chi/u\ll1$) for a screened Coulomb potential:

$$\frac{d\sigma_{sc}^{(1)}}{d\omega}=\frac{4Z^2\alpha^3}{3m^2\omega}\left\{\frac{\omega^2}{\varepsilon^2}\left(\ln\frac{1}{\gamma\vartheta_1}+\frac{1}{3}\right)+2\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\right.$$

$$\times\left(\ln\frac{1}{\gamma\vartheta_1}+\frac{7}{12}\right)+\frac{2}{5}\left(\frac{\chi}{u}\right)^2\left[\frac{\omega^2}{\varepsilon^2}\left(32\ln\frac{1}{\gamma\vartheta_1}-\frac{199}{15}\right)\right.$$

$$\left.\left.+\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\left(43\ln\frac{1}{\gamma\vartheta_1}-\frac{251}{15}\right)\right]\right\}. \quad (3.28)$$

In the absence of a field, the cross section (3.28) is the same as the standard cross section in a screened potential (see Ref. 5, for example). This is the first calculation of the corrections to power-law accuracy.

In strong fields ($\chi/u\gg1$), contributions to the integrals (3.25), (3.26) come from

$$w^2\tau^2\sim\eta^{-2}\sim\theta^2\sim\frac{1}{\gamma^2}\left(\frac{\chi}{u}\right)^{2/3}\gg\frac{1}{\gamma^2}.$$

Therefore, up to terms $\sim(u/\chi)^{2/3}$, we can neglect the term linear in $\tau$ which appears in the argument of the exponential ($\rho(\tau)\approx ibw^2\tau^3/3$), and we can omit the term $\omega^2/\gamma^2\varepsilon'^2$ in square brackets in (3.25) and (3.27). We can then rotate the integration contour in $\tau$ by an angle $-\pi/6$, so that $\tau\to e^{-i\pi/6}\tau(i\tau^3\to\tau^3)$, and transform to the variable $\rho$. The integrals over $\eta$ and $\vartheta$ in (3.25) and (3.26) are conveniently calculated via an exponential parametrization. With $\chi/u\gg1$, we find after some fairly tedious calculations that

$$\frac{d\sigma_{sc}^{(1)}}{d\omega}=\frac{2Z^2\alpha^3\Gamma(^1/_3)}{5m^2\omega}\left(\frac{u}{3\chi}\right)^{2/3}\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\left[\ln\left(\frac{1}{\gamma\vartheta_1}\left(\frac{\chi}{u}\right)^{1/3}\right)+D_{sc}\right], \quad (3.29)$$

where $D_{sc}=2.3008$. The emission inhibition factor $(u/\chi)^{2/3}$ in (3.29) occurs, as was shown in the Introduction, because of a decrease in the photon production length. The factor $(\chi/u)^{1/3}$ in the argument of the logarithm is associated with the increase in effective emission angles $\vartheta_{\text{eff}} \sim \gamma^{-1}(\chi/u)^{1/3}$. Note that since the quantum parameter $\chi \propto 1/m^3$, the cross section (3.29) is independent of the mass of the radiating particle.

## 4. RADIATION NEAR A NUCLEUS

In a purely Coulomb radiation problem, the scattering cross section for momentum transfer close to the minimum depends not only on the angular difference $\boldsymbol{\vartheta}' - \boldsymbol{\vartheta} = \boldsymbol{\theta}$ (on the transverse momentum transfer $q_\perp \approx \varepsilon\theta$), but on the radiation angle itself as well (on the longitudinal momentum transfer[4] $q_\parallel \approx (u\varepsilon/2)(1/\gamma^2 + \vartheta^2)$). This is connected with the fact that for such values of momentum transfer, the emission of a photon influences the particle scattering.[5] In that event, the scattering cross section is conveniently given in the form

$$\sigma(\boldsymbol{\vartheta}', \boldsymbol{\vartheta}) = \sigma(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}') = \sigma(\boldsymbol{\theta}, (\boldsymbol{\vartheta}+\boldsymbol{\vartheta}')/2)$$
$$= \sigma(\boldsymbol{\theta}, \boldsymbol{\vartheta}+\boldsymbol{\theta}/2) = \sigma(\boldsymbol{\theta}, \boldsymbol{\vartheta}_1) + \tilde{\sigma}, \tag{4.1}$$

where

$$\tilde{\sigma} = \sigma(\boldsymbol{\theta}, \boldsymbol{\vartheta}+\boldsymbol{\theta}/2) - \sigma(\boldsymbol{\theta}, \boldsymbol{\vartheta}_1). \tag{4.2}$$

Here $\sigma(\boldsymbol{\theta}, \vartheta_1)$ is the scattering cross section in the screened potential (3.1). We assume that $\vartheta_1$ is small enough that $\vartheta_1 \ll \vartheta_{\text{eff}}$ (3.5), and large enough that when $\theta \gtrsim \vartheta_1$, we can neglect the longitudinal momentum transfer $q_\perp \approx \varepsilon\theta \gg q_\parallel$:

$$\vartheta_1 \ll \vartheta_{\text{eff}} \approx \frac{1}{\gamma}\left[1 + \left(\frac{\chi}{u}\right)^{1/3}\right],$$
$$\vartheta_1 \gg \frac{1}{\varepsilon} q_\parallel(\vartheta_{\text{eff}}) \approx \frac{u}{2}\left(\frac{1}{\gamma^2} + \vartheta^2_{\text{eff}}\right). \tag{4.3}$$

The right-hand (collision) side of Eq. (2.13) may be represented by a sum of a screened cross section and $\tilde{\sigma}$. In the first Born approximation, these terms contribute linearly and independently to the radiation cross section. We calculated the corresponding radiation cross section in a screened potential above. On the other side with $\tilde{\sigma}$, we make use of the fact that when $\theta > \vartheta_1$, $\tilde{\sigma}$ falls off as $\vartheta_1^2/\theta^2$, and we may expand the contributing terms in powers of $\theta/\vartheta \sim \theta/\vartheta_{\text{eff}}$:

$$V_\mu(\boldsymbol{\vartheta}') \approx V_\mu(\boldsymbol{\vartheta}) + \boldsymbol{\theta}\frac{\partial V_\mu}{\partial \boldsymbol{\vartheta}} + \frac{1}{2}\theta_i\theta_j\frac{\partial^2 V_\mu}{\partial \vartheta_i \partial \vartheta_j} + \dots,$$

$$\tilde{\sigma} \approx \tilde{\sigma}(\boldsymbol{\theta}, \boldsymbol{\vartheta}) + \frac{1}{2}\boldsymbol{\theta}\frac{\partial \tilde{\sigma}(\boldsymbol{\theta}, \boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}} + \dots. \tag{4.4}$$

Substituting this expansion into (2.13), we obtain a Fokker-Planck type of equation:

$$\frac{\partial V_\mu}{\partial \tau} + ib\frac{\boldsymbol{\vartheta}^2}{2}V_\mu - \mathbf{w}\frac{\partial V_\mu}{\partial \boldsymbol{\vartheta}} = \frac{1}{4}\frac{\partial}{\partial \boldsymbol{\vartheta}}\left(\frac{\partial V_\mu}{\partial \boldsymbol{\vartheta}}\tilde{\theta}^2(\boldsymbol{\vartheta})\right),$$

where

$$\tilde{\theta}^2(\boldsymbol{\vartheta}) = n\int \tilde{\sigma}(\theta^2, \boldsymbol{\vartheta})\theta^2 d^2\theta. \tag{4.5}$$

For a nuclear potential, $\tilde{\theta}^2(\vartheta)$ is of the form ($|q^2| = \varepsilon^2\theta^2 + q_\parallel^2$)

$$\tilde{\theta}^2(\boldsymbol{\vartheta}) = \frac{4Z^2\alpha^2 n\pi}{\varepsilon^2}\int d\theta^2\theta^2\left[\frac{1}{(\theta^2+\delta^2\zeta^2)^2} - \frac{1}{(\theta^2+\vartheta_1^2)^2}\right]$$
$$= \frac{8Z^2\alpha^2 n\pi}{\varepsilon^2}\ln\frac{\vartheta_1}{\delta\zeta}, \quad \delta = \frac{u}{2\gamma^2}, \quad \zeta = 1 + \gamma^2\boldsymbol{\vartheta}^2. \tag{4.6}$$

Let $\mathbf{v} = \boldsymbol{\vartheta} + \mathbf{w}\tau$, and

$$2\pi n\frac{Z^2\alpha^2}{\varepsilon^2}\ln\frac{\vartheta_1}{\delta\zeta} \equiv q\ln\frac{\vartheta_1}{\delta\zeta} \equiv qL(\boldsymbol{\vartheta}). \tag{4.7}$$

Then Eq. (4.5) takes the form

$$\frac{\partial V_\mu(\mathbf{v}, \tau)}{\partial \tau} + \frac{ib}{2}\boldsymbol{\vartheta}^2(\tau)V_\mu(\mathbf{v}, \tau) = q\frac{\partial}{\partial \mathbf{v}}\left[\frac{\partial V_\mu}{\partial \mathbf{v}}L(\boldsymbol{\vartheta}(\tau))\right]. \tag{4.8}$$

We next find the zeroth approximation to the solution of (4.8):

$$\frac{\partial V_\mu^{(0)}(\mathbf{v}, \tau)}{\partial \tau} + i\frac{b}{2}\boldsymbol{\vartheta}^2(\tau)V_\mu^{(0)}(\mathbf{v}, \tau) = 0,$$

$$V_0^{(0)}(\mathbf{v}, \tau) = \Phi(\mathbf{v}, \tau) = \exp\left\{-\frac{ib}{2}\int_0^\tau d\tau'\boldsymbol{\vartheta}^2(\tau')\right\}$$

$$= \exp\left\{-\frac{ib\tau}{2}\left(\mathbf{v}^2 - \mathbf{v}\mathbf{w}\tau + \frac{w^2\tau^2}{3}\right)\right\},$$

$$\mathbf{V}^{(0)} = \mathbf{v}\Phi(\mathbf{v}, \tau). \tag{4.9}$$

Substitution of this solution into Eq. (2.12) gives the spectral and angular distribution of emission probability in a constant field. In the first Born approximation, we seek a solution of Eq. (4.8) in the form

$$V_\mu(\mathbf{v}, \tau) = V_\mu^{(0)}(\mathbf{v}, \tau) + V_\mu^{(1)}(\mathbf{v}, \tau),$$

$$V_\mu^{(1)}(\mathbf{v}, \tau) = v_\mu(\mathbf{v}, \tau)\Phi(\mathbf{v}, \tau). \tag{4.10}$$

The equation for $v_\mu$ then takes the form

$$\frac{\partial v_\mu}{\partial \tau} = q\Phi^{-1}\frac{\partial}{\partial \mathbf{v}}\left[\frac{\partial V_\mu^{(0)}}{\partial \mathbf{v}}L(\mathbf{v} - \mathbf{w}\tau)\right], \tag{4.11}$$

and, with the initial data $v_\mu(\boldsymbol{\vartheta}, 0) = 0$, we obtain its solution simply by integrating the right-hand side of this equation:

$$v_\mu(\mathbf{v}, \tau) = q\int_0^\tau d\tau'\Phi^{-1}(\mathbf{v}, \tau')\frac{\partial}{\partial \mathbf{v}}\left[\frac{\partial V_\mu^{(0)}(\mathbf{v}, \tau')}{\partial \mathbf{v}}L(\mathbf{v} - \mathbf{w}\tau')\right]. \tag{4.12}$$

Substituting the solution (4.12) first into (4.10) and then into (2.12), we obtain the following expression:

$$\frac{dW}{d\omega} = \frac{\alpha q\omega}{(2\pi)^2}\operatorname{Re}\int_0^\infty d\tau e^{-ia\tau/2}\int d^2\mathbf{v}\Phi(\mathbf{v}, \tau)$$
$$\times\int_0^\tau d\tau'\Phi^{-1}(\mathbf{v}, \tau')\left\{\frac{\omega^2}{\gamma^2\varepsilon'^2}\frac{\partial}{\partial \mathbf{v}}\left[\frac{\partial\Phi(\mathbf{v}, \tau')}{\partial \mathbf{v}}L(\mathbf{v} - \mathbf{w}\tau')\right]\right.$$
$$\left.+\left(1 + \frac{\varepsilon^2}{\varepsilon'^2}\right)(\mathbf{v} - \mathbf{w}\tau)\frac{\partial}{\partial v_i}\left[\frac{\partial \mathbf{v}\Phi(\mathbf{v}, \tau')}{\partial v_i}L(\mathbf{v} - \mathbf{w}\tau')\right]\right\}. \tag{4.13}$$

Note that

$$\Phi(\mathbf{v}, \tau)\Phi^{-1}(\mathbf{v}, \tau') = \Phi(\mathbf{v} - \mathbf{w}\tau', \tau - \tau'). \tag{4.14}$$

Integrating (4.13) by parts and making use of (4.14), we may write (4.13) in the symmetric form

$$\frac{dW}{d\omega}=-\frac{\alpha q\omega}{(2\pi)^2}\operatorname{Re}\int_0^\infty d\tau\, e^{-ia\tau/2}\int d^2\vartheta\, L(\vartheta)\int_0^\tau d\tau'$$

$$\times\left\{\frac{\omega^2}{\gamma^2\varepsilon'^2}\left[\frac{\partial}{\partial\vartheta}\Phi(\vartheta,\tau-\tau')\right]\left[\frac{\partial}{\partial\vartheta}\Phi(\vartheta+\mathbf{w}\tau',\tau')\right]\right.$$

$$+\left(1+\frac{\varepsilon^2}{\varepsilon'^2}\right)\sum_i\left[\frac{\partial}{\partial\vartheta_i}(\vartheta-\mathbf{w}(\tau-\tau'))\Phi(\vartheta,\tau-\tau')\right]\left[\frac{\partial}{\partial\vartheta_i}(\vartheta+\mathbf{w}\tau')\right.$$

$$\left.\left.\times\Phi(\vartheta+\mathbf{w}\tau',\tau')\right]\right\}. \qquad (4.15)$$

Making the change of variables $\tau-\tau'=(1-\mu)\tau/2$, $\tau'=(1+\mu)\tau/2$, and letting $\tau\to2\tau$, and $\vartheta=\boldsymbol{\theta}-\mu\mathbf{w}\tau$, we obtain a correction to the emission probability spectrum of the form

$$\frac{dW}{d\omega}=-\frac{\alpha q\omega}{\pi^2}\operatorname{Re}\int_0^\infty d\tau\,\tau e^{-\rho(\tau)}\int_0^1 d\mu$$

$$\times\int d^2\theta e^{-ib\tau\theta^2}\ln\frac{\delta(1+\gamma^2(\theta-\mathbf{w}\tau)^2)}{\vartheta_1}$$

$$\times\left\{b^2\tau^2(1-\mu^2)\left[R_1+R_2(\theta^2-w^2\tau^2)\right]\right.$$

$$\times\left(\theta^2-\mathbf{w}\theta\mu\tau+\frac{\mu^2-1}{4}w^2\tau^2\right)-2R_2$$

$$\left.+ib\tau R_2[2\theta(\theta+\mathbf{w}\mu\tau)+(1-\mu^2)w^2\tau^2]\right\},$$

$$(4.16)$$

where

$\rho(\tau)=i(a\tau+b\omega^2\tau^3/3)$, $R_1=\omega^2/\varepsilon'^2\gamma_2$, $R_2=1+\varepsilon^2/\varepsilon'^2$.

Expanding the expressions in (4.16) in powers of $w\tau$ as $\mathbf{w}\to0$, and performing some straightforward but tedious calculations, we obtain the following correction to the emission cross section for the weak-field case ($\chi/u\ll1$):

$$\frac{d\tilde\sigma}{d\omega}=\frac{4Z^2\alpha^3}{3m^2\omega}\left\{\frac{\omega^2}{\varepsilon^2}\left(\ln\frac{\vartheta_1}{\delta}-\frac{5}{6}\right)+2\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\left(\ln\frac{\vartheta_1}{\delta}-\frac{13}{12}\right)\right.$$

$$+\frac{2}{5}\left(\frac{\chi}{u}\right)^2\left[\frac{\omega^2}{\varepsilon^2}\left(32\ln\frac{\vartheta_1}{\delta}-\frac{662}{105}\right)\right.$$

$$\left.\left.+\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\left(43\ln\frac{\vartheta_1}{\delta}-\frac{3\,967}{420}\right)\right]\right\}. \qquad (4.17)$$

Combining this result with the cross section (3.28) for the screened potential, we obtain the emission cross section at a nucleus, with $\chi/u\ll1$, and taking corrections $\sim(\chi/u)^2$ into account:

$$\frac{d\sigma}{d\omega}=\frac{4Z^2\alpha^3}{3m^2\varepsilon^2\omega}\left\{[2(\varepsilon^2+\varepsilon'^2)+\omega^2]\left(\ln\frac{2\gamma}{u}-\frac{1}{2}\right)+\frac{2}{5}\left(\frac{\chi}{u}\right)^2\right.$$

$$\left.\times\left[\omega^2\left(32\ln\frac{2\gamma}{u}-\frac{137}{7}\right)+(\varepsilon^2+\varepsilon'^2)\left(43\ln\frac{2\gamma}{u}-\frac{733}{28}\right)\right]\right\}.$$

$$(4.18)$$

To logarithmic accuracy, (4.18) is the same as the cross section calculated in Ref. 7. This is the first time corrections have been obtained to power-law accuracy.

In a strong external field, the calculation is the same as in the screened case (see the discussion preceding Eq. (3.29)). For $\chi/u\gg1$, the correction $d\tilde\sigma/d\omega$ to the emission section takes the form

$$\frac{d\tilde\sigma}{d\omega}=\frac{2Z^2\alpha^3\Gamma(^1/_3)}{5m^2\omega}\left(\frac{u}{3\chi}\right)^{2/3}\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\left[\ln\left(\frac{\vartheta_1}{\delta}\left(\frac{u}{\chi}\right)^{2/3}\right)+\tilde D\right],$$

$$(4.19)$$

where $\tilde D=-3.0131$. The argument of the logarithm in (4.19) determines the limit of total screening of the potential during emission. When there is total screening, we must have

$$\left(\frac{\chi}{u}\right)^{2/3}\frac{\delta}{\vartheta_1}=\left(\frac{\chi}{u}\right)^{2/3}q_{min}a\ll1, \qquad (4.20)$$

where $q_{\min}=\varepsilon\delta=\omega m^2/2\varepsilon(\varepsilon-\omega)$, and $a$ is the characteristic radius of influence of the potential. For example, in an amorphous substance under standard conditions, $a\sim Z^{-1/3}a_0$ (where $a_0$ is the Born radius), and in oriented single crystals, this length is determined by the amplitude of thermal vibrations. Equation (4.20) implies that in the absence of an external field, a totally screened field ($q_{\min}\,a\ll1$) can become unscreened. Note that (4.20) depends little on the photon frequency ($\omega^{1/3}$), and is independent of the particle mass. Combining (4.19) with the asymptotic expression (3.29) for the emission cross section in a screened potential, we obtain for the emission cross section at a nucleus in an external field, when $\chi/u\gg1$,

$$\frac{d\sigma_{nose}}{d\omega}=\frac{2Z^2\alpha^3\Gamma(^1/_3)}{5m^2\omega}\left(\frac{u}{3\chi}\right)^{2/3}\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)$$

$$\times\left[\ln\left(\frac{2\gamma}{u}\left(\frac{u}{\chi}\right)^{2/3}\right)+D_{nose}\right];$$

$$D_{nose}=D_{sc}+\tilde D, \qquad (4.21)$$

where $D_{\rm nose}=-0.7123$. The cross section (4.21), like (3.29), is independent of the mass of the radiating particle. The argument of the logarithm obtained in Ref. 7 differs from the one in Eq. (4.21) by an additional factor $(u/\chi)^{1/3}$, which is easily seen to be proportional to the particle mass.

## 5. BREMSSTRAHLUNG FROM COLLIDING $e^+e^-$ BEAMS

As particle energies increase in colliding-beam accelerators, the intrinsic fields due to the beams themselves become all the stronger, and this reduces the beam dimensions significantly, as dictated by collision efficiency (luminosity). Thus, the increase in the parameter $\chi$ in this situation is due both to an increase in actual particle energy and to a considerable increase in the beam field strength (the self-action of each beam is $\sim1/\gamma^2$) in the collision region. As was already pointed out in the Introduction, the effect of an external field on bremsstrahlung can manifest itself fairly early, when $\varepsilon\chi/q_0\sim4\gamma^2\chi/u\gg1$. In that event, the spectrum of equivalent photons changes in the vicinity of minimum momentum transfer, so that the lower bound on effective momentum transfer increases as

$$\tilde q\sim\frac{q_0}{\gamma}\left(\frac{\varepsilon\chi}{q_0}\right)^{1/3}. \qquad (5.1)$$

Then so long as the radiation vertex doesn't change, the virtual-photon characteristic frequencies are unaffected; for small momentum transfers, these are determined by the kinematic relation $qp'\approx kp$, so that

$$q_0 \approx \frac{\omega\varepsilon}{4(\varepsilon-\omega)}\left(\frac{1}{\gamma^2}+\vartheta^2\right)=q_0'\zeta,$$

$$q_0'=\omega m^2/4\varepsilon(\varepsilon-\omega)=u\varepsilon/4\gamma^2,\quad \zeta=1+\gamma^2\vartheta^2. \tag{5.2}$$

For this case ($\chi/u \ll 1, q_0 \sim q_0'$), the process has been studied in Refs. 4 and 6. The following results was obtained, with $4\gamma^2\chi/u \gg 1$, to logarithmic accuracy in Ref. 4 and to power-law accuracy in Ref. 6:

$$\frac{d\sigma}{d\omega}=\frac{4\alpha^3\varepsilon'}{m^2\omega\varepsilon}\left\{\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}-\frac{2}{3}\right)\left[\ln\frac{m}{q_{min}\left(4\gamma^2\frac{\chi}{u}\right)^{1/3}}-\frac{4-2C-\ln 3}{6}\right]+\frac{1}{27}\right\},\quad q_{min}=\frac{m^3\omega}{4\varepsilon^2(\varepsilon-\omega)}=\frac{q_0'}{\gamma}. \tag{5.3}$$

When the parameter $\chi/u$ is not small, the radiation vertex does change, and as indicated by the foregoing analysis, the characteristic radiation angles increase as $\vartheta_{eff} \sim \gamma^{-1}(\chi/u)^{1/3}$, resulting in an increase in the characteristic frequencies of virtual photons at small momentum transfers:

$$q_0 \sim q_0'(1+\gamma^2\vartheta^2_{eff}) \sim q_0'(\chi/u)^{2/3}. \tag{5.4}$$

The upper bound on effective momentum transfer increases for the same reason: $q_{max} \sim \varepsilon\vartheta_{eff} \sim m(\chi/u)^{1/3}$.

As a result, for $\chi/u \gg 1$, the argument of the logarithm takes the form

$$\frac{q_{max}}{\tilde{q}} \sim \frac{m(\chi/u)^{1/3}}{(q_0/\gamma)(\chi\varepsilon/q_0)^{1/3}} \sim \frac{1}{u^{1/3}}\left(\frac{\varepsilon}{q_0}\right)^{2/3} \sim \frac{\gamma^{4/3}}{u^{5/9}\chi^{4/9}}. \tag{5.5}$$

Making use of the foregoing analysis, we can estimate the characteristic impact parameters $\rho$ responsible for the lower bound on effective momentum transfer:

$$\rho_{eff} \sim \frac{1}{\tilde{q}} \sim \frac{\gamma}{q_0(\varepsilon\chi/q_0)^{1/3}} \sim \frac{1}{m}\frac{\gamma^{4/3}}{u^{2/9}\chi^{7/9}}. \tag{5.6}$$

**The cutoff at small momentum transfer can be accounted for not only by the external field, but by a number of other factors as well.[11] In particular, it has been shown that for several-GeV electron-positron beams colliding in a storage ring, the dominant cutoff is due to the smallness of the transverse beam dimensions (see Ref. 12 for experimental details, and Ref. 13 for a theortical analysis).** When the latter effect is taken into account, the lower bound on effective momentum transfer must be taken to be $\max(1/\sigma, 1/\rho_{eff})$. Thus, if the transverse beam dimensions $\sigma$ are greater than the characteristic impact parameters given by Eq. (5.6), the finiteness of these dimensions may be neglected; otherwise ($\rho_{eff} > \sigma$), the lower bound on momentum transfer is governed by beam size. The coefficient of the logarithm comes from Eq. (4.21), assuming there that $Z = 1$. As a result, for $\chi/u \gg 1$, we have the following expression for the colliding-beam emission cross section to logarithmic accuracy:

$$\frac{d\sigma}{d\omega}=\frac{2\alpha^3\Gamma(1/3)}{5m^2\omega}\left(\frac{u}{3\chi}\right)^{2/3}\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\ln\Delta, \tag{5.7}$$

where

$$\Delta=\begin{cases}\dfrac{\gamma^{4/3}}{u^{5/9}\chi^{4/9}}, & \dfrac{1}{m}\dfrac{\gamma^{4/3}}{u^{2/9}\chi^{7/9}}<\sigma\\[2ex] m\sigma(\chi/u)^{1/3}, & \dfrac{1}{m}\dfrac{\gamma^{4/3}}{u^{2/9}\chi^{7/9}}>\sigma\end{cases} \tag{5.8}$$

Equation (5.7) in local in character ($\chi=\chi(\rho)$). In order to calculate global characteristics of the radiation, it is necessary to carry out the appropriate averaging, taking account of the distribution of particles in the beams.

1) In this paper, we employ units such that $\hbar = c = 1$.
2) This follows from the quasiclassical theory of radiation (see Ref. 5, for example.
3) Diffraction by the nucleus, which is significant at angles $\vartheta_{eff} \gtrsim \vartheta_2 \approx (Z^{1/3}\alpha\gamma)^{-1}$, is not taken into account.
4) The present treatment is for a nucleus in its ground state.
5) This question is discussed in detail in Ref. 5.


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