Conditions for validity of the Townsend-Shockley impact-ionization law in semiconductors

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According to the Shockley picture, impurity centers are ionized by fast electrons that have reached ionization energy without colliding even once with scatterers. For a mathematical realization of this picture it is necessary to find for the kinetic equation a solution that is needle-shaped at high energies. For deformation scattering of electrons by optical phonons, Keldysh obtained such a solution, but it is shown that the conditions for the validity of his solutions are necessary but not sufficient. An additional condition for the realizability of a needle-shaped Keldysh distribution is established. If this condition is not met, the number of fast electrons is determined by the distribution of medium-energy electrons. It is shown that if this is a quasiequilibrium distribution, a needle-shaped beam is formed in the fast-electron region. The features of formation of a needle-shaped distribution function are investigated for various mechanisms of energy and momentum relaxation on phonons and impurities. When elastic scattering predominates, the distribution function is weakly anisotropic rather than needle-shaped, but the Townsend-Shockley distribution can be realized. This case is considered from a most general viewpoint.

1. The free carriers in a semiconductor placed in a strong electric field **E** are accelerated by this field and can acquire enough energy for impact ionization of impurity centers, excitons, or other bound states of an electron. The ionization coefficient is proportional in this case to the electron distribution function for an ionization energy ε_i that usually exceeds significantly the average electron energy. The dependence of the ionization coefficient on the electric field is determined by how the electron acquires the ionization energy, since it loses energy in large or small batches on scattering. In the latter case, the energy relaxation reduces to electron diffusion in energy. The distribution function is then determined by the balance between the Joule heating and the diffusive electric relaxation, and takes the form[1]

$$f_0(\varepsilon)=A\exp\left(-\int^{\varepsilon}\frac{d\varepsilon'}{e^2E_iE_k\theta_{ik}(\varepsilon')}\right),\tag{1}$$

where the heating function $\theta(\varepsilon)$ (Ref. 2) depends on the energy and momentum relaxation mechanisms.

Shockley,[3] contradicting this scenario, proposed on the basis of a gas-ionization theory developed by Townsend, a different picture of the process. The electron beam is accelerated by the electric field and strives to reach high energies, but is continuously weakened by collisions with scatterers. The colliding electron is knocked out of the beam. The ionization is effected by an electron that reached ionization energy without experiencing even one collision with the scatterers. In this picutre, the number of high-energy electrons is determined by the mean free path $l(\varepsilon)$ and has the following dependence on the electric field:

$$f_0(\varepsilon)=A\exp\left(-\int^{\varepsilon}\frac{d\varepsilon'}{eEl(\varepsilon')}\right),\quad E=|\mathbf{E}|.\tag{2}$$

The validity limits of each of these ionization laws were given in Keldysh's known paper.[4] Electron scattering by the deformation potential of the optical phonons was considered, and this made it possible to write the kinetic equation in the form

$$e\mathbf{E}\frac{\partial f_\mathbf{p}}{\partial\mathbf{p}}+\frac{f_\mathbf{p}2^{1/2}}{l_{ph}m^{1/2}}\left[(\varepsilon_p-\hbar\omega)^{1/2}\frac{e^{\beta}}{2\,\mathrm{ch}\,\beta}+(\varepsilon_p+\hbar\omega)^{1/2}\frac{e^{-\beta}}{2\,\mathrm{ch}\,\beta}\right]$$
$$=\frac{2^{1/2}}{l_{ph}m^{1/2}}\left[(\varepsilon_p-\hbar\omega)^{1/2}\frac{e^{-\beta}}{2\,\mathrm{ch}\,\beta}f_0(\varepsilon_p-\hbar\omega)+(\varepsilon_p+\hbar\omega)^{1/2}\frac{e^{\beta}}{2\,\mathrm{ch}\,\beta}f_0(\varepsilon_p+\hbar\omega)\right].\tag{3}$$

Here **p** is the electron momentum, ε_p its energy, m its effective mass, ω the phonon frequency, $\beta=\hbar\omega/2T$, T the temperature in energy units, $f_\mathbf{p}$ the electron momentum distribution function, and $f_0(\varepsilon_p)$ the same function averaged over a constant-energy surface.

It is assumed here and elsewhere that

$$T\ll\hbar\omega\ll\varepsilon_i.\tag{4}$$

As shown in Ref. 4, the influence of the electric field on the ionization is noticeable only when the energy eEl acquired by the electron over the mean free path exceeds the thermal energy T. If, however, this energy exceeds also the inelasticity energy $\hbar\omega$ lost by the electron on emission of a phonon, diffusion in time takes place and leads to the ionization law (1).

Keldysh obtained the Townsend-Shockley ionization law (2) for fields meeting the conditions

$$T\ll eEl\ll\hbar\omega\ll\varepsilon_i.\tag{5}$$

In this case, the solution of (3) is the needle-like distribution function

$$f_\mathbf{p}=A_0(\varepsilon_p)(1-s_0\cos\vartheta)^{-1}\exp\left(-\frac{\hbar\omega}{eEl}s_0\right)\exp\left(-\frac{\varepsilon_p}{eEl}s_0\right),$$
$$\cos\vartheta=\frac{\mathbf{pE}}{pE},\tag{6}$$

where $A_0(\varepsilon_p)$ is a weaker-than-exponential function of the energy, and the parameter s_0 that satisfies the self-consistency equation differs extremely little from unity:

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$$s_0=1-2\exp\left(-2e^{\hbar\omega/eEl}\right). \tag{7}$$

It is important that for high energies, on the order of ε_i, the realization of such a distribution function is based on the sensitive balance between the departure of electrons losing energy by phonon emission and the arrival of electrons having high energies or accelerated in the electric field. Such a balanced needle-like distribution differs in its concept from the simple Shockley picture.[3]

We shall show now that conditions (5) are necessary but not sufficient for the solution (6), and one more condition must be added. This is simplest to show in the following manner. We solve, according to Ref. 4, Eq. (3) by treating the right-hand side as an inhomogeneity:

$$f_{\mathbf{p}}=\int_0^\infty dt\,\frac{2^{1/2}}{l_{ph}m^{1/2}}\Big[(\varepsilon_{\mathbf{p}-e\mathbf{E}t}-\hbar\omega)^{1/2}\frac{e^{-\beta}}{2\,\mathrm{ch}\,\beta}f_0(\varepsilon_{\mathbf{p}-e\mathbf{E}t}-\hbar\omega)$$
$$+(\varepsilon_{\mathbf{p}-e\mathbf{E}t}+\hbar\omega)^{1/2}\frac{e^{\beta}}{2\,\mathrm{ch}\,\beta}f_0(\varepsilon_{\mathbf{p}-e\mathbf{E}t}+\hbar\omega)\Big]\exp\left(-\frac{2^{1/2}}{l_{ph}m^{1/2}}\right)$$
$$\times\int_0^t dt'\Big[(\varepsilon_{\mathbf{p}-e\mathbf{E}t'}-\hbar\omega)^{1/2}\frac{e^{\beta}}{2\,\mathrm{ch}\,\beta}+(\varepsilon_{\mathbf{p}-e\mathbf{E}t'}+\hbar\omega)^{1/2}\frac{e^{-\beta}}{2\,\mathrm{ch}\,\beta}\Big]. \tag{8}$$

It should be noted, *in passim*, that the solution can take the form (8) only for angles outside the range $0<\sin\vartheta<(\hbar\omega/\varepsilon_p)^{1/2}$, whereas within this range the solution has a different form. To arrive at the approximate expression (6) from the exact one (8) we need confine ourselves to the argument of the exponential to terms linear in t. The sum of these terms is $tl_{ph}{}^{-1}m^{1/2}(2\varepsilon)^{1/2}(1-s_0\cos\vartheta)$, and specifies the t interval in which the integral of (8) converges. Expanding the quadratic terms, we determine the conditions under which they are small compared with the accounted-for term (6). The sought inequality is rewritten in the form

$$\frac{\hbar\omega}{\varepsilon}\ll4\frac{\hbar\omega}{eEl}\exp\left(-2e^{\hbar\omega/eEl}\right). \tag{9}$$

The inequality (9) is quite stringent: the right hand-side of (9) is equal to $3\cdot10^{-6}$ even at $\hbar\omega/eEl=2$, i.e., for the limit of inequality (5).

Keldysh[4] considered also a combination of elastic scattering by optical phonons and elastic scattering by acoustic phonons. The latter is identical with scattering by pointlike impurities with a δ-function potential and will be designated as such hereafter. The left-hand side of (3) acquires accordingly an additional term $(2^{1/2}\varepsilon_p{}^{1/2}/l_{im}m^{1/2})(f_{\mathbf{p}}-f_0(\varepsilon_p))$. It is important to note that in the solution (6) the expression for the parameter s_0 is now different:

$$s_0=1-2\exp(-2)\left[\frac{l}{l_{im}}+\frac{l}{l_{ph}}e^{-\hbar\omega/eEl}\right]^{-1},$$
$$\frac{1}{l}=\frac{1}{l_{im}}+\frac{1}{l_{ph}}. \tag{10}$$

Its influence on the structure of the needle can be substantial, even for weak impurity scattering, under the condition

$$e^{-\hbar\omega/eEl}\ll\frac{l_{ph}}{l_{im}}\ll1. \tag{11}$$

The expression for the parameter s_0 is correspondingly simplified. The additional condition for the existence of the solution (6) takes in this case the form

$$\frac{eEl}{\varepsilon}\ll4\exp\left(-2\frac{l}{l_{im}}\right). \tag{12}$$

It can be seen that the presence of elastic scattering by pointlike impurities makes it easier to obtain a self-consistent needle-like distribution of the electrons at high energies.

2. We consider now some modification of the Keldysh model—let the deformation scattering by the optical phonons be accompanied by scattering from ionized impurites. To this end we must add to (3) a term that describes the elastic impurity scattering by the Coulomb potential. This is predominantly small-angle scattering, so that the expression can be simplified in the manner used by Landau to simplify the expression for electron-electron collisions[5]:

$$J(f_{\mathbf{p}})=-\frac{\partial}{\partial p_i}\left[\left(\frac{n_{im}e^4m^{1/2}}{16\pi\cdot2^{1/2}\varkappa^2\varepsilon_p{}^{1/2}}\ln\frac{2m\varepsilon_pr_D{}^2}{\hbar^2}\right)\times\left(\delta_{ik}-\frac{p_ip_k}{\mathbf{p}^2}\right)\frac{\partial f_{\mathbf{p}}}{\partial p_k}\right]. \tag{13}$$

Here n_{im} is the impurity density, $\varkappa$ the dielectric constant, and r_D the Debye radius. Expression (13), naturally conserves the energy, meaning also the modulus of the momentum, and is identical with

$$J(f_{\mathbf{p}})=-\left(\frac{n_{im}e^4}{32\pi\cdot2^{1/2}\varkappa^2m^{1/2}\varepsilon_p{}^{3/2}}\ln\frac{2m\varepsilon_pr_D{}^2}{\hbar^2}\right)\times\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f_{\mathbf{p}}}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2f_{\mathbf{p}}}{\partial\varphi^2}\right]. \tag{14}$$

ϑ and φ are angles in a spherical coordinate frame in momentum space. Choosing the z axis along $\mathbf{E}$, we obtain the dependence of $f_{\mathbf{p}}$ on φ.

We assume impurity scattering to be weak compared with phonon scattering:

$$\alpha=\frac{n_{im}e^4l_{ph}}{32\pi\varkappa^2\varepsilon_p{}^2}\ln\frac{2m\varepsilon_pr_D{}^2}{\hbar^2}\ll1. \tag{15}$$

The impurity-scattering term contains thus a small quantity α that is multiplied, however, by a derivative with respect to $\cos\vartheta$, the latter substantial at small angles. Transforming the function $f_0(\varepsilon_p)$:

$$f_0(\varepsilon)=\exp\left(-\int_{\hbar\omega}^{\varepsilon}\frac{s(\varepsilon')}{eEl_{ph}}d\varepsilon'\right) \tag{16}$$

and neglecting, as before, terms small in eEl_{ph}/ε, we arrive at a simplified equation for the distribution function:

$$f_{\mathbf{p}}=\chi_{\mathbf{p}}\exp\left(-\int_{\hbar\omega}^{\varepsilon}\frac{s(\varepsilon')}{eEl_{ph}}d\varepsilon'\right), \tag{17}$$

$$\alpha\frac{\partial\chi}{\partial x}+(1-xs(\varepsilon))\chi=\exp\left(-\frac{\hbar\omega}{eEl_{ph}}s(\varepsilon)\right),\qquad x=\cos\vartheta. \tag{18}$$

The definitions (16) and (17) are made self-consistent by the equality

$$\frac{1}{2}\int_{-1}^{1}\chi(\varepsilon,x)\,dx=1. \tag{19}$$

The electric-field interval of interest to us is specified by inequalities (5). It can be shown that the structure of the needle-like distribution function is determined by weak impurity scattering, inasmuch as at small angles its role increases. The function $s(\varepsilon)$ becomes then larger than unity:

$$s(\varepsilon)=\left[1-\left(\alpha\frac{\hbar\omega}{eEl_{ph}}\right)^{1/2}\right]^{-1}. \tag{20}$$

Of course, no such self-consistent solution is obtained for arbitrarily weak impurity scattering, since for $\alpha = 0$ it must go over into the solution (6). The solution is valid when α, albeit small, still exceeds a certain value. Comparison of the discarded derivative with respect to x, which contains the small parameter eEl_{ph}/ε, with the accounted-for derivative yields

$$\alpha\gg\frac{(eEl_{ph})(4\hbar\omega)^{1/3}}{\varepsilon^{4/3}[1-(\alpha\hbar\omega/eEl_{ph})^{1/2}]^{2/3}}. \tag{21}$$

When weak scattering by ionized impurities is added to the phonon scattering, this inequality is the only condition supplementing (5). The condition (9) is not required in this case.

3. We consider now the Keldysh model in the case when conditions (5) are met, but the condition (9) or (12) is not. As shown by expression (8), we cannot confine ourselves now to consideration of high energy, and must know the nature of the distribution function at energies of order $\hbar\omega$ and lower. For dispersion-free phonons of energy lower than $\hbar\omega$, electron-phonon collisions do not lead at all to relaxation—a region called passive is formed. The Keldysh model is in a certain sense not closed. The simplest situation, at first glance, is one in which there is not relaxation at all in the passive region. It was considered by Dmitriev and Tsendin.[6] Under these conditions, a needle-like distribution function is formed even in the passive region in view of the acceleration of the electron by the electric field and of the absence of relaxation. According to Ref. 6, the needle-like distribution extends also to the high-energy region, leading to the Townsend-Shockley ionization law.

We consider now the opposite situation: we assume that effective energy and momentum relaxation mechanisms are active at low energies, so that the distribution function has a rather smooth form, say,

$$f_{\mathbf{p}}=\frac{n\hbar^3 2^{1/2}\pi^{3/2}}{m^{3/2}T_e^{3/2}}\exp\left(-\frac{\varepsilon_p}{T_e}\right), \tag{22}$$

where n is the electron density, and the effective temperature T_e is taken to be of the order of the equilibrium temperature. The anisotropic part, if present in the distribution function, is assumed small and is left out. A Boltzmann distribution function was assumed to simplify the calculations, but the main results that follow do not depend on the actual form of the function, and T_e yields simply the average energy scale. Expressions (8) and (22) lead at high energies to the following needle-like expression for the distribution function:

$$f_{\mathbf{p}}=\frac{n\hbar^3 2^{3/2}\pi^2}{m^{3/2}T_e^{1/2}eE}\left\{\frac{\exp[-\hbar\omega(1/T-1/T_e)]}{l_{ph}}(\varepsilon_p\sin^2\vartheta-\hbar\omega)^{1/2}\right.$$
$$\left.+\frac{\exp[(-\hbar\omega/T_e)]}{l_{ph}}(\varepsilon_p\sin^2\vartheta+\hbar\omega)^{1/2}+\frac{\varepsilon_p^{1/2}\sin\vartheta}{l_{im}}\right\}$$
$$\times\exp\left(-\frac{\varepsilon_p}{T_e}\sin^2\vartheta-\frac{\varepsilon_p}{eEl}\cos\vartheta\right). \tag{23}$$

This distribution is indeed needle-shaped if $T_e \ll eEl$. As already noted, expression (8) is valid only for angles for which $\sin\vartheta > (\hbar\omega/\varepsilon_p)^{1/2}$. For smaller angles, the distribution is given by the ionic expression

$$f_{\mathbf{p}}=\frac{n\hbar^3(2\pi^3)^{1/2}}{m^{3/2}T_e^{1/2}eE}\exp\left(-\frac{\varepsilon_p}{eEl}\cos\vartheta\right)\left\{\left[\frac{\exp(-\hbar\omega/T)}{l_{ph}}\right.\right.$$
$$\left.+\frac{\exp(-2\hbar\omega/T_e)}{l_{ph}}\left(\frac{2\pi\hbar\omega}{T_e}\right)^{1/2}+\frac{\exp(-\hbar\omega/T_e)}{l_{im}}\left(\frac{\pi\hbar\omega}{T_e}\right)^{1/2}\right]$$
$$+\frac{\exp[(-\varepsilon_p/T_e)\sin^2\vartheta]}{T_e}\int\limits_{(\hbar\omega-\varepsilon_p\sin^2\vartheta)}^{\varepsilon_p\cos\vartheta}\left\{\frac{\exp[-\hbar\omega(1/T-1/T_e)]}{l_{ph}}\right.$$
$$\times(\varepsilon'+\varepsilon_p\sin^2\vartheta-\hbar\omega)^{1/2}+\frac{\exp(-\hbar\omega/T_e)}{l_{ph}}(\varepsilon'+\varepsilon_p\sin^2\vartheta+\hbar\omega)^{1/2}$$
$$\left.\left.+\frac{(\varepsilon'+\varepsilon_p\sin^2\vartheta)^{1/2}}{l_{im}}\right\}(\varepsilon')^{-1/2}d\varepsilon'. \tag{24}$$

We present also an expression for the averaged distribution function:

$$f_0(\varepsilon_p)=\frac{n\hbar^3 2^{-3/2}\pi^{3/2}}{(mT_e)^{3/2}eE\varepsilon_p}\exp\left(-\frac{\varepsilon_p}{eEl}\right)\left\{\frac{\exp(-\hbar\omega/T)}{l_{ph}}[\hbar\omega T_e\right.$$
$$+T_e^{3/2}(\pi\hbar\omega)^{1/2}+T_e^2\pi]+\frac{2^{1/2}\exp(-2\hbar\omega/T_e)}{l_{ph}}[(\hbar\omega)^{3/2}(\pi T_e)^{1/2}$$
$$+2\hbar\omega T_e+2(\pi\hbar\omega)^{1/2}T_e^{3/2}]+\frac{\exp(-\hbar\omega/T_e)}{l_{im}}[(\hbar\omega)^{3/2}(\pi T_e)^{1/2}$$
$$\left.+2\hbar\omega T_e+2(\pi\hbar\omega)^{1/2}T_e^{3/2}]\right\}. \tag{25}$$

The distribution function (23)–(25) is the mathematical realization of the Shockley picture. The power of this needle-like distribution function is determined by the values of $f_{\mathbf{p}}$ at low energies, and differs thereby from the balanced Keldysh distribution.

The pre-exponential factor for the balanced Keldysh distribution was also calculated.[4] This factor can be determined by taking into account both the correction term of the expansion in $s(\varepsilon)$ and the correction to (6) in terms of the parameter eEl/ε. In this case $A_0(\varepsilon_p)$ turns out to be a power function of the energy:

$$A_0(\varepsilon_p)=(2\pi)^{-3}n\left(\frac{\hbar}{m\omega}\right)^{3/2}(4e)^{\hbar\omega/2eEl_{ph}}\left(\frac{\varepsilon}{\hbar\omega}\right)^{\hbar\omega/2eEl_{ph}}. \tag{26}$$

We see that the total power of an electron beam drawn out by the strong electric field from the quasiequilibrium distribution function is small compared to the power of the balanced beam that would correspond to the same value of the electric field if such a beam could be formed.

4. In ionic semiconductors, the interaction of electrons with optical phonons is of the polarization type, and the kinetic equation is of the form

$$e\mathbf{E}\frac{\partial f_{\mathbf{p}}}{\partial\mathbf{p}}+\frac{2\pi}{\hbar}\sum_{\mathbf{q}}\frac{|C_q|^2}{2\,\mathrm{ch}\,\beta}\{[f_{\mathbf{p}}e^{\beta}-f_{\mathbf{p}-\hbar\mathbf{q}}e^{-\beta}]\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\hbar\mathbf{q}}-\hbar\omega),$$
$$+[f_{\mathbf{p}}e^{-\beta}-f_{\mathbf{p}+\hbar\mathbf{q}}e^{\beta}]\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}+\hbar\mathbf{q}}+\hbar\omega)\}=0, \tag{27}$$
$$|C_q|^2=\frac{2\pi e^2\hbar\omega}{V\mathbf{q}^2}\left(\frac{1}{\varkappa_\infty}-\frac{1}{\varkappa_0}\right),$$

where V is the total volume, and $\varkappa_0$ and $\varkappa_\infty$ are the dielectric constants at zero and infinite frequencies, respectively. The mean free path in such an interaction is not constant but depends on the electron energy:

$$\frac{1}{l_{ph}(\varepsilon_p)}=\frac{me^2\omega(1/\varkappa_\infty-1/\varkappa_0)}{\hbar\varepsilon_p}\ln\frac{4\varepsilon_p}{\hbar\omega}. \qquad (28)$$

To solve this equation, Chuenkov[7] attempted to generalize the Keldysh method and reached the conclusion that the conditions (5) are sufficient to find a distribution of type (6) that leads to the Townsend-Shockley ionization law. This conclusion, however, is based on an unjustified, in our opinion, modification of the collision integral. We shall show in Appendix 1 that it is impossible to find for polarized electron-phonon interaction a self-consistent needle-shaped distribution function of type (6). We arrive by the same token at a situation similar to that when the additional condition (9) is not met for the deformation interaction; just as in the latter case, to determine the distribution function at high energies we must know the distribution function at low energies. The latter was obtained in the already cited paper[16] for the case when there is no relaxation at all in the passive region. If, on the contrary, the relaxation at low energies is substantial and leads to a quasiequilibrium distribution function, we obtain at high energies a distribution function of type (23); some modifications of the latter are due to the energy dependence of the mean free path.

If scattering by pointlike impurities is present in addition to the electron-phonon scattering, a balanced needle-shaped distribution can be obtained. We assume that the scattering by the impurities is weak, so that its contribution is significant not so much for the determination of the mean free path as for the formation of the needle-shaped distribution structure. Since, however, the polarization interaction itself scatters predominantly into small angles, the conditions for such a treatment of the impurity scattering of electrons differ from (11) and take the form

$$1\ll\frac{l_{im}}{l(\varepsilon)}\ll\frac{\hbar\omega}{eEl(\omega)}. \qquad (29)$$

When conditions (5), (29), and of course also the supplementary condition (12), are met [all the expressions contain $l(\varepsilon)$], the self-consistent distribution (6) that leads to relation (2) is realized.

5. At low temperatures, energy and momentum scattering by acoustic phonons can become stronger than scattering by optical phonons. It is necessary to include in the kinetic equation (27) the constant $|C_q|^2=\Lambda^2\hbar\mathbf{q}^2/2\rho\omega_q V$ (Λ is the deformation-potential constant and ρ is the density) of the deformation electron-phonon interaction, and take into account the dependence of the frequency ω_q of the acoustic phonon on the wave vector $\mathbf{q}$, viz., $\omega_q=wq$, where w is the speed of sound. This calls for modification of the conditions (4) and (5), and it is readily understood that the role of the phonon energy $\hbar\omega$ should be assumed under these conditions by the energy $2w(2m\varepsilon)^{1/2}$. We regard this substitution as made in the text that follows.

We consider temperatures that are not infralow, i.e., $T\gg mw^2$. All the experimental situations must then be divided into two classes, depending on the relation between T and $2w(2m\varepsilon)^{1/2}$. In a situation of the first class ($T\gg 2w(2m\varepsilon)^{1/2}$), for electrons interacting with phonons of arbitrary wave vectors, the number of phonons is large and they have a Rayleigh distribution function. The electron mean free path is therefore independent of energy. If the energy acquired by an electron over the mean free path in situations of the first class exceeds the thermal energy, it exceeds all the more the inelasticity energy for the emission of a phonon of any possible frequency. It is thus possible to meet the conditions for diffusion scattering of the energy, but not the condition (5). In diffusion scattering of the energy the distribution function is the known Davydov function,[1] which yields Eq. (1) for the ionization coefficient.

In situations of the second class ($T\ll 2w(2m\varepsilon)^{1/2}$) the electron mean free path is determined by spontaneous phonon emission and is therefore energy-dependent:

$$\frac{1}{l_{ph}(\varepsilon_p)}=\frac{2^{1/2}\Lambda^2m^{5/2}\varepsilon_p^{1/2}}{3\pi\rho w\hbar^4}. \qquad (30)$$

This was noted in Chuenkov's article.[8] It is also stated there, however, that in diffusion energy scattering the same length should be contained in the definition of the heating function. As a result, the expression for the distribution function differs from that of Davydov.

It is stated in the same article that when conditions (5) are met it is poissible to find a solution of type (6). It is shown in Appendix 2, however, that no self-consistent needle-shaped solutions can be found for energy and momentum scattering by acoustic phonons.

We verify now that a needle-shaped function of type (23) cannot be obtained for such an electron-phonon interaction. From conditions (5) it can be determined that the electrons are strongly heated at low energies. These electrons have a Davydov distribution function with an average energy proportional to the electric field and much higher than the temperature. It is easily shown that this energy also exceeds $eEl_{ph}(\varepsilon)$, so that a function which is of this type at low energies cannot change into the function (23) which is needle-shaped at high energies. Thus, in experimental situations of either class, the thermal dependence of the "tail" of the distribution function changes immediately into a dependence such as (1) in an electric field.

All the foregoing is valid only for pure phonon scattering. Elastic scattering by impurities can lead to formation of the balanced needle-shaped distribution

$$f_p=\frac{l(\varepsilon)}{l_{im}}\frac{1}{1-s(\varepsilon)\cos\vartheta}\exp\left(-\int_0^\varepsilon\frac{s(\varepsilon')}{eEl(\varepsilon')}d\varepsilon'\right). \qquad (31)$$

When conditions (5) are met we obtain for $s(\varepsilon)$ a solution similar to (10):

$$s(\varepsilon)=1-2\exp[-2l(\varepsilon)/l_{im}]. \qquad (32)$$

We have assumed the impurity scattering to be weak, but for the needle structure it is more substantial than the phonon scattering if

$$1\gg\frac{l_{ph}(\varepsilon)}{l_{im}}\gg\frac{[eEl(\varepsilon)]^3}{(8mw^2\varepsilon)^{3/2}}. \qquad (33)$$

Besides the conditions indicated, this needle-shaped distribution is produced, naturally, only if the additional condition (12) with an energy-dependent mean free path is met.

6. We show now that when elastic scattering prevails, $l_{im}\ll l_{ph}$, the Townsend-Shockley ionization condition can be met even though the distribution is not needle-shaped but is weakly anisotropic. Fro the case of energy scattering by optical phonons this is shown in Gribnikov's article.[9] For energy scattering by acoustic phonons this statement was made earlier by Chuenkov,[8] but we show in Appendix 2 that its mathematical derivation is not quite correct.

We consider an arbitrary impurity-scattering potential and an energy relaxation produced by any interaction of electrons with optical, acoustic, or piezoacoustic phonons. The part of the distribution function that is antisymmetric in the momentum is expressed in the usual fashion in terms of $f_0(\varepsilon_p)$ (Refs. 1 and 2):

$$f_p^- = -eEl(\varepsilon_p)\cos\vartheta\frac{df_0(\varepsilon_p)}{d\varepsilon_p}. \quad (34)$$

Averaging the kinetic equation over a constant-energy surface, we arrive at an equation for $f_0(\varepsilon_p)$:

$$-\frac{1}{\varepsilon^{1/2}}\frac{d}{d\varepsilon}\left(\frac{2^{1/2}e^2E^2l(\varepsilon)\varepsilon}{3m^{1/2}}\frac{df_0(\varepsilon)}{d\varepsilon}\right)+\frac{(2\varepsilon)^{1/2}}{l_{ph}(\varepsilon)m^{1/2}}f_0(\varepsilon) - \int_0^{(8m\varepsilon)^{1/2}/\hbar}\frac{dq}{2\pi}\frac{q|C_q|^2Vm^{1/2}}{\hbar^2(2\varepsilon)^{1/2}}f_0(\varepsilon+\hbar\omega_q)=0. \quad (35)$$

This equation shows that $f_0(\varepsilon_p)$ depends on the mean free path $l^{1/2}(\varepsilon_p)l_{ph}^{1/2}(\varepsilon_p)$ and that this path is contained in the conditions for the applicability of both the strongly elastic and of the diffuse cases of energy relaxation. Diffuse energy scattering corresponds to expansion of the electron-phonon collision operator in terms of the phonon energy. The conditions (5) change into

$$T \ll eEl^{1/2}(\varepsilon_i)l_{ph}^{1/2}(\varepsilon_i) \ll \left\{\begin{matrix}\hbar\omega \\ 2w(2m\varepsilon_i)^{1/2}\end{matrix}\right\} \ll \varepsilon_i. \quad (36)$$

When these conditions are met, the arrival term of the collision operator—the last term of (35)—is small. For optical phonons it is small exponentially, $\exp(-\hbar\omega/eEl^{1/2}(\varepsilon)l_{ph}^{1/2}(\varepsilon))$, (Ref. 9), and for acoustic phonons only in power-law fashion, $(e^2E^2l(\varepsilon)l_{ph}(\varepsilon)/8mw^2\varepsilon)^{3/2}$. The solution of (35) is then

$$f_0(\varepsilon)=A\exp\left(-\int^{\varepsilon}\frac{3^{1/2}}{eEl^{1/2}(\varepsilon')l_{ph}^{1/2}(\varepsilon')}d\varepsilon'\right). \quad (37)$$

Substituting (37) in (34) we see that f_p^- is $(l_{im}/l_{ph})^{1/2}$ times smaller than $f_0(\varepsilon_p)$. Note that the factor $3^{1/2}$ in (37) does not permit a direct "conversion" of the weakly anisotropic equation (37) into Eq. (2) that follows from the needle-shaped distribution.

Equation (35) has a general character analogous to that of the heating equation in the theory of "hot" electrons, and can, in analogy with the latter, be generalized to the case of anisotropic scattering; it is convenient to use for this purpose the method developed by Gurevich and Katilyus.[10]

7. The ionization coefficent η is determined by integrating the ionization probability $\sigma(\varepsilon)$, the density of states, and the distribution function $f_0(\varepsilon)$ over all energies exceeding the ionization energy

$$\eta=\frac{2^{1/2}m^{3/2}}{nv_d\pi^2\hbar^3}\int_{\varepsilon_i}^{\infty}\sigma(\varepsilon)f_0(\varepsilon)\varepsilon^{1/2}d\varepsilon, \quad (38)$$

where v_d is the electron drift velocity. The energy dependence of the probability near the threshold is a power-law function of the excess of the energy above the threshold, $\sigma(\varepsilon)=\sigma_0[(\varepsilon-\varepsilon_i)/\varepsilon_i]^r$, where r is an integer.[4] Since the probability varies over energy scales of the order of ε_i, and the distribution function decreases rapidly over much smaller energy scales, this expression can be used at all energies.

The ionization coefficient for a self-consistent needle-shaped distribution function was calculated by Keldysh.[4] We present an expression for the distribution function (25):

$$\eta=\frac{\sigma_0\Gamma(r+1)\hbar\omega}{2\pi^{1/2}v_dT_e^{1/2}\varepsilon_i^{1/2}}\left(\frac{eEl}{\varepsilon_i}\right)^2\left[\frac{l}{l_{ph}}\exp\left(-\frac{\hbar\omega}{T}\right)+\frac{l}{l_{ph}}\left(\frac{2\pi\hbar\omega}{T_e}\right)^{1/2}\exp\left(-2\frac{\hbar\omega}{T_e}\right)+\frac{l}{l_{im}}\left(\frac{\pi\hbar\omega}{T_e}\right)^{1/2}\exp\left(-\frac{\hbar\omega}{T_e}\right)\right]\times\exp\left(-\frac{\varepsilon_i}{eEl}\right) \quad (39)$$

The ionization coefficient is thus governed by the Townsend-Shockley dependence in three cases: 1) when a needle-shaped distribution function is formed, 2) when a strong field "draws out" the needle from the region of average energies, and 3) in a superposition of strong elastic scattering of a pulse and of essentially inelastic scattering of the energy. In the first case the formation of a needle-shaped distribution is possible only if the additional conditions (9) and (12) or (13) are met. In the second case a needle can be formed only for definite functions in the low-energy region. The features of the third case are determined by Eq. (37). The determination of the conditions under which each of these cases is realized was in fact the aim of the present article.

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APPENDIX 1

The collision integral for electrons with optical phonons is transformed into ($x=\cos\vartheta$):

$$J(f_p)=\frac{(2\varepsilon_p)^{1/2}}{l_{ph}(\varepsilon_p)m^{1/2}}[f(\varepsilon_p,x)-f(\varepsilon_p+\hbar\omega,x)] + \frac{(2\varepsilon_p)^{1/2}}{2l_{ph}(\varepsilon_p)m^{1/2}\ln(4\varepsilon_p/\hbar\omega)} \times\int_{-1}^{1}\frac{dx'}{|x-x'|}[f(\varepsilon_p+\hbar\omega,x)-f(\varepsilon_p+\hbar\omega,x')].$$

If the solution is sought in the form (17), disregarding terms small in eEl_{ph}/ε_p, we get

$$(1-\exp(-s\hbar\omega/eEl_{ph})-sx)\chi(x) \quad (A1)$$

$$=\frac{\exp(-s\hbar\omega/eEl_{ph})}{2\ln(4\varepsilon/\hbar\omega)}\int_{-1}^{1}\frac{dx'}{|x-x'|}[\chi(x')-\chi(x)]. \quad (A2)$$

This equation can be expressed an an expansion of $\chi(x)$ in Legendre polynomials:

$$\chi(x)=\sum_{n=0}^{\infty}\chi_nP_n(x),\quad [1-\exp(-s\hbar\omega/eEl_{ph})-sx]\chi(x) =-\frac{\exp(-s\hbar\omega/eEl_{ph})}{\ln(4\varepsilon/\hbar\omega)}\sum_{n=1}^{\infty}b_n\chi_nP_n(x), \quad (A3)$$

$$b_n=\frac{1}{2}\int_{-1}^{1}\frac{(1-P_n(x'))}{(1-x')}dx'. \quad (A4)$$

Chuenkov[7] proposed to approximate the coefficients b_n by a constant b. This approximation yields a solution of (A3) in the form (6). The coefficients b_n can be calculated exactly

$$b_n=\sum_{k=1}^{n}\frac{1}{k}$$

and it can be seen that they increase without limit, albeit slowly, with the number m. Therefore the approximation of Ref. 7, in our opinion, does not agree with the problem and cannot serve as the basis for any conclusions concerning the solution of Eq. (A3). Moreover, one can verify directly from the form of (A2) that a solution of the form $c/(a-sx)$ does not satisfy this equation. For not very small s, i.e., such that $s\hbar\omega/eEl_{ph}$ is not small, the exponential in the right-hand side of (A2) is small, and for a solution to exist it is necessary that $\chi(x)$ increase steeply at $x\sim 1$. The form of the integral in the right-hand side of the equation, however, cannot make such a solution self-consistent. This means that $s\approx 1$ is not consistent with those eigenvalues of this homogeneous equation for which a nontrivial homogeneous solution exist. We were similarly unable to find for the kinetic equation (27) a solution that leads to Eq. (2)

APPENDIX 2

For electron scattering by acoustic phonons, we seek the solution of the kinetic equation in the form[17]

$$(1-sx)\chi(x)=\frac{l_{ph}\Lambda^2m^{1/2}}{2(2\pi)^2(2\varepsilon_p)^{1/2}\rho w}\int d^3q\,q\exp(-s\hbar wq/eEl_{ph})\times\chi_{\mathbf{p}+\hbar\mathbf{q}}\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}+\hbar\mathbf{q}}+\hbar wq). \quad (A5)$$

The dependence of the phonon frequency on the wave vector assigns an important role in the arrival integral term [the right-hand side of Eq. (A5)] to the small wave vectors

$$\hbar q_0=\frac{eEl_{ph}}{ws}\ll(8m\varepsilon_p)^{1/2}.$$

This relation was not taken into account in Chuenkov's article.[8]

We rewrite (A5) in simpler form

$$(1-sx)\chi(x)=\int_0^1 dy\,y^2e^{-\xi sy}\int_{-\pi}^{\pi}\frac{d\varphi}{2\pi}\chi(x-2xy+2y(1-y^2)^{1/2}(1-x^2)^{1/2}\cos\varphi),$$

$$\xi=(8mw^2\varepsilon)^{1/2}/eEl_{ph}. \quad (A6)$$

For values of s that are not anomalously small, i.e., such that ξs is large, the right-hand side contains a small factor $(\xi s)^{-3}$. A solution exists if the increase of the function $\chi(x)$ at $x\sim 1$ can compensate for this smallness. It can be verified, however, that a function of the form $c/(a-sx)$, which was used in Ref. 8, does not satisfy (A6). From the mean-value theorem it can be concluded that $\chi(x)$ does not increase at all at $x\sim 1$. This means that $s\approx 1$ does not correspond to those eigenvalues at which a nonzero solution of (A6) is possible.

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