Effective-medium approximation in localization theory

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An effective method is proposed for the description of the Anderson metal-insulator transition in a disordered d-dimensional system. The method becomes exact in the limit of large d. The density-density correlator is calculated. In the dielectric region this correlator decreases exponentially at large distances, with the localization length in the critical region inversely proportional to the square root of the proximity to the transition point. The correlator has a power-law decrease at the transition point. In the metallic region, the density-density correlation has a diffuse form with a diffusion coefficient that decreases exponentially as the transition point is approached.

1. INTRODUCTION

A useful model for the investigation of Anderson localization and for metal-insulator transitions is the granulated disordered metal model. The use of the supersymmetry model[1] permits the problem to be reduced to a study of the supermatrix σ model for a lattice. Notwithstanding the analogy with spin models, investigation of the metal-insulator transition within the framework of the supermatrix σ model is not simple. This is primarily due to the inapplicability of the mean-field approximation that explains the main features of the transition in ordinary spin systems. The supermatrix σ model was recently[2–4] investigated for a Bethe lattice, where an exact solution was successfully obtained. It was shown with the aid of explicit calculations that a metal-insulator Anderson transition exists, and the correlation functions were obtained. Although the results of the calculations were physically quite reasonable, their validity for real d-dimensional lattices still remains questionable.

We propose below an approximate method that permits a description of the properties of the considered σ model that corresponds to the model of a d-dimensional granulated metal. The method consists of singling out two sites and calculating the interaction between them exactly, while the interaction with the remaining sites is replaced by interaction with an effective medium. Self-consistency conditions are specified for this medium. An analysis is carried out of a perturbation theory that permits separation of a class of diagrams whose sum yields in fact the described approximation. The approximation considered becomes exact in the limit of high dimensionality of the space. The density-density correlator is calculated within the framework of this approximation. It is shown than an Anderson metal-insulator transition exists. The diffusion coefficient in the metallic region decreases exponentially as the transition is approached. The density correlator in the dielectric region decreases exponentially with distance. The localization length is inversely proportional to the square root of the distance to the transition point. The fundamental integral equation that describes the character of the transition coincides with the corresponding equation obtained for the model with a Bethe lattice. At the same time, the density-density correlators have different forms for d-dimensional space and for the Bethe lattice.

2. PERTURBATION THEORY. NEED FOR PARTIAL SUMMATION

The kinetics of a system of metal granules is described by the supermatrix σ model for the lattice. The effective Lagrangian in this model is of the form[1–3]

$$F=-\gamma_0\Big[\sum_{i,j}J_{ij}\,\mathrm{STr}\,Q_iQ_j-\frac{i}{4}(\omega+i\delta)\pi\nu V\sum_i\mathrm{STr}\,\Lambda Q_i\Big]. \tag{1}$$

The first term in (1) describes the interaction of the granules, the second term is the Lagrangian of the isolated granules, ω and ν denote the frequency and density of states in the granules, Str is a supertrace, V is the volume of one granule, and the parameter γ_0 takes on values 2 or 1, depending on whether scattering with spin flip is taken into account or not. The supermatrices Q and Λ in (1) have a dimensionality 8×8 and are equal to

$$Q=UQ_0\bar{U},\quad Q_0=\begin{pmatrix}\cos\hat{\theta} & i\sin\hat{\theta}\\ -i\sin\hat{\theta} & -\cos\hat{\theta}\end{pmatrix},$$

$$U=\begin{pmatrix}u & 0\\ 0 & v\end{pmatrix},\quad \Lambda=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad u\bar{u}=1,\quad v\bar{v}=1. \tag{2}$$

The structure of the matrices u, v, and $\hat{\theta}$ depends on the presence of symmetry with respect to time reversal and of central symmetry. A superior bar denotes the charge conjugate of a matrix. All the necessary equations can be found in the review by Efetov.[1]

The calculation of the density-density correlator $K(\mathbf{r}_1,\mathbf{r}_2)$ reduces in the model considered to calculation of the correlation function

$$K(\mathbf{r}_1,\mathbf{r}_2)=-2\pi^2\nu^2\gamma_0\int(Q_{13}^{12})_{r_1}(Q_{31}^{21})_{r_2}\exp(-F[Q])\prod_i dQ_i, \tag{3}$$

where $F[Q]$ is defined in Eq. (1). The superscripts in (3) number the blocks explicitly set apart in (2), and the subscripts number the elements in these blocks.

The model described by expressions (1)–(3) is valid for any type of lattice. The most widespread method of investi-

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gating a σ model of type (1)–(3) is perturbation theory in the limit of large J_{ij}, a theory corresponding to low-temperature expansions in spin models. One method of obtaining the perturbation-theory series is parametrization of the matrix Q, say with the aid of the following expression:

$$Q=\Lambda(1+iH)(1-iH)^{-1}, \quad H=\begin{pmatrix} 0 & B \\ \bar{B} & 0 \end{pmatrix}. \tag{4}$$

Next, expanding in expressions (1) and (3) in powers of H (retaining in the exponential the terms quadratic in H), and calculating the Gaussian integrals, we can find all terms of the series. Such a procedure is used in fact in the renormalization-group method in a space of dimensionality $2+\varepsilon$ (Refs. 5, 6 and 1). Such an expansion, however, leads to errors due to replacing by infinity all the limits of integration over the elements of matrix B in (4). A more correct expansion method is proposed in Ref. 7. This method was used in Ref. 8 to investigate the model defined by (1)–(3). The key point was the separation of the mean value $\langle Q \rangle = \Lambda$ and its expansion in powers of the deviations from the mean. We rewrite accordingly F [Eq. (1)] in the form

$$F=-2\tilde{J}\sum_i \mathrm{STr}\,\Lambda Q_i+F_1,$$

$$F_1=-\sum_{i,j} J_{ij}\,\mathrm{STr}(Q_i-\Lambda)(Q_j-\Lambda),$$

$$J=J-i(\omega+i\delta)/8, \quad J=\sum_j J_{ij}. \tag{5}$$

Regarding the second term of (5) as small and expanding with respect to it in the integral (3), we can write down the perturbation-theory series. The first-order terms were written out explicitly in Ref. 8. (Owing to the omission of the contribution of one of the diagrams, it is incorrectly stated in Ref. 8 that the calculation results contradict the prediction of the one-parameter renormalization group.) The next terms of the series turned out to be small compared with the preceding ones in the limit of large J_{ij} or of the large interaction radius.

Notwithstanding the decrease of the first terms of the perturbation-theory with increase of order, starting with a certain number that depends on the value of J_{ij}, the perturbation-theory series terms begin to increase. The reason for this increase is the non-compactness of the group of matrices Q, and no such increases occur in compact σ models. The importance of the non-compactness was discussed already in Refs. 2 and 3. An increase of the terms of the perturbation-theory series is observed in the calculation of any of the mean values of the product of several matrices Q. To estimate the magnitude of the terms of the perturbation theory series we choose any two neighboring sites 1 and 2 and calculate the integral

$$I=\int \mathrm{STr}[(Q_1-\Lambda)k\Lambda]\mathrm{STr}[(Q_2-\Lambda)k\Lambda] \times\exp(-F[Q])\prod_i dQ_i. \tag{6}$$

A definition of the matrix k can be found in Refs. 1, 2, and 3.

Expanding in terms of F_1 in (6), we can write down all the terms of the series. It is convenient to represent each J_{ij} in the graphs by a thin line joining the sites i and j. Let us estimate the contribution of graph a in Fig. 1. This graph consists of n lines joining the sites i and j, and corresponds to the expression

$$I_n=\frac{(2J_{12})^n}{n!}\int \mathrm{STr}[(Q_1-\Lambda)k\Lambda]\mathrm{STr}[(Q_2-\Lambda)k\Lambda]\{\mathrm{STr}[(Q_1-\Lambda)(Q_2-\Lambda)]\}^n \exp[2\mathcal{J}(\mathrm{STr}\,\Lambda Q_1+\mathrm{STr}\,\Lambda Q_2)]\,dQ_1\,dQ_2. \tag{7}$$

The calculation in (7) must be separately continued for an orthogonal, unitary, and symplectic ensemble. The growth mechanism of the perturbation-theory series terms is the same for all three ensembles; we confine ourselves therefore only to the unitary one. Integrating over the matrices u and v [Eq. (2)] (the explicit form of these matrices and the definition of θ_1 can be found in Refs. 1–3 and separating only the largest significant region $\theta_1 \gg 1$, we get

$$L_n \approx \frac{(4J_{12})^n(-1)^n \mathrm{ch}^2 8\mathcal{J}}{(n-1)!}\left[\int_0^\infty \exp(n\theta_1-8\mathcal{J}\,\mathrm{ch}\,\theta_1)\,d\theta_1\right]^2. \tag{8}$$

The region $\theta_1 \gg 1$ makes the main contribution to L_n at large $n \gg \tilde{J}$. In this limit, the integral in (8) can be calculated by the saddle-point method.

The saddle-point value is

$$\theta_{1s} \approx \ln(n/4\mathcal{J}). \tag{9}$$

Integrating in (8) near the saddle-point (9) we get

$$L_n=(-1)^n(n-1)!(J_{12}/\mathcal{J})^n(4\mathcal{J})^{-2n}\mathrm{ch}^2 8\mathcal{J}. \tag{10}$$

The value of $|L_n|$ increases with increase of n if n is large enough. An essential condition of this growth is noncompactness of the model, as a result of which a nontrivial saddle-point exists at large n. Calculation by expanding (4) in terms of H corresponds to calculation near $\hat{\theta}=0$, which does not allow account to be taken of the saddle point (9).

A similar calculation can also be carried out for models that are invariant to time reversal. The perturbation-theory series terms increase also in calculations of other correlators, different from I [Eq. (6)]. Of course, besides the diagram 1*a*, there exist also other graphs of the same order in $\tilde{J}^{-1}$. It will be shown below that at high dimensionality d of space these graphs have additional smallness in terms of d^{-1}. It is difficult to assume, however, that they can cancel out the obtained increasing term also at $d \sim 1$, since their contribution should depend on the lattice geometry. The contribution of graph a of Fig. 1, however, does not depend on the geometry.

The difficulty with the divergence of the perturbation-theory series can be circumvented by partial summation over repeated lines and by converting to graphs that contain

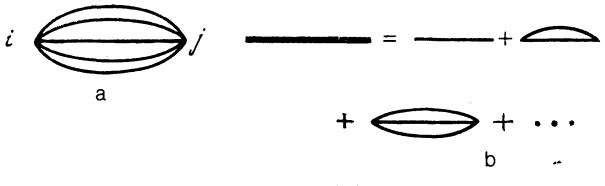

FIG. 1.

only effective lines. Denoting the effective line by a thick bar, we have for it the equality illustrated by graph *b* of Fig. 1. Carrying out such a summation in all the perturbation-theory graphs, we can change to a new expansion whose graphs contain only effective lines. An analogous partial summation is also the starting point of the calculation, from convenience considerations, of the partition function in the Ising model.[9] In the localization problem, however, this partial summation is essential.

The need for summing the sequence of graphs *b* of Fig. 1 can be arrived at also with the aid of a "high-temperature" expansion, in which the second term of (1) is the principal one, and first is the perturbation. In this case the singled-out sequence of diagrams diverges most as $\omega \to 0$, since the nth order contains the quantity $n!\omega^{-2n}$.

Of course, partial summation does not solve the problem exactly, and some approximations are necessary. We propose in the next section an approximation that describes the Anderson transition and becomes exact in the limit of high dimensionality of space.

3. PRINCIPAL APPROXIMATION

Carrying out the partial summation described in the preceding section, we can represent the function $-K(\mathbf{r}_1,\mathbf{r}_2)/2\pi^2\nu^2$ in the form of a sum of graphs, each of which contains only thick lines. Typical graphs are shown in Fig. 2. To calculate the contribution of any particular graph it is necessary to set each line with ends at the sites i and j in correspondence with the expression $\exp(2J_{ij}\mathrm{Str}Q_iQ_j)$. Each black circle at site i corresponds to $\exp[\frac{1}{4}i(\omega+i\delta)\pi\nu V\,\mathrm{Str}\,\Lambda Q_i]$. The left-hand light circle differs from the dark one by an additional factor $(Q^{12}_{13})_i$, and the right one by a factor $-(Q^{21}_{31})_j$. After writing down the corresponding expression we must integrate over Q in each site.

The graphs of Fig. 2 correspond to the case of a simple cubic d-dimensional lattice with nearest-neighbor interaction; this is the case studied below. The length of each segment on any graph is equal to edge length of the elementary cube, and any two sites can be connected by not more than one such segment.

The graphs obtained in this manner can be classified in accordance with the number of the closed loops. For example, graph 2*a* contains no loops, while graph 2*b* contains one loop. Denoting by $K_n(\mathbf{r}_1,\mathbf{r}_2)$ the sum of all the n-loop graphs, we represent the exact density-density correlator $K(\mathbf{r}_1,\mathbf{r}_2)$ in the form

$$K(\mathbf{r}_1,\mathbf{r}_2)=\sum_{n=0}^{\infty}K_n(\mathbf{r}_1,\mathbf{r}_2). \tag{11}$$

It is impossible to calculate $K_n(\mathbf{r}_1,\mathbf{r}_2)$ for arbitrary n. The simplest approximation accounts in Eq. (11) for only the zero-loop graphs $K_0(\mathbf{r}_1,\mathbf{r}_2)$. In a real three-dimensional space this approximation cannot be checked, since the diagrams with loops are not small compared with those without loops. The approximation becomes exact, however, in the limit of high dimensionality of space.

To check this statement it would be necessary to estimate the contribution of the graphs with loops and compared with that of graphs without loops. The explicit calculations are quite difficult. It is simpler to recall the initial perturbation theory in F_1, to which graphs with thin lines correspond, and see when merging several lines into one is more expedient than formation of a loop. By way of example we compare graph *b* of Fig. 3 with graph *a* of this figure, the latter obtained from the former by merging points 3 and 1. The cause of the difference is that graph 3*b* contains one more integration with respect to Q. Changing over to variable $\hat{\theta}$ of (2) gives rise to an additional Jacobian. The volume element in terms of these variables, say in the unitary model, is proportional to[1,2]

$$\delta\Omega\sim d\lambda d\lambda_1/(\lambda_1-\lambda)^2,\quad \lambda_1=\mathrm{ch}\,\theta_1,\quad \lambda=\cos\theta.$$

If J in (5) is small, the significant λ_i are large and are of the order of J^{-1} (we consider the limit $\omega\to 0$). Each additional integration with respect to Q introduces a small J. This statement remains in force for an orthogonal or a symplectic ensemble. It can be verified, by comparing more complicated graphs, that for small J it is more expedient to merge the lines than to force loops. This permits the validity of the zero-loop approximation to be expressed ultimately in the form

$$J\ll 1. \tag{12}$$

It will be shown below that the Anderson transition occurs in the region in which (12) is valid if the dimensionality d of the space is high.

It remains now to sum the contributions of the zero-loop graphs. We note for this purpose that the graphs for $K_0(\mathbf{r}_1,\mathbf{r}_2)$ take the form of trees that grow out of sites of broken lines joining the points $\mathbf{r}_1$ and $\mathbf{r}_2$ (see, e.g., Fig. 2*a*). In the approximation considered we assume that the trees and branches are not linked with one another, so that all the trees are independent. From each site on the broken line can grow $m-1$ trees, where m is connected with the dimensionality of space by the relation $m=2d-1$. Each branch of the tree can have m branches of its own.

The structure of the trees obtained allows us to write an integral equation that determines their contribution to $\psi(Q)$, where Q pertains to the base of the trees

$$\psi(Q)=\int N(Q,Q')Z(Q')\psi(Q')\,dQ', \tag{13}$$

where

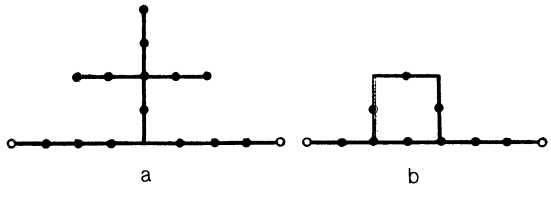

FIG. 2.

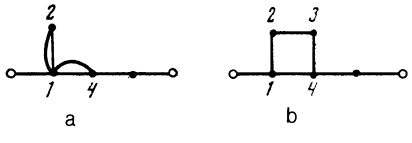

FIG. 3.

$$N(Q, Q')=\exp\left(\tfrac{1}{4}\alpha\,\mathrm{STr}\,QQ'\right),$$

$$Z(Q)=\psi^{n-1}(Q)\exp\left(\tfrac{1}{4}\beta\,\mathrm{STr}\,\Lambda Q\right),$$

$$\alpha=8\gamma_0 J_{12},\quad \beta=-i\gamma_0(\omega+i\delta)\pi\nu V.$$

Equation (13) coincides with the corresponding equation written for a Bethe lattice.[2-4]

The correlator $K_0(\mathbf{r}_1,\mathbf{r}_2)$ can be conveniently calculated by introducing the function $P(\mathbf{r},Q)$, $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, which differs from $K(\mathbf{r}_1,\mathbf{r}_2)$ by the absence of the factor $-2\pi^2\nu^2\gamma_0 Z(Q)\psi(Q)Q_{13}^{12}$ at the point $\mathbf{r}_1$ and by the fact that no integration with respect to Q is carried out at the point $\mathbf{r}_2$. In accordance with this definition we have

$$K_0(\mathbf{r}_1,\mathbf{r}_2)\equiv K(\mathbf{r})=-2\pi^2\nu^2\gamma_0\int Q_{13}^{12}P_{31}(\mathbf{r},Q)Z(Q)\psi(Q)\,dQ. \tag{14}$$

In the zero-loop approximation we can obtain for $P(\mathbf{r},Q)$ the equation

$$P(\mathbf{r},Q)-\sum_{\mathbf{r}'}W(\mathbf{r}-\mathbf{r}')\int N(Q,Q')P(\mathbf{r}',Q')Z(Q')\,dQ'$$
$$+m\int N_2(Q,Q')P(\mathbf{r},Q')Z(Q')\,dQ'=\delta(\mathbf{r})Q^{21}\psi(Q), \tag{15}$$

where

$$N_2(Q,Q')=\int N(Q,Q'')N(Q'',Q')Z(Q'')\,dQ'',$$

$$W(\mathbf{r}-\mathbf{r}')=\begin{cases}1, & |\mathbf{r}-\mathbf{r}'|=1,\\ 0, & |\mathbf{r}-\mathbf{r}'|\neq 1.\end{cases}$$

The third term in the left-hand side of (15) takes into account the fact that two segments of a broken line cannot coincide. Equation (15) differs somewhat in form from the corresponding equation for the Bethe lattice.[2,3] The reason is that the geometry of a d-dimensional lattice differs from that of a Bethe lattice even at large d.

Equation (13) can be obtained from (1) by using the following self-consistency procedure. We single out in (1) two arbitrary sites and describes the interaction between them, as before, by the first term of (1). We replace the interaction of these sites with the remaining ones by an interaction with a certain effective medium. If site 2 corresponds to a matrix Q', then the interaction Y with the medium depends only on Q and is proportional to m. The self-consistency condition takes then the form

$$\exp Y(Q)=\int N(Q,Q')\exp\left[mY(Q')+\tfrac{1}{4}\beta\,\mathrm{STr}\,\Lambda Q'\right]dQ'.$$

Putting $\psi(Q)=\exp[Y(q)]$ we arrive at Eq. (13).

In kinetics it is necessary to write the self-consistency condition for quantities that characterize transitions between globules. This is precisely why it is necessary to set apart not less than two sites, since single-site quantities describe only the density of states. This approximation is close in meaning to the effective-medium approximation used in percolation theory,[10] and can play the same role in localization theory. It is reasonable to call it likewise the effective-medium approximation.

The foregoing approximation is similar to the Bethe-Peierls approximation used in the theory of phase transitions. The Bethe-Peierls approximation becomes exact on the Bethe lattice. It is therefore not surprising that Eq. (13) coincides with the corresponding equation obtained for the model with the Bethe lattice.

4. CALCULATION OF THE DENSITY CORRELATOR

The remainder of the calculation of the correlator $K(\mathbf{r})$ is similar to that given in Ref. 3. The essential difference is that (15) is not a recursion equation. To solve it we must change to the Fourier representation

$$P_{\mathbf{k}}(Q)=V\sum_{\mathbf{r}}P(\mathbf{r},Q)e^{i\mathbf{k}\mathbf{r}}.$$

As a result of this transformation we obtain

$$\tilde{P}_{\mathbf{k}}(Q)-W(\mathbf{k})\hat{M}\tilde{P}_{\mathbf{k}}(Q)+m\hat{M}^2\tilde{P}_{\mathbf{k}}(Q)=iQ^{21}\psi(Q)[Z(Q)]^{1/2},$$
$$\tilde{P}_{\mathbf{k}}(Q)=i[Z(Q)]^{1/2}P_{\mathbf{k}}(Q). \tag{16}$$

The operator $\hat{M}$ acts on an arbitrary function $\varphi(Q)$ in the following manner:

$$\hat{M}\varphi(Q)\equiv\int N(Q,Q')[Z(Q)Z(Q')]^{1/2}\varphi(Q')\,dQ'. \tag{17}$$

Next, expanding $\tilde{P}(\mathbf{k},Q)$, as in Ref. 3, in the eigenfunctions $\varphi_E(Q)$ of an operator $\hat{M}$ such that

$$\hat{M}\varphi_E(Q)=E\varphi_E(Q), \tag{18}$$

we reduce the correlator $K(\mathbf{k})$ to the form

$$K(\mathbf{k})=2\pi^2\nu^2\gamma_0 V\sum_E B_E[1-EW(\mathbf{k})+mE^2]^{-1}, \tag{19}$$

$$B_E=-\int\mathrm{STr}[k\varphi_E^{+}(Q)Q^{21}]\psi(Q)[Z(Q)]^{1/2}\,dQ$$
$$\times\int Q_{13}^{12}[\varphi_E(Q)]_{31}\psi(Q)[Z(Q)]^{1/2}\,dQ. \tag{19a}$$

Equations (13), (19), and (19a) coincide with the corresponding equations of Ref. 3. The entire difference between the considered d-dimensional lattice and the Bethe lattice is contained in Eq. (13). The behavior of the density correlator $K(\mathbf{r})$ is determined by the form of the solution $\psi(Q)$ of Eq. (13).

It was shown in Refs. 2 and 3 that there exists a critical point such that at $\alpha<\alpha_c$ there exists a solution $\psi(Q)=1$ in the limit as $\omega\to 0$. At $\alpha>\alpha_c$ we get a nontrivial solution $\psi(Q)$ that decreases as $\theta_1\to\infty$. The quantity α_c is the boundary between the metallic ($\alpha>\alpha_c$) and insulating ($\alpha<\alpha_c$) regions and is defined by equations that take, when the inequality (12) [or the equivalent $\alpha m\ll 1$] is taken into account, the form

$$\frac{2^{3/2}m}{\pi}\left(\frac{\alpha_c}{2\pi}\right)^{1/2}\ln\frac{\gamma}{\alpha_c}=1\ \text{(orthogonal ensemble)}$$
$$m\left(\frac{\alpha_c}{2\pi}\right)^{1/2}\ln\frac{2}{\alpha_c}=1\ \text{(unitary ensemble)}$$
$$\frac{3}{4}m\left(\frac{\alpha_c}{2\pi}\right)^{1/2}\ln\frac{2}{\alpha_c}=1\ \text{(symplectic ensemble)}. \tag{20}$$

Equation (21) has the usual diffusive form. The diffusion coefficient D is expressed in terms of the solution ψ of Eq.

In the first equation of (20), γ is a number of the order of unity. Note that transition points α_c (20) lie in the applicability region (12) of the zero-loop approximation if $m \gg 1$. Equations (13), (18), (19), and (19a) yield an explicit expression for the density correlators in both the metallic and the insulating regions.

In the metallic region, the spectrum of the operator $\hat{M}$ (17) is discrete.[2-4] In the limit of low frequencies, just as in the Bethe-lattice models, it suffices to retain in the sum over E in (19) only the term with largest eigenvalue E_0. The equality of Eqs. (17) and (18) to the corresponding equations of Refs. 2 and 3 permits the use of the value of E_0 obtained in these references. Of greatest interest is the behavior of the density correlator in the region of small k and ω. Expanding in (19) with respect to k and retaining in E_0 only the zeroth and first terms of the expansion in ω, we get

$$K(\mathbf{k}) = 4\pi\nu/(D\mathbf{k}^2 - i\omega). \tag{21}$$

(13), which depends only on the matrix elements θ, θ_1 and θ_2 [Eq. (2)]. The explicit form of the diffusion coefficient depends on the model considered, but is equal in all cases to the diffusion coefficient for a Bethe lattice. Using the results of Refs. 2 and 3 we have

$$D = \frac{mb^2}{2\pi\nu\gamma_0} \int \left[\left(\frac{\partial \psi_0}{\partial \theta} \right)^2 + \sum_i \left(\frac{\partial \psi_0}{\partial \theta_i} \right)^2 \right] \psi_0^{m-1} J \, d\theta \prod_i d\theta_i, \tag{22}$$

where $i = 1$ and 2 for the orthogonal and symplectic models, $i = 1$ for the unitary model, and b is the distance between the nearest sites. The function ψ_0 in (22) is the solution of Eq. (13), taken for $\omega = 0$. In the orthogonal model, the Jacobian J is equal to

$$J = [\mathrm{ch}\,(\theta_1 - \theta_2) - \cos\theta]^{-2} [\mathrm{ch}\,(\theta_1 + \theta_2) - \cos\theta]^{-2} \sin^3\theta \,\mathrm{sh}\,\theta_1 \,\mathrm{sh}\,\theta_2, \tag{22a}$$

and the integration in (22) is over the region $0 < \theta < \pi$, $\theta_1 > 0$, $\theta_2 > 0$.

The Jacobian of the symplectic model is obtained from (22) by making the interchanges $\sinh \rightleftarrows \sin$ and $\cosh \rightleftarrows \cos$, and the integration region is defined by the inequalities $0 < \theta_1 < \pi$, $0 < \theta_2 < \pi/2$, $\theta > 0$.

Finally, in the unitary model

$$J = (\mathrm{ch}\,\theta_1 - \cos\theta)^{-2}, \quad \theta_1 > 0, \quad 0 < \theta < \pi. \tag{22b}$$

Equation (22) permits calculation of the diffusion coefficient D for all α (13) in the metallic region.

If α is large enough, the solution ψ of Eq. (13) takes the rather simple form

$$\psi(Q) = \exp\left(\tfrac{1}{4}\alpha\, \mathrm{STr}\, \Lambda Q\right). \tag{23}$$

Substituting (23) in (13) and estimating the resultant errors, we can find the validity region of the solution (23):

$$\alpha \gg 1/m^2. \tag{24}$$

Substituting (23) in (22) and integrating, we get

$$D = \alpha b^2/\pi\nu\gamma_0 = 8J_{12}b^2/\pi\nu. \tag{25}$$

Equation (25) is none other than the classical diffusion coefficient for the considered system of granules. In the derivation of Eq. (13) it is essential to use the condition (12), which corresponds to the case $\alpha \ll 1/m$. Of course, Eq. (25) is valid also in the region $\alpha \gtrsim 1/m$. In the latter, however, the effective-mass approximation does not lead to correct expressions for the quantum corrections to the classical expression (25). On the other hand, to calculate the quantum corrections in the region $\alpha \ll 1/m$ it suffices to solve Eq. (13) more accurately. The explicit form of the diffusion coefficient near the transition point α_c was obtained with Bethe-lattice models in Refs. 2 and 3. The equality of Eqs. (13) and (22) to the corresponding equations of Refs. 2 and 3 allows us to write for the dependence of the diffusion coefficient on α for small $\alpha - \alpha_c$;

$$D = p(\alpha - \alpha_c)^{-1/2} \exp[-q(\alpha - \alpha_c)^{-1/2}], \tag{26}$$

where p and q are parameters that depend on α_c and on the symmetry of the considered model. The function (26) is very steep, and as the transition point is approached the diffusion coefficient D decreases more rapidly than any power of $\alpha - \alpha_c$. Equation (26) is valid for any high dimensionality d, although the values of p, q, and α_c are different in each case.

In the dielectric region, $\alpha < \alpha_c$, the eigenfunctions φ of the operator $\hat{M}$ (17) have a continuous spectrum. To calculate the correlator $K(\mathbf{k})$ with the aid of Eqs. (19) and (1a) we can use also the results of Refs. 2 and 3, where an expression was obtained for (19a) in the limit of low frequencies:

$$B_E = -\varepsilon^2 a(\varepsilon)/i\omega\pi\nu\gamma_0,$$

where $a(\varepsilon)$ is a function of a certain variable ε for which an explicit expression is given in Refs. 2 and 3 in the form of an integral of a combination of the functions ψ and φ_E.

After some transformations similar to those carried out in Refs. 2 and 3 we obtain

$$K(\mathbf{k}) = \frac{2\pi\nu}{-i\omega} \int_0^\infty \frac{a^2(\varepsilon)\varepsilon^2 \, d\varepsilon}{1 - W(\mathbf{k})E(\varepsilon) + mE^2(\varepsilon)}, \tag{27}$$

where $E(\varepsilon) = \Gamma_\varepsilon(\alpha)$. The form of the functions $\Gamma_\varepsilon(\alpha)$ was obtained in Refs. 2 and 3 [in Ref. 2 this function is designated $\Gamma_{0\varepsilon}(\alpha)$]. Equation (27) solves in principle the problem of calculating the density correlator in the dielectric region.

An important quantity that characterizes the decrease of the wave function is the function $p_\infty(\mathbf{r}, \mathscr{E})$ (see, e.g., Ref. 12)

$$p_\infty(\mathbf{r}, \mathscr{E}) = \left\langle \sum_{discr} \delta(\mathscr{E} - \mathscr{E}_n) \, |u_n(0) u_n(\mathbf{r})|^2 \right\rangle, \tag{28}$$

where u_n and $\mathscr{E}_n$ are the eigenfunctions and energies of the initial Schrödinger equation, the angle brackets denote averaging over the impurities, and the summation is over the discrete levels. In the granule system considered, the dependence on the energy $\mathscr{E}$ reduces to a dependence on α, which is a function of $\mathscr{E}$. We replace accordingly the argument $\mathscr{L}$ in the function p_∞ by the parameter α. Knowledge of the density correlator permits calculation of $p_\infty(\mathbf{r}, \alpha)$:

$$p_\infty(\mathbf{r}, \alpha) = K(\mathbf{r}, t \to \infty), \tag{29}$$

where $\tilde{K}(\mathbf{r},t)$ is the density correlator in the coordinate and temporal representation.

In the limit of large distances, $r \gg b$, the calculation or $p_\infty(\mathbf{r},\alpha)$ is greatly simplified, since only small values of ε become significant. Expanding in terms of ε and $\mathbf{k}$ in (27), and using (29) and the fact that $a(\varepsilon)$ is finite for $\varepsilon \to 0$, we get

$$p_\infty(r,\alpha) = \frac{2\pi\nu[\Gamma_0(\alpha)]^{1/2} b a^2(0)}{c^{1/2}} \times \int_0^\infty [\varepsilon^2 + (4l)^{-2}]^{(d-2)/2} G\{r[\varepsilon^2 + (4l)^{-2}]^{1/2}\} \varepsilon^2 \, d\varepsilon, \tag{30}$$

where

$$c = -\frac{m}{2} \frac{\partial^2 \Gamma_\varepsilon(\alpha)}{\partial \varepsilon^2}\Big|_{\varepsilon=0}, \qquad (4l)^{-2} = \frac{1 - m\Gamma_0(\alpha)}{\Gamma_0(\alpha) b^2}. \tag{30a}$$

The function $G(\rho)$ in (30) satisfies the equation

$$l = \text{const}(\alpha_c - \alpha)^{-1/2}, \tag{31}$$

where Δ_d is the d-dimensional Laplacian.

Solution of the equation $1 - m\Gamma_0(\alpha)$ yields the transition point α_c (20).[2-4] The length l in (30) therefore becomes large as the transition point is approached:

$$l = \text{const}(\alpha_c - \alpha)^{-1/2}. \tag{32}$$

In the regions $r \gg l$ and $b \ll r \ll l$ it is possible to obtain explicit asymptotics of the function $p_\infty(r,\alpha)$. For $r \gg l$ the integral (30) is governed by large values of the argument $r(\varepsilon^2 + l^{-2})^{1/2}$ of the function G and by the small quantities $\varepsilon \sim (lr)^{-1} \ll l^{-2}$. Expanding $(\varepsilon^2 + l^{-2})^{1/2}$ in powers of ϵ and using the asymptotic form at $\rho \gg 1$:

$$G(\rho) \sim e^{-\rho} \rho^{(1-d)/2},$$

we get

$$p_\infty(r,\alpha) = \text{const}\, r^{-(d+2)/2} l^{-d/2} e^{-r/4l}. \tag{33}$$

Comparison of (33) with (28) leads to the conclusion that the wave functions are localized, with l the localization length. Equation (33) with l as defined in (30a) is valid for all $\alpha < \alpha_c$.

In the region $b \ll r \ll l$, which exists if $|\alpha_c - \alpha| \ll \alpha_c$, the quantities l^{-2} in the integral of (30) can be neglected. In this limit we obtain

$$p_\infty(r,\alpha) = Ar^{-d-1}, \qquad A = 2\pi\nu c^{-1/2} \Gamma_0^{1/2}(\alpha) a^2(0) \int_0^\infty x^d G(x)\, dx. \tag{34}$$

Equations (34) and (28) show that at distances much shorter than the localization length the wave functions undergo a power-law decrease. At the transition point α_c the localization length becomes infinite, and this leads to a power-law decrease of the wave functions for all distances $r \gg b$.

The function $p_\infty(r,\alpha) r^{d-1} S_d$, where S_d is the volume of a d-dimensional sphere of unit radius, is proportional at $r \gg b$ to the probability that a particle located at the origin at the instant $t = 0$ will end up at a distance r after an infinite time. The quantity $p_\infty(0,\alpha)$ determines the probability of remaining in place and coincides, in the considered effective-medium approximation, with the corresponding value in Bethe-lattice models. As shown in Refs. 2 and 3, $p_\infty(0\alpha)$ remains finite at the transition point α_c. The integral of $p_\infty(r,\alpha)$ over the volume converges for all $\alpha \leqslant \alpha_c$ and remains finite as $\alpha \to \alpha_c$. The moments of this quantity, however increase without limit when $\alpha \to \alpha_c$, as seen directly from (34). The second moment determines the polarizability

$$\delta_{\alpha\beta}\varkappa = e^2 \int r_\alpha r_\beta p_\infty(r,\alpha)\, dV, \tag{35}$$

where e is the electron charge.

To calculate the integral in (35) it is convenient to transform to the momentum representation. Using Eqs. (27) and (29) and recognizing that small ε are significant at $\alpha_c - \alpha \ll \alpha_c$, we get

$$\varkappa = 4\pi^2 \nu a^2(0) d^{1/2} c^{-3/2} b l. \tag{36}$$

Expression (36) shows that the polarizability near the transition point is proportional to the localization length l whose growth is described by Eq. (32). The dielectric constant coincides in this region with the polarizability.

5. DISCUSSION OF RESULTS

The effective-medium approximation developed in the preceding sections has made it possible to describe completely the kinetics of a quantum particle in a granulated disordered metal. The existence of an Anderson metal-insulator transition was proved in this approximation and the density correlator was calculated. In the metallic region this correlator has the usual diffusive form (21), while in the dielectric region it is described by expression (27), which is also the usual one for localized states. In the insulator region we obtained the function $p_\infty(r,\alpha)$ (28), (29), whose exponential decrease over distances r larger than the localization length l proves the exponential localization of the states. In the metallic region, $p_\infty(r,\alpha)$ vanishes, as seen directly from (21) and (29). Comparison with (28) leads to the conclusion that there is no discrete spectrum of localized levels against the background of the continuum of the conducting states. The foregoing properties do not call for time-reversal symmetry or central symmetry, and agree with the universally accepted premises of localization theory.

The approximation developed becomes exact in the limit of high dimensionality d of the space. Equation (13) agrees with the corresponding equation obtained in Bethe-lattice models. At the same time, the form of the density-density correlator for d-dimensional space differs from that for the Bethe lattice. This difference is not a specific feature of the localization problem, and should exist in usual spin models. The main quantity that determines the properties of the effective medium is the function $\psi(Q)$. This function can be expressed in principle in the form

$$\psi(Q) = \int \exp(\mathrm{STr}\, hQ) R(h)\, dh. \tag{37}$$

Equation (37) can be regarded as a generalized Laplace transformation. The integration extends over all the matrices h that satisfy the conditions $h = \bar{h}$, $h^+ = khk$ and $h\Lambda = \Lambda h$. The function $\psi(Q)$ depends only on the matrix $\hat{\theta}$. Therefore if the matrix h is represented in the form $h = Vh_d\bar{V}$, where V has the same structure as U [Eq. (2)],

and h_d is the diagonal part, R should depend only on h_d. In the limiting case of a pure metal, ψ takes the form (23) and $R(h)$ reduces to a δ-function. In this limit, the interaction of Q_l with the remaining Q_j is replaced by an interaction with an effective field h_0, which is described by a matrix that takes in the diagonalized representation the form $h_{0d} = \Lambda\alpha/4$. If α does not meet the condition (14), one can speak not of an effective field h_0, but only of a nontrivial distribution $R(h)$ of the fields [Eq. (37)]. This statement is valid also near the transition point α_c, so that the order parameter should be regarded as a function. This property is formally the consequence of the noncompactness of the group of supermatrices Q. For any compact group, the role of the order parameter would be assumed by the zeroth harmonics in the expansion of ψ in terms of generalized spherical function (see the last paper in Ref. 2).

In the region (24), h_0 is proportional to the diffusion coefficient. It can be assumed that the need for using the distribution of the effective fields corresponds to the need for considering not the conductivity of a finite voltage, but a distribution of conductivities. Only in the limit of weak disorder does this distribution become narrow and one can speak of the conductivity of the volume. Such a distribution must be broad in a strong-disorder region. An attempt to develop a theory in which distributions are used has already been made.[13] It is difficult to assess the validity of the proposed theory, since it is not free of strong assumptions and contains some contradictions, yet the undertaking of such an investigation seems reasonable.

The critical behavior of the distribution coefficient D (26) and of the dielectric constant $\varkappa$ (36) does not agree with the one-parameter-scaling hypothesis.[14] The discrepancy may be due precisely to the fact that on going from one scale to another it is necessary to renormalize the distribution of the conductances, and the very concept of a conductance that depends little on the change of the configurations is meaningful only in the limit of long ranges. It is seen from (6) that the expression for the dielectric constant contains not only the localization length but also the distance b between the granules. Of course, the effective-medium approximation is quantitatively valid only for large d. It appears that there exists a critical dimensionality d_c below which the fluctuations become significant and the results are in some way changed. It is unlikely, however, that the length b can be eliminated from the theory and a transition made to one-parameter scaling. A possible cause of the deviation from the self-consistent theory[15] was discussed in Ref. 2. All the arguments there can be repeated also for the effective-medium approximation and it can be shown that artificial compacting of the group of supermatrices Q would lead to the results of Ref. 15. It is not yet clear whether such a compacting is related in any way to a physical situation.

Note also the discrepancy between our results and those of an investigation of a continual σ-model by the renormalization-group method in a space of dimensionality $2+\varepsilon$. What is important is that this contradiction exists within the framework of one and the same model describable by the Lagrangian (1). The reasons for this are not yet clear. It can only be suggested that some nonperturbative terms make an important contribution in a space of dimensionality $2+\varepsilon$, for otherwise it is impossible to discern the difference between compact and noncompact models, a difference of importance to the results above.

An interesting conclusion concerning the behavior of the wave functions $u_n(r)$ can be drawn from Eq. (34). Comparing (33) and (34) with (28) we can posit a power-law decrease of the localized-states wave functions in the region $|\mathbf{r}-\mathbf{r}_0| \ll l$, where $\mathbf{r}_0$ is the localization center. At $|\mathbf{r}-\mathbf{r}_0| \gg l$, the power-law decrease (34) gives way to an exponential one (33), with a power-law pre-exponential factor in (33). At the transition point the wave functions obey strictly a power law. In the metallic region the form of the correlator K (19) also permits a qualitative description of the form of the wave functions. At $|\mathbf{r}-\mathbf{r}_0| \ll \xi$, where ξ is a characteristic length proportional to the distance ΔE between the eigenvalue E_n of Eq. (18), the wave functions have exactly the same power-law decrease as in the localized region at $|\mathbf{r}-\mathbf{r}_0| \ll l$. The quantity ΔE is inversely proportional to the size of that region of the parameters of the matrix $\hat{\theta}$, in which the function ψ differs substantially from zero. From this we can obtain the estimate $\xi \sim (\alpha-\alpha_c)^{-1/2}$. At $|\mathbf{r}-\mathbf{r}_0| \gtrsim \xi$ the characteristic amplitude of the wave function should stop decreasing. With increasing distance from the transition point α_c, the region of the power-law decrease becomes narrower. At $\alpha \gg \alpha_c$ where perturbation theory can be used, the amplitudes of the wave functions varies little in space.

A power-law decrease of the wave function for not very large distances was recently deduced from a numerical calculation.[16] The data were reduced there under the assumption that the exponent of the power-law decrease of the wave functions depends on the impurity density. It would be of interest to reduce the same data by starting from the assumption that a power-law decrease takes place in the region $r < \xi$ but the wave functions do not decrease in the region $r > \xi$.

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