Electromagnetic scattering from statistically rough surfaces

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Using macroscopic electrodynamics, we treat electromagnetic scattering from a rough interface between two uniform media by recasting the problem as the solution of a nonsingular equation for the scattering T matrix; this procedure is valid to any order in the dielectric constant of the boundary layer. We solve for the T matrix in the linear and quadratic approximations to the surface roughness h, and we obtain expressions for the scattering indicatrices and reflection coefficients which satisfy energy conservation requirements for all angles of incidence, any polarization of the incident and scattered waves, and any surface-roughness spectrum. We also derive expressions for the surface impedance, the dispersion relations for s- and p-polarized electromagnetic surface waves, and the Brewster angle to terms of order h^2. We finally analyze the applicability of the expressions obtained.

1. INTRODUCTION

Recent experimental discoveries of new phenomena in the optics of rough surfaces such as stimulated Raman scattering,[1] enhanced second harmonic generation,[2] and the displacement, broadening, and splitting of the dispersion curve for electromagnetic surface waves (ESW),[3] have all served to stimulate the further development of theory. Results obtained recently in the limit of low surface roughness include expressions for the scattering indicatrices of electromagnetic waves,[4–7] the reflection coefficients[8–15] $r_{\alpha\beta}$ [$\alpha(\beta) = s,p$ is the polarization state of the reflected (incident) wave], the effective surface impedance[16–18] $\zeta_{\alpha\beta}$, and the ESW dispersion relation on a rough surface.[3,7,19–23] The results obtained by different methods are often in disagreement, however, and are either incomplete or valid only in limiting cases. For example, the expressions for $r_{\alpha\beta}$ derived in Ref. 8 applied only to perfect conductors ($\varepsilon_2 \to \infty$); those in Ref. 9 were for impedance boundary conditions, which hold[24] when $|\varepsilon_2| \gg \varepsilon_1$; those in Ref. 10 were for the limiting case $|\varepsilon_2 - \varepsilon_1| \ll \varepsilon_1$ (x-ray region of the spectrum) and a gently sloping surface (ε_1 and ε_2 are the dielectric constants of the surfaces in contact). Expressions were obtained in Ref. 11 for the reflection coefficients from a rough interface between two media with arbitrary ε_j, but only for the r_{ss} and r_{ps} components, while in Refs. 12–15, they were obtained only for the diagonal components r_{ss} and r_{pp}. The results in Refs. 11–13 and 14, 15 are mutually inconsistent, and the equations for the ESW dispersion relation at a rough surface[22,23] are not in agreement with those obtained elsewhere.[3,7,19–21]

The fundamental assumption usually made is that the Rayleigh hypothesis[25] holds; its limited domain of applicability has been demonstrated only for deterministic surfaces,[26–28] and remains an open question for statistically rough ones.[21] Attempts to avoid this assumption result in mathematically improper expressions, even to first order in the roughness[29,30] (see also the review in Ref. 23). Thus, there is presently no unique and consistent method of calculating the quantities of interest for the optics of rough surfaces, and no determination has been made of the domain of applicability of the existing expressions.

Our objective in the present work is to develop a systematic theory of the scattering of light by statistically rough surfaces which is free of improper expressions for perturbations of any order, without resorting to the Rayleigh hypothesis. The paper is organized as follows. Section 2 presents a statement of the problem, and an exact solution for the components of the diffracted field in terms of the scattering T-matrix. The T-matrix is shown to satisfy an integral equation analogous to the conventional equations of quantum scattering theory. Section 3 contains a solution of the equation for the T-matrix to terms linear in the amplitude of the roughness, and a derivation of expressions for the scattering indicatrices and reflection coefficients which are valid for uniform media with any dielectric constant; Section 4 does the same for the quadratic approximation. These expressions are shown to conserve energy at all angles of incidence, for any incident or reflected polarization, and for surface roughness with an arbitrary spectrum, if the reflecting medium is nondissipative and opaque. Limiting expressions for $r_{\alpha\beta}$ are presented for the cases $|\varepsilon_2 - \varepsilon_1| \ll \varepsilon_1$ and $|\varepsilon_2| \gg \varepsilon_1$. In Section 5, we derive both the effective surface impedance and dispersion relations for p- and s-polarized electromagnetic surface waves. Expressions for the reflection coefficients are given in a form consistent with analyticity of the S-matrix, and which correctly determines the locations of poles and zeroes. The conditions for applicability of the results obtained is discussed in Sec. 6, and we conclude with a summary of research accomplished.

2. STATEMENT OF THE PROBLEM

The equations of macroscopic electrodynamics governing the propagation of a monochromatic electromagnetic wave $\mathbf{E}(\mathbf{r})e^{-i\omega t}$ in a medium with dielectric constant

$$\varepsilon(\mathbf{r}) = \varepsilon_1\theta(z-h(\rho)) + \varepsilon_2\theta(h(\rho)-z) \tag{2.1}$$

containing an interface at $z = h(\mathbf{p})$, where $\mathbf{p} = (x,y)$ is a two-dimensional vector lying in the plane $z = 0$ and $\theta(z)$ is the unit step function, are

$$(\mathrm{rot\,rot} - k_0^2\varepsilon_z)E(\mathbf{r}) = v(\mathbf{r})E(\mathbf{r}). \tag{2.2}$$

Here $k_0 = \omega/c$ is the wave vector of the electromagnetic wave in vacuum, and ε_z is the dielectric constant of the medium with an unperturbed interface,

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$$\varepsilon_z=\varepsilon_1\theta(z)+\varepsilon_2\theta(-z);$$

$v(\mathbf{r})$ is the perturbation induced by surface roughness,

$$v(\mathbf{r})=k_0^2\Delta\varepsilon(\mathbf{r})\equiv k_0^2[\varepsilon(\mathbf{r})-\varepsilon_z]. \tag{2.3}$$

Assuming that the surfaces in contact are uniform and have no spatial dispersion, we may put $\varepsilon_{1,2}=\varepsilon_{1,2}(\omega)$. In the absence of perturbations, the linearly independent solutions of the homogeneous form of Eq. (2.2) are

$$\mathbf{E}_{j\gamma}(\mathbf{r})=\mathbf{E}_{j\gamma}(\mathbf{b},z)e^{i\mathbf{b}\rho}, \tag{2.4}$$

where the subscript $j=1,2$ indicates the medium containing the incident plane wave, and $\gamma=s,p$ indicates its polarization. Assuming with no loss of generality that the incident wave is in the medium with $j=1$, we may expand the general solution of the inhomogeneous form of Eq. (2.2) for $z>h_m=\max\{h(\rho)\}$ in plane waves,

$$\mathbf{E}(\mathbf{r})=[C_{0s}\mathbf{E}_{1s}(\mathbf{b}_0,z)+C_{0p}\mathbf{E}_{1p}(\mathbf{b}_0,z)]e^{i\mathbf{b}_0\rho}+i\int\frac{d^2b}{2\eta_1}e^{i(\mathbf{b}\rho+\eta_1 z)}$$
$$\cdot\{[\hat{\mathbf{s}}E_{ss}(\mathbf{b},\mathbf{b}_0)+\hat{\mathbf{p}}E_{ps}(\mathbf{b},\mathbf{b}_0)]C_{0s}$$
$$+[\hat{\mathbf{s}}E_{sp}(\mathbf{b},\mathbf{b}_0)+\mathbf{p}E_{pp}(\mathbf{b},\mathbf{b}_0)]C_{0p}\}, \tag{2.5}$$

where C_{0s} and C_{0p} are independent constants, $\mathbf{b}_0$ ($\mathbf{b}$) is the projection of the incident (scattered) wave vector on the plane $z=0$,

$$\eta_{j0}=(k_j^2-\mathbf{b}_0^2)^{1/2},\quad \eta_j=(k_j^2-\mathbf{b}^2)^{1/2}$$
$$(\mathrm{Re},\,\mathrm{Im}\,\eta_{j0},\,\eta_j\geqslant 0) \tag{2.6}$$

is the projection of the wave vector on the normal to the plane $z=0$ in medium $j=1,2$, and $k_j=k_0\varepsilon_j^{1/2}$ is the propagation constant in medium j; $\hat{\mathbf{s}}$ and $\hat{\mathbf{p}}$ are unit polarization vectors for the scattered wave:

$$\hat{\mathbf{s}}=[\mathbf{b}\hat{\mathbf{z}}],\quad \hat{\mathbf{p}}=(\mathbf{b}\hat{\mathbf{z}}-\eta_1\hat{\mathbf{b}})/k_1, \tag{2.7}$$

and finally $E_{\alpha\beta}(\mathbf{b},\mathbf{b}_0)$ specifies the desired scattered field components with polarization α and wave-vector projection $\mathbf{b}$, given an incident wave with polarization β and wave-vector projection $\mathbf{b}_0$. These components are nonzero only if the perturbation $v(\mathbf{r})$ is nonzero as well.

Making use of the solution[31,32] of the equation

$$(\mathrm{rot\,rot}-k_0^2\varepsilon_z)G(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}') \tag{2.8}$$

for the Green's function $G(\mathbf{r},\mathbf{r}')$ for a medium with a plane interface, we can express the general form of Eq. (2.2) in integral form as

$$\mathbf{E}(\mathbf{r})=\mathbf{E}_0(\mathbf{r})+\int d^3\mathbf{r}'\,G(\mathbf{r},\mathbf{r}')v(\mathbf{r}')\mathbf{E}(\mathbf{r}'), \tag{2.9}$$

where $\mathbf{E}_0(\mathbf{r})$ is the general solution of the homogeneous equation. Equation (2.9) was used in Ref. 29 for an iterative solution, where the perturbation

$$v(\mathbf{r})=(k_2^2-k_1^2)\left[h(\rho)\delta(z)-\frac{h^2(\rho)}{2}\frac{d}{dz}\delta(z)+\dots\right] \tag{2.10}$$

and the field $\mathbf{E}(\mathbf{r})$ were expanded in powers of h. Due to the singular behavior of the Green's function at $\mathbf{r}=\mathbf{r}'$, however $[G(\mathbf{r},\mathbf{r})\sim\delta(\mathbf{r}-\mathbf{r}')$ as $\mathbf{r}\to\mathbf{r}']$, and the discontinuous behavior of (2.4) at $z=0$, mathematically improper expressions such as products of delta functions arise in field components which are discontinuous at $z=0$, even in the linear approximation $[\sim h(\rho)]$.[29,30,23]

In order to avoid these difficulties, only field components continuous at $z=0$ were used in Refs. 33–35 to construct the Green's function. Even then, improper expressions are only eliminated from the linear term $\mathbf{E}^{(1)}\sim h(\rho)$; they remain in all subsequent terms, due to discontinuities in the derivatives of the field components with respect to z at $z=0$. Furthermore, because of the preferred status conferred upon the normal to the plane $z=0$, the equations in Refs. 33–35 are not covariant, which greatly complicates any subsequent calculations.

When written in the form (2.9), then, the equation for the field is ill-suited to further analysis. It can be transformed, however, by isolating the singular term at the plane $z=z'$:

$$G(\mathbf{r},\mathbf{r}')=-\frac{\hat{\mathbf{z}}\hat{\mathbf{z}}}{k_0^2\varepsilon_z}\delta(\mathbf{r}-\mathbf{r}')+G'(\mathbf{r},\mathbf{r}'), \tag{2.11}$$

where the second term, $G'(\mathbf{r},\mathbf{r}')$, is finite at $\mathbf{r}=\mathbf{r}'$. Here and below we use the dyadic notation $\mathbf{ab}$ to represent second-rank tensors; $\hat{\mathbf{z}}$ is the unit normal to the plane $z=0$.

The form taken by the singular term in (2.11) reflects the specifics of the zeroth-order problem, namely that the interface is planar. The same representation of the Green's function was used in Ref. 34 for a layered medium. In the treatments of electromagnetic wave propagation in randomly inhomogeneous media reported in Refs. 36 and 37, the singular term extracted from the Green's function was of the form $-(\hat{\mathbf{r}}\hat{\mathbf{r}}\dots)\delta(\dots)$, due to the spherical form of the excluded region. The way in which the form of the singular term depends on the form of the excluded region is discussed in Ref. 38.

Substituting (2.11) into (2.9) and transforming, we obtain the following solution for the desired field components $E_{\alpha\beta}(\mathbf{b},\mathbf{b}_0)$:

$$E_{\alpha\beta}(\mathbf{b},\mathbf{b}_0)=t_\alpha(\mathbf{b})t_\beta(\mathbf{b}_0)\iint_{-\infty}^{\infty}dz_1\,dz_2\,\mathbf{X}_{1\alpha}^-(\mathbf{b},z_1)T$$
$$\cdot(\mathbf{b},\mathbf{b}_0,z_1,z_2)\mathbf{X}_{1\beta}^+(\mathbf{b}_0,z_2), \tag{2.12}$$

where T is the scattering operator, which satisfies the equation

$$T(\mathbf{b},\mathbf{b}_0,z_1,z_2)=v(\mathbf{b}-\mathbf{b}_0,z_1)\delta(z_1-z_2)\hat{1}+\int d^2\mathbf{b}'\,dz'\,v(\mathbf{b}-\mathbf{b}',z_1)$$
$$\cdot G_0(\mathbf{b}',z_1,z')T(\mathbf{b}',\mathbf{b}_0,z',z_2), \tag{2.13}$$

or in symbolic form

$$T=v+vG_0T. \tag{2.14}$$

The notation used in (2.12) and (2.13) is as follows: $v(\mathbf{q},z)$ is the Fourier transform of the perturbation $v(\mathbf{r})$, $G_0(\mathbf{b},z,z')$ is the transformed nonsingular Green's function

$$G_0(\mathbf{b},z,z')=\frac{i}{2\eta_1}\sum_{\gamma=s,p} t_\gamma(\mathbf{b})\,[\mathbf{X}_{2\gamma}^+(\mathbf{b},z)\mathbf{X}_{1\gamma}^-(\mathbf{b},z')\theta(z-z') + \mathbf{X}_{1\gamma}^+(\mathbf{b},z)\mathbf{X}_{2\gamma}^-(\mathbf{b},z')\theta(z'-z)], \qquad (2.15)$$

$t_\gamma(\mathbf{b})$ is the (Fresnel) index of refraction at the plane interface:

$$t_s(\mathbf{b})=\frac{2\eta_1}{\eta_1+\eta_2}, \quad t_p(\mathbf{b})=\frac{2\eta_1(\varepsilon_1\varepsilon_2)^{1/2}}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}; \qquad (2.16)$$

$\mathbf{X}_{j\gamma}^{\pm}(\mathbf{b},z)$ represents the transformed solutions of the homogeneous form of Eq. (2.2):

$$\mathbf{X}_{js}^{\pm}(\mathbf{b},z)=\mathbf{E}_{js}(\mathbf{b},z),$$
$$\mathbf{X}_{jp}^{\pm}(\mathbf{b},z)=\varepsilon_z(\varepsilon_1\varepsilon_2)^{-1/2}\mathbf{E}_{jz}(\mathbf{b},z)\pm\mathbf{E}_{jb}(\mathbf{b},z), \qquad (2.17)$$

where $\mathbf{E}_{jz}(\mathbf{b},z)$ and $E_{jb}(b,z)$ are the $\hat{\mathbf{z}}$- and $\hat{\mathbf{b}}$-components of the field of the p-polarized wave $\mathbf{E}_{jp}=\mathbf{E}_{jz}+\mathbf{E}_{jb}$.

The form in which Eqs. (2.12) and (2.15) have been written entails a choice of unit normalization for the amplitude of the $\mathbf{E}_{j\gamma}(\mathbf{b},z)$ at $z=0$:

$$\mathbf{E}_{js}(\mathbf{b},0)=\hat{\mathbf{s}}, \quad \mathbf{E}_{1p}(\mathbf{b},0)=\hat{\mathbf{p}}_{2+}, \quad \mathbf{E}_{2p}(\mathbf{b},0)=\hat{\mathbf{p}}_{1-}, \qquad (2.18)$$

where

$$\hat{\mathbf{p}}_{j\pm}=(b\hat{\mathbf{z}}\pm\eta_j\hat{\mathbf{b}})/k_j$$

are the unit vectors for a p-polarized wave traversing the interface at $z=0$. From (2.17) and (2.16), we then have

$$\mathbf{X}_{js}^{\pm}(\mathbf{b},0)=\hat{\mathbf{s}}, \quad \mathbf{X}_{1p}^{\pm}(\mathbf{b},0)=\frac{b}{k_1}\hat{\mathbf{z}}\pm\frac{\eta_2}{k_2}\hat{\mathbf{b}},$$

$$\mathbf{X}_{2p}^{\pm}(\mathbf{b},0)=\frac{b}{k_2}\hat{\mathbf{z}}\mp\frac{\eta_1}{k_1}\hat{\mathbf{b}}. \qquad (2.19)$$

The functions $\mathbf{E}_{j\gamma}(\mathbf{b},z)$ in (2.5) are normalized to the amplitude of the incident wave

Equations (2.12) and (2.13) reduce the problem of electromagnetic wave diffraction at an arbitrary corrugated interface between two uniform media to the solution of the integral equation (2.14) for the scattering operator T, which is in the standard form encountered in quantum scattering theory,[39] solution methods are well-developed for this problem.[39] The Green's function (2.15) which enters into (2.14) is nonsingular at $z=z'$, while the fields $\mathbf{X}_{j\gamma}^{\pm}(\mathbf{b},z)$ of which it is composed are continuous at $z=0$, which follows directly from the continuity at $z=0$ of $\mathbf{E}_{js}$, $\mathbf{E}_{jb}$, and $\mathbf{D}_{jz}=\varepsilon_z\mathbf{E}_{jz}$, the components which enter into (2.17). The solution (2.12) and Eq. (2.14) have been written in covariant form, and are mathematically well-behaved to any order in the perturbation v.

We next derive Eq. (2.13) to first and second order in the amplitude of the roughness $h(\boldsymbol{\rho})$.

3. LINEAR APPROXIMATION: ANGULAR SPECTRUM

Restricting attention to the linear term in the expansion of the perturbation $v(\mathbf{r})$, we obtain from (2.13) an expression for the scattering operator T,

$$T(\mathbf{b},\mathbf{b}_0,z_1,z_2)=(k_2^2-k_1^2)h_{\mathbf{b}-\mathbf{b}_0}\delta(z_1)\delta(z_2)\hat{1}+\ldots, \qquad (3.1)$$

where $h_{\mathbf{q}}$ is the Fourier transform of the surface profile $h(\boldsymbol{\rho})$.

Substituting (3.1) into (2.12), we obtain a solution for the field components of the scattered wave in the linear approximation,

$$E_{\alpha\beta}(\mathbf{b},\mathbf{b}_0)=(k_2^2-k_1^2)t_\alpha(\mathbf{b})t_\beta(\mathbf{b}_0)\cdot[\mathbf{X}_{1\alpha}^-(\mathbf{b},0)\mathbf{X}_{1\beta}^+(\mathbf{b}_0,0)]h_{\mathbf{b}-\mathbf{b}_0}+\ldots, \qquad (3.2)$$

which is the same as the existing solution by virtue of the normalization (2.19). Equations (3.2) and (2.5) enable one to calculate the Poynting vector of the scattered wave directly, and averaging this over an ensemble of rough surfaces, using

$$\langle h_{\mathbf{q}_1}h_{\mathbf{q}_2}^*\rangle=S(\mathbf{q}_1)\delta(\mathbf{q}_1-\mathbf{q}_2), \qquad (3.3)$$

where angle brackets denote an ensemble average, and $S(\mathbf{q})$ is the spectral density of the rough surface, we obtain the angular spectrum of the scattered power $dP/d\Omega$, normalized to the z-component of the incident power P_z:

$$\frac{1}{P_z}\frac{dP}{d\Omega}=\frac{k_1^2|k_2^2-k_1^2|^2}{4\eta_1\eta_{10}}\left\{|t_s|^2\left|(\hat{\mathbf{s}}\hat{\mathbf{s}}_0)t_{s0}C_{0s}+(\hat{\mathbf{s}}\hat{\mathbf{b}}_0)\frac{\eta_{20}}{k_2}t_{p0}C_{0p}\right|^2 + |t_p|^2\left|-\hat{\mathbf{b}}\hat{\mathbf{s}}_0\frac{\eta_2}{k_2}t_{s0}C_{0s}+\left[\frac{bb_0}{k_1^2}-\frac{\eta_2\eta_{20}}{k_2^2}\hat{\mathbf{b}}\hat{\mathbf{b}}_0\right]t_{p0}C_{0p}\right|^2\right\}\cdot S(\mathbf{b}-\mathbf{b}_0), \qquad (3.4)$$

where $t_\gamma=t_\gamma(\mathbf{b})$, $t_{\gamma 0}=t_\gamma(\mathbf{b}_0)$, and the $C_{0\gamma}$ are normalized by

$$|C_{0s}|^2+|C_{0p}|^2=1. \qquad (3.5)$$

Equation (3.4) is the same as the results in Refs. 4 and 5.

4. QUADRATIC APPROXIMATION: REFLECTION COEFFICIENTS

To calculate the reflection coefficients for an electromagnetic wave at a rough surface, it is necessary to determine the diffracted field components $E_{\alpha\beta}(\mathbf{b},\mathbf{b}_0)$ up to terms quadratic in the surface profile $h(\boldsymbol{\rho})$. We solve Eq. (2.13) for the scattering operator T by iterating up to terms $\sim v^2$,

$$T(\mathbf{b},\mathbf{b}_0,z_1,z_2)=v(\mathbf{b}-\mathbf{b}_0,z_1)\delta(z_1-z_2)\hat{1}+\int d^2\mathbf{b}'\,v(\mathbf{b}-\mathbf{b}',z_1)\cdot G_0(\mathbf{b}',z_1,z_2)v(\mathbf{b}'-\mathbf{b}_0,z_2)+\ldots \qquad (4.1)$$

and substituting the result into (2.12). The scattering operator $T(\mathbf{b},\mathbf{b}_0,z_1,z_2)$ is nonzero for $|z_1|\leqslant h_m$, $|z_2|\leqslant h_m$, where $v(\mathbf{q},z)\neq 0$. The characteristic scale for changes in the functions $\mathbf{X}_{j\gamma}^{\pm}(\mathbf{b},z)$ depends on the coordinate dependence of the fields $E_{j\gamma}(\mathbf{b},z)\sim e^{\pm i\eta_1 z}$, $e^{\pm i\eta_2 z}$ which, according to (2.17), comprise them. Thus, when

$$\eta_{j0}h_m\ll 1, \quad \eta_j h_m\ll 1, \quad \eta_j' h_m\ll 1, \qquad (4.2)$$

where

$$j=1,\ 2,\quad \eta_j'=(k_j^2-\mathbf{b}'^2)^{1/2}\quad (\mathrm{Re},\ \mathrm{Im}\,\eta_j'\geqslant 0),\tag{4.3}$$

the functions $\mathbf{X}_{j\gamma}{}^{\pm}(\mathbf{b},z)$ vary only slightly over the range of integration of (2.12), and they can be expanded as power series in z.

Keeping only linear terms of $\mathbf{X}_{j\gamma}{}^{\pm}(\mathbf{b},z)$ in the first iteration for T, stopping at the zeroth approximation to (2.19) for the second iteration, and bearing in mind that

$$\int_{-\infty}^{\infty} dz\, z^n v(\mathbf{r}) = \frac{k_2^2-k_1^2}{n+1} h^{n+1}(\boldsymbol{\rho}),\tag{4.4}$$

the result for the second approximation to the scattered field components $E_{\alpha\beta}{}^{(2)}(\mathbf{b},\mathbf{b}_0)\sim h^2$ is

$$\begin{aligned} E_{\alpha\beta}^{(2)}(\mathbf{b},\mathbf{b}_0) = t_\alpha t_{\beta 0}\Bigg\{\Bigg[\left(\frac{d\mathbf{X}_{1\alpha}{}^{-}(\mathbf{b},0)}{dz}\mathbf{X}_{1\beta}{}^{+}(\mathbf{b}_0,0)\right) \\ +\left(\mathbf{X}_{1\alpha}{}^{-}(\mathbf{b},0)\frac{d\mathbf{X}_{1\beta}{}^{+}(\mathbf{b}_0,0)}{dz}\right)\Bigg]\frac{(k_2^2-k_1^2)}{2}[h^2(\boldsymbol{\rho})]_{\mathbf{b}-\mathbf{b}_0} \\ +(k_2^2-k_1^2)^2\int d^2\mathbf{b}'\, h_{\mathbf{b}-\mathbf{b}'}h_{\mathbf{b}'-\mathbf{b}_0} \\ \cdot[\mathbf{X}_{1\alpha}{}^{-}(\mathbf{b},0)G_0{}^{+}(\mathbf{b}',0,0)\mathbf{X}_{1\beta}{}^{+}(\mathbf{b}_0,0)]\Bigg\}.\end{aligned}\tag{4.5}$$

In (4.5), $G_0{}^{+}$ denotes the symmetric part of the Green's function (2.15), obtained by replacing the function $\theta(z)$ by 1/2. The antisymmetric part $G_0{}^{-}(\mathbf{b},z_1,z_2)\sim\mathrm{sign}(z_1-z_2)$ makes a contribution $E_{\alpha\beta}\sim h^3$ to the solution, and is not considered in the present approximation. The first two addends in Eq. (4.5) (those not involved with the integral) and given in a condensed symbolic form. For a more explicit representation, it should be noted that the derivatives $d\,\mathbf{X}_{j\gamma}{}^{\pm}(\mathbf{b},z)/dz$ are discontinuous at $z=0$. Additional integral terms in the profile function $h(\boldsymbol{\rho})$ of the form $[h(\boldsymbol{\rho})-\ldots]$ then appear in this equation, and these tend to zero upon subsequent rough-surface ensemble averaging; they are therefore omitted from (4.5).

Averaging (4.5) over an ensemble of rough surfaces with (3.3) taken into account, and substituting the result into (2.5), we obtain for the reflection coefficients $r_{\alpha\beta}$ from an initial state with polarization β to a final state with polarization α, to terms of order h^2,

$$\begin{aligned} r_{ss}=r_{s0}\Big\{1-2\eta_{10}\eta_{20}\sigma^2-2\eta_{10}(k_1^2-k_2^2) \\ \cdot\int\frac{d^2\mathbf{b}}{2\eta_1}S(\mathbf{b}-\mathbf{b}_0)\left[t_s-t_p\frac{\mathbf{b}^2}{k_1k_2}(\hat{\mathbf{b}}\hat{\mathbf{s}}_0)^2\right],\end{aligned}\tag{4.6}$$

$$\begin{aligned} r_{pp}=r_{p0}\Big\{1-2\eta_{10}\eta_{20}\sigma^2-2\eta_{10}(k_1^2-k_2^2)\int\frac{d^2\mathbf{b}}{2\eta_1}S(\mathbf{b}-\mathbf{b}_0) \\ \cdot\left[t_s-\frac{(\varepsilon_1-\varepsilon_2)(\varepsilon_1\varepsilon_2)^{1/2}}{\varepsilon_2{}^2\eta_{10}{}^2-\varepsilon_1{}^2\eta_{20}{}^2}t_p\left(\frac{\eta_1 b_0}{k_1^2}+\frac{\eta_{20}b}{k_2^2}\hat{\mathbf{b}}\hat{\mathbf{b}}_0\right)\right. \\ \left.\cdot(\eta_2 b_0-\eta_{20}b\hat{\mathbf{b}}\hat{\mathbf{b}}_0)\right]\Big\},\end{aligned}\tag{4.7}$$

$$\begin{aligned} r_{ps}=-r_{sp}=\frac{(k_1^2-k_2^2)^2}{2\eta_{10}k_1k_2^2}t_{s0}t_{p0} \\ \cdot\int\frac{d^2\mathbf{b}}{2\eta_1}S(\mathbf{b}-\mathbf{b}_0)t_p\mathbf{b}\hat{\mathbf{s}}_0[\eta_2 b_0-\eta_{20}b\hat{\mathbf{b}}\hat{\mathbf{b}}_0],\end{aligned}\tag{4.8}$$

where $\sigma^2=\langle h^2\rangle$ is the mean squared surface roughness, and the $r_{\alpha 0}$ are the smooth-surface reflection coefficients:

$$r_{s0}=\frac{\eta_{10}-\eta_{20}}{\eta_{10}+\eta_{20}},\qquad r_{p0}=\frac{\varepsilon_2\eta_{10}-\varepsilon_1\eta_{20}}{\varepsilon_2\eta_{10}+\varepsilon_1\eta_{20}}.\tag{4.9}$$

Equations (4.6)–(4.8) are quite general in nature, and hold for any homogeneous media in contact, which may have complex dielectric constants $\varepsilon_1(\omega)$ and $\varepsilon_2(\omega)$, arbitrary angles of incidence, and an arbitrary surface-roughness spectrum. The applicability of these equations is governed by the inequalities (4.2) and the requirement that corrections of order σ^2 in (4.6) and (4.7) be small compared to unity.

In the general case of anisotropic rough surfaces, the reflection coefficient matrix $r_{\alpha\beta}$ is nondiagonal, with $r_{ps}=-r_{sp}$. For isotropic surfaces, the cross terms r_{sp} and r_{ps} go to zero. The degree to which the nondiagonal components $r_{\alpha\beta}$ differ from zero is thus a measure of the anisotropy of the surface roughness spectrum.

The equations for the angular spectrum of the scattered wave (3.4) and the reflection coefficients (4.6)–(4.8) satisfy the energy conservation law

$$x_z{}^c+x_z{}^d=1\tag{4.10}$$

for all angles of incidence $0\leqslant\theta_0\leqslant\pi/2$, any incident polarization state, and any type of rough-surface spectrum, if only the media are nonabsorptive, $\mathrm{Im}\,\varepsilon_{1,2}=0$, and medium 2 is opaque.

The quantities $x_z{}^c$ and $x_z{}^d$ in (4.10) denote the total fraction of the power reflected in the direction of the normal to the plane $z=0$ for the coherent and incoherent components of the diffracted wave, respectively:

$$x_z{}^c=S_z{}^{\mathrm{ref}}/P_z{}^{\mathrm{inc}}=|r_{ss}C_{0s}+r_{sp}C_{0p}|^2+|r_{ps}C_{0s}+r_{pp}C_{0p}|^2,\tag{4.11}$$

$$x_z{}^d=\frac{1}{P_z{}^{\mathrm{inc}}}\int_0^{\pi/2}d\theta\int_0^{2\pi}d\varphi\sin\theta\cos\theta\frac{dP}{d\Omega},\tag{4.12}$$

where the coefficients $C_{0\alpha}$ in (4.11) are related by the normalization (3.5), and the angular spectrum $dP/d\Omega$ is given by (3.4). The fact that energy is conserved is an indication that Eqs. (3.4) and (4.6)–(4.8) are mutually consistent, and is a necessary condition for their validity.

In the limit $|\varepsilon_2-\varepsilon_1|\ll\varepsilon_1$, with a mildly sloping rough surface, if we expand (4.6)-(4.8) up to linear terms in $(\varepsilon_2-\varepsilon_1)$ and neglect terms $\sim(\mathbf{b}-\mathbf{b}_0)^2$, we obtain the asymptotic expressions of Ref. 10:

$$\begin{aligned} r_{\alpha\alpha}=r_{\alpha 0}\left[1-2\eta_{10}\int d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)(\eta_1-\eta_2+\eta_{20})\right], \\ r_{sp}=r_{ps}=0.\end{aligned}\tag{4.13}$$

Conversely, for good conductors with $|\varepsilon_2|\gg\varepsilon_1$, expanding (4.6)–(4.8) in powers of $\varepsilon_2{}^{-1/2}$, neglecting terms $\sim 1/\varepsilon_2$, and keeping the resonant behavior of the denominator intact, we obtain

$$r_{ss}=r_{s0}\Big\{1-2\eta_{10}\int\frac{d^2\mathbf{q}\,S(\mathbf{q})}{\eta_1+\varepsilon_1\eta_2/\varepsilon_2}\cdot\Big[\eta_1{}^2+(\mathbf{q}\hat{\mathbf{s}}_0)^2+\frac{\eta_1}{k_2}\Big(k_1{}^2+\frac{q^2}{2}\Big)+\ldots\Big]\Big\},$$

$$r_{pp}=r_{p0}\Big\{1-\frac{2\eta_{10}}{\eta_{10}{}^2-\varepsilon_1{}^2\eta_{20}{}^2/\varepsilon_2{}^2}\int\frac{d^2\mathbf{q}\,S(\mathbf{q})}{\eta_1+\varepsilon_1\eta_2/\varepsilon_2}\cdot\Big[\eta_1{}^2\eta_{10}{}^2+k_1{}^2(\mathbf{q}\hat{\mathbf{b}}_0)^2+\frac{\eta_1k_1{}^2}{k_2}\Big(\eta_{10}{}^2+\frac{q^2}{2}\Big)+\ldots\Big]\Big\},$$

$$r_{sp}=-r_{ps}=\frac{2\eta_{10}k_1}{\eta_{10}-\varepsilon_1\eta_{20}/\varepsilon_2}\cdot\int\frac{d^2\mathbf{q}\,S(\mathbf{q})}{\eta_1+\varepsilon_1\eta_2/\varepsilon_2}(\mathbf{q}\hat{\mathbf{s}}_0)(\mathbf{q}\hat{\mathbf{b}}_0)\Big(1-\frac{\eta_{10}}{\eta_{20}}+\ldots\Big).\tag{4.14}$$

For perfect conductors ($\varepsilon_2\to\infty$), equations (4.14) go into the asymptotic expressions obtained in Ref. 8.

Transforming from reflection coefficients to surface impedance [see Eq. (5.7) below] and making use of the expansion (4.14), we obtain the first terms in the expansion of the effective surface impedance in powers of the impedance of the medium[18], $(\varepsilon_1/\varepsilon_2)^{1/2}$.

5. SURFACE IMPEDANCE; POLES AND ZEROES OF $r_{\alpha\beta}$; DISPERSION RELATION FOR ELECTROMAGNETIC SURFACE WAVES

The foregoing equations (4.6)–(4.8) for the reflection coefficients form the basis for the derivation of other quantities pertinent to the optics of rough surfaces, such as the effective surface impedance $\zeta_{\alpha\beta}$, the ESW dispersion relation, and the shift in the Brewster angle. These topics are treated in the present section.

When p-polarized electromagnetic waves are reflected at the Brewster angle $\theta_0=\theta_B$ from a perfectly uniform interface between two dielectric media, the reflection coefficient r_{p0} of (4.9) vanishes if

$$\varepsilon_2\eta_{10}-\varepsilon_1\eta_{20}=0.\tag{5.1}$$

The solution of (5.1) is

$$b_0=k_0[\varepsilon_1\varepsilon_2/(\varepsilon_1+\varepsilon_2)]^{1/2}.\tag{5.2}$$

For reflection from a rough surface, $r_{pp}{}^B=r_{pp}|_{\theta_0=\theta_B}$ is nonzero in general. Using (5.1), Eq. (4.7) gives

$$r_{pp}{}^B=\frac{b_0}{2}k_0{}^2(\varepsilon_1-\varepsilon_2)^2\cdot\int\frac{d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}\Big(\frac{\eta_1}{k_1}+\frac{\mathbf{b}\hat{\mathbf{b}}_0}{k_2}\Big)\Big(\frac{\eta_2}{k_2}-\frac{\mathbf{b}\hat{\mathbf{b}}_0}{k_1}\Big),\tag{5.3}$$

where b_0 is given by (5.2). The difference between $r_{pp}{}^B\sim\sigma^2$ and zero is a measure of the roughness of the interface. This is exactly the range of angles for which polarimetric control methods are most sensitive to surface quality.[40]

We note in this regard an incorrect result in Ref. 41, where a correction of order σ^4, rather than σ^2 as in (5.3), was applied to the fundamental equation of ellipsometry, significantly degrading the result.

Equating (4.6) and (4.7) to zero, we obtain the equations which determine the locations of zeroes of the coefficients $r_{\alpha\alpha}$ in the complex plane:

$$\eta_{10}-\eta_{20}=k_0{}^2(\varepsilon_1-\varepsilon_2)^2\int\frac{d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}[\eta_1\eta_2+(\mathbf{b}\hat{\mathbf{b}}_0)^2],\tag{5.4}$$

$$\varepsilon_2\eta_{10}-\varepsilon_1\eta_{20}=(\varepsilon_1-\varepsilon_2)^2\int\frac{d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}(\mathbf{b}\mathbf{b}_0-\eta_{10}\eta_2)(\mathbf{b}\mathbf{b}_0+\eta_1\eta_{20}).\tag{5.5}$$

Terms of order σ^4 have been discarded from these equations.

Other authors[16–18] have calculated the effective surface impedance $\zeta_{\alpha\beta}$ of a rough surface in the impedance approximation, which is only applicable[24] when $|\varepsilon_2|\gg\varepsilon_1$. Below, we derive expressions giving $\zeta_{\alpha\beta}$ for media in contact, having arbitrary ε_j. Defining $\zeta_{\alpha\beta}$ through the equation at the boundary $z=0$

$$\varepsilon_1{}^{1/2}E_\alpha{}^c(\boldsymbol{\rho},0)=\zeta_{\alpha\beta}[\hat{\mathbf{z}}\mathbf{H}^c(\boldsymbol{\rho},0)]_\beta,\tag{5.6}$$

($\alpha,\beta=s,p$), where $\mathbf{E}^c\ldots\mathbf{H}(\mathbf{r})\rangle$ are the coherent components of the fields, substituting (2.5), and solving for $\zeta_{\alpha\beta}$, we obtain the effective surface impedance of a rough surface in terms of the reflection coefficients $r_{\alpha\beta}$:

$$\begin{gathered}\zeta_{ss}=(k_1/\eta_{10}Z)[(1+r_{ss})(1+r_{pp})-r_{sp}r_{ps}],\\ \zeta_{sb}=2r_{sp}/Z,\\ \zeta_{bb}=(\eta_{10}/k_1Z)[(1-r_{ss})(1-r_{pp})-r_{sp}r_{ps}],\\ \zeta_{bs}=-2r_{ps}/Z,\\ Z=(1-r_{ss})(1+r_{pp})+r_{sp}r_{ps}.\end{gathered}\tag{5.7}$$

Substituting (4.6)–(4.8) into (5.7) and retaining terms of order σ^2, we have

$$\zeta_{ss}=\frac{k_1}{\eta_{20}}\Big\{1+\frac{(k_2{}^2-k_1{}^2)}{\eta_{20}}\int d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)\cdot\Big[\eta_{20}+\eta_1-\eta_2-\frac{(\varepsilon_1-\varepsilon_2)(\mathbf{b}\hat{\mathbf{s}}_0)^2}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}\Big]\Big\},$$

$$\zeta_{bb}=\frac{k_1\eta_{20}}{k_2{}^2}\Big\{1+\frac{1}{\eta_{20}}\int d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)\Big[\Big(\frac{\varepsilon_2}{\varepsilon_1}\eta_{10}{}^2-\frac{\varepsilon_1}{\varepsilon_2}\eta_{20}{}^2\Big)\cdot(\eta_{20}+\eta_1-\eta_2)-\frac{(\varepsilon_1-\varepsilon_2)^2}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}\Big(\frac{\eta_1b_0}{\varepsilon_1}+\frac{\eta_{20}}{\varepsilon_2}\mathbf{b}\hat{\mathbf{b}}_0\Big)(\eta_2b_0-\eta_{20}\mathbf{b}\hat{\mathbf{b}}_0)\Big]\Big\},$$

$$\zeta_{sb}=\zeta_{bs}=\frac{k_1(\varepsilon_1-\varepsilon_2)^2}{\eta_{20}\varepsilon_2}\int\frac{d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}\mathbf{b}\hat{\mathbf{s}}_0[\eta_{20}\mathbf{b}\hat{\mathbf{b}}_0-\eta_2b_0].\tag{5.8}$$

Equations (5.8) are valid for arbitrary ε_j; they are the same as the result in Ref. 44, and for $|\varepsilon_2|\gg\varepsilon_1$ they are equivalent to the asymptotic equations obtained in Refs. 17 and 18.

Equations (4.6)–(4.8) for the reflection coefficients determine the positions of the zeroes of $r_{\alpha\beta}$ only, and not the poles. It is well known that the latter are given by the ESW dispersion relation at the rough surface. To produce the dispersion relation, we solve Eq. (5.7) for $r_{\alpha\beta}$, obtaining

$$r_{ss}=\frac{1}{R}\left[\left(\frac{\eta_{10}}{k_1}\zeta_{ss}-1\right)\left(1+\frac{k_1}{\eta_{10}}\zeta_{bb}\right)-\zeta_{sb}\zeta_{bs}\right],\quad r_{sp}=\frac{2}{R}\zeta_{sb},\tag{5.9}$$

$$r_{pp}=\frac{1}{R}\left[\left(\frac{\eta_{10}}{k_1}\zeta_{ss}+1\right)\left(1-\frac{k_1}{\eta_{10}}\zeta_{bb}\right)+\zeta_{sb}\zeta_{bs}\right],\quad r_{ps}=-\frac{2}{R}\zeta_{bs},$$
$$R=\left(1+\frac{\eta_{10}}{k_1}\zeta_{ss}\right)\left(1+\frac{k_1}{\eta_{10}}\zeta_{bb}\right)-\zeta_{sb}\zeta_{bs}.\tag{5.10}$$

If we neglect cross terms $\zeta_{sb}\zeta_{bs}$ in (5.10), which are of order σ^4, the equation for the poles of $r_{\alpha\beta}$, $R=0$, separates into two. Substituting (5.8) and neglecting terms $\sim\sigma^4$, we obtain

$$\eta_{10}+\eta_{20}=-k_0^2(\varepsilon_1-\varepsilon_2)^2\int\frac{d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}[\eta_1\eta_2+(\mathbf{b}\hat{\mathbf{b}}_0)^2],\tag{5.11}$$

$$\varepsilon_2\eta_{10}+\varepsilon_1\eta_{20}=-(\varepsilon_1-\varepsilon_2)^2\int\frac{d^2\mathbf{b}\,S(\mathbf{b}-\mathbf{b}_0)}{\varepsilon_2\eta_1+\varepsilon_1\eta_2}(\mathbf{b}\mathbf{b}_0+\eta_{10}\eta_2)(\mathbf{b}\mathbf{b}_0+\eta_1\eta_{20}).\tag{5.12}$$

Equation (5.11), if it has solutions, is the dispersion equation for s-polarized electromagnetic surface waves induced by surface roughness; it is of the same form as that obtained in Ref. 21.

Equation (5.12) is the dispersion equation for p-polarized electromagnetic surface waves at a rough surface, and corresponds to the analogous equation derived in Refs. 3, 7, 19–21; it is not consistent, however, with Refs. 22 and 23.

The derivation of the dispersion equations (5.11) and (5.12) relied on the concept of impedance. We can directly derive these same equations diagrammatically,[42] starting with Eq. (2.14) for the scattering T-matrix, by writing the perturbation v in the linear approximation (2.10), averaging the iterative solution (2.14) over an ensemble of rough surfaces, assigning all unreduced diagrams to the mass operator, and summing reduced diagrams. If we then calculate the mass operator to order σ^2, we obtain equations equivalent to (5.11) and (5.12) for the poles of the $\langle T\rangle$-matrix.

Equations (5.11) and (5.12) reduce to the equations for the locations of the zeroes, (5.4) and (5.5), when we make the replacement $\eta_{10}\to-\eta_{10}$, i.e., to every root of (5.4), (5.5) with $\eta_{10}=\eta_{1c}$ in the complex η_{10} plane, there corresponds a root of (5.11), (5.12) with $\eta_{10}=-\eta_{10}$. This symmetry of the locations of poles and zeroes of the reflection coefficients is a manifestation of the general properties of analytic scattering matrices.[43] If we neglect cross terms $\zeta_{sb}\zeta_{bs}\sim\sigma^4$ in (5.9) and (5.10), and making use of (5.8), the reflection coefficients become

$$r_{ss}=\frac{\eta_{10}(1+I_s)-\eta_{20}}{\eta_{10}(1+I_s)+\eta_{20}},\quad r_{pp}=\frac{\varepsilon_2\eta_{10}-\varepsilon_1\eta_{20}(1+I_p)}{\varepsilon_2\eta_{10}+\varepsilon_1\eta_{20}(1+I_p)},$$
$$r_{sp}=-r_{ps}=\frac{2\varepsilon_2\eta_{10}\eta_{20}\zeta_{sb}}{[\eta_{10}(1+I_s)+\eta_{20}][\varepsilon_2\eta_{10}+\varepsilon_1\eta_{20}(1+I_p)]},\tag{5.13}$$

where I_s and I_p are the integral terms in curly brackets in (5.8) corresponding to ζ_{ss} and ζ_{bb} respectively. For a smooth surface, $I_s=I_p=0$, and (5.13) reduces to (4.9). In contrast to the original equations (4.6)–(4.8), the expressions comprising (5.13) satisfy the analyticity requirements for the scattering matrix,[43] and correctly determine the location of poles and zeroes of the reflection coefficients for electromagnetic waves scattered by a rough surface.

6. APPLICABILITY

The applicability of the foregoing expressions for the angular spectrum (3.4) and reflection coefficients (4.6)–(4.8) is constrained by the inequalities (4.2), in which we can put $h_m\sim\sigma$ for purposes of estimation. The physical justification for these conditions derives directly from their derivation: the characteristic transverse roughness scale σ must be small compared with the typical transverse scale of irregularities for all fields both in medium 1 and medium 2—for the incident and specularly reflected waves [first condition of (4.2)], for scattered waves [second condition of (4.2)], and for all waves in some intermediate state [third condition of (4.2)].

In medium 1, the first inequality of (4.2) reduces to the conventional Rayleigh condition

$$k_1\sigma\cos\theta_0\ll1,\tag{6.1}$$

and in medium 2, it reduces to

$$k_1\sigma(\varepsilon-\sin^2\theta_0)^{1/2}\ll1,\tag{6.2}$$

where $\varepsilon=\varepsilon_2/\varepsilon_1$ is relative dielectric constant. As a rule, (6.2) is a more rigorous constraint than (6.1). For example, for metals in the near- and mid-infrared, where $|\varepsilon|$ is large, (6.2) reduces to $\sigma\ll\delta$, where δ is the skin depth.

The second inequality of (4.2) is reduced to (6.1), (6.2) by replacing the angle of incidence θ_0 with the scattering angle θ.

$$k_1\sigma\cos\theta\ll1,\ k_1\sigma(\varepsilon-\sin^2\theta)^{1/2}\ll1.\tag{6.3}$$

The form taken by the third inequality of (4.2) depends on the nature of the surface roughness. The characteristic scale of variations in the variable b' is governed by the weight assigned to the intermediate states of the integral terms in (4.1), (4.5) or (4.6)–(4.8). This weighting factor is the perturbation $v(\mathbf{q},z)$ in (4.1), the profile variation $h_\mathbf{q}$ in (4.5), or the spectral density $S(\mathbf{q})$ in (4.6)–(4.8). For single-scale rough surfaces, a typical value of q is determined by the correlation length l, such that $ql\lesssim1$. If we then consider only the coherent component of the diffracted wave and let

$\mathbf{b}' = \mathbf{b}_0 + \mathbf{q}$ in (4.2), we transform the third inequality of (4.2) for $j = 1$ and 2 to the form

$$\sigma[\eta_{10}^2-2(\mathbf{q}\mathbf{b}_0)-q^2]^{1/2}\ll 1, \tag{6.4}$$

$$\sigma[\eta_{20}^2-2(\mathbf{q}\mathbf{b}_0)-q^2]^{1/2}\ll 1. \tag{6.5}$$

For typical values $q \sim 1/\mathbf{l}$, (6.4) reduces to

$$k_1\sigma[\cos^2\theta_0-2(k_1 l)^{-1}\sin\theta_0-(k_1 l)^{-2}]^{1/2}\ll 1. \tag{6.6}$$

This implies that for large-scale roughness $k_1\mathbf{l} \gg 1$ and moderately large angles of incidence, with $\cos\theta_0 > (2/k_1 l)^{1/2}$, (6.4) reduces to (6.1), and for grazing propagation, with $\cos\theta_0 < (2/k_1 l)^{1/2}$, we obtain instead of (6.1) the stricter constraint

$$\sigma(2/k_1 l)^{1/2}\ll 1. \tag{6.7}$$

For fine-scale roughness $k_1 l \ll 1$ and arbitrary angles of incidence, (6.6) gives the constraint $\sigma/l \ll 1$.

A similar evaluation of the left-hand side of (6.5) leads to the following constraints. For large-scale roughness $k_1 l \gg 1$ and

$$k_1 l|\varepsilon-\sin^2\theta_0|>1, \tag{6.8}$$

(6.5) reduces to (6.2). Condition (6.8) is violated in the neighborhood of the critical angle $\theta_0 \sim \theta_c = \arcsin \varepsilon^{1/2}$, upon reflection of a wave from an optically less dense dielectric medium with $0 < \varepsilon < 1$. If then the converse inequality $k_1 l\,|\varepsilon - \sin^2\theta_0| < 1$ is satisfied, (6.5) will reduce to (6.7).

For fine-scale roughness $k_1 l \ll 1$, and if $k_1 l(\varepsilon - \sin^2\theta_0)^{1/2} > 1$, (6.5) reduces to (6.2), and conversely, if $k_1 l(\varepsilon - \sin^2\theta_0)^{1/2} < 1$, it reduces to the condition $\sigma/l \ll 1$.

It must be noted that (6.2), (6.3), and (6.5) are violated by perfectly conducting media. Formally, this occurs because the perturbation (2.3) diverges, and there is a radical change of behavior in the zeroth-order fields at the interface $z = 0$. For example, the z-component of the displacement vector will no longer be continuous: it will be a discontinuous function of z. The way in which the problem of light scattering by a rough surface is posed is then fundamentally altered.[45]

However, notwithstanding the violation of conditions (6.2), (6.3), and (6.5), the expressions comprising (3.4) and (4.6)–(4.8) remain in force for a surface with only minor roughness [see (4.14)], and these are consistent with the corresponding expressions in Refs. 8 and 9, where a basis system consisting of field functions in perfect conductors was used from the outset.

CONCLUSION

In this paper, the diffraction of electromagnetic waves by a rough interface between two homogeneous isotropic media has been reduced to the solution of a conventional potential-scattering problem of quantum mechanics.[39] We have formulated Eq. (2.14) for the scattering T-matrix, which in contrast to approaches developed in Refs. 29, 33, and 35, is mathematically well-posed to any order of perturbation theory, is valid for any amplitude of roughness $h(\mathbf{p})$, and does not rely on the Rayleigh hypothesis. The solution for the components of the diffracted field $E_{\alpha\beta}(\mathbf{b},\mathbf{b}_0)$ is given in the form of matrix elements (2.12) of the scattering operator T. The fields $\mathbf{X}_{\gamma j}^{\pm}(\mathbf{b}, z)$ which enter into the solution (2.12) and the Green's function (2.15) vary smoothly over the interface separating the media at $z = 0$. This is our main result.

A direct iterative solution of the equation for the T-matrix leads to a series expansion of the diffracted field in powers of the perturbation of the dielectric constant, and not to the usual expansion in powers of the roughness amplitude $h(\mathbf{p})$. Each iterate of (2.14) is then a nonlinear and nonanalytic function of the surface profile. Other approaches do not make use of terms with nonanalytic profile dependences of the form $(h_1 - h_2)|h_1 - h_2|$, where $h_j = h(\mathbf{p}_j)$ [see the discussion following (4.5)].

In Secs. 3 and 4, we presented the linear and quadratic approximations, respectively, to a conventional solution in powers of the roughness amplitude. Averaging over an ensemble of rough surfaces provides expressions for the angular spectrum (3.4) and the reflection coefficients (4.6)–(4.8). In the quadratic approximation ($\sim h^2$), these equations satisfy the energy conservation condition (4.10) for all incident and scattered polarizations, all angles of incidence $0 \leqslant \theta_0 \leqslant \pi/2$, and any surface-roughness spectrum. From the general expressions (4.6)–(4.8), which hold for any contiguous media, we have obtained expressions for the reflection coefficients $r_{\alpha\beta}$ in the limits $|\varepsilon_2 - \varepsilon_1| \ll \varepsilon_1$ (4.13) and $|\varepsilon_2| \gg \varepsilon_1$ (4.14), and these are consistent with previous results obtained by independent means. A comparison of the general expressions for $r_{\alpha\beta}$ with existing results[11–15,44] indicates that over the whole space, i.e., for all incident β-polarizations and reflected α-polarizations ($\alpha, \beta = s, p$), the only results consistent with (4.6)–(4.8) are those in Ref. 44, if the media in the latter are assumed isotropic, and the corresponding equations are reduced to the form of (4.6)–(4.8). The results in the other papers are incomplete and only partially consistent with (4.6)–(4.8): all agree with (4.6) for the coefficient r_{ss}; for r_{pp}, Eq. (4.7) is in agreement with Refs. 12 and 13, and in disagreement with Refs. 14 and 15; for r_{ps}, Eq. (4.8) is in agreement with Ref. 11.

The applicability of our results is restricted by an overall requirement that the characteristic transverse scale of surface roughness be small compared with the characteristic transverse scale of all field irregularities in the two contiguous media.

The iterative solution method for the T-matrix employed in Secs. 3 and 4 leads to a representation for the solution which does not satisfy the general requirements of S-matrix analyticity. As a result, the derived reflection coefficients give incorrect pole locations [in (4.6)–(4.8), the poles are located on the plane interface]. Section 5 provides a transformation of the equations for the reflection coefficients to the form (5.13), which has the contract analyticity properties and the proper symmetry of pole and zero locations; the derivation makes use of the concept of impedance (5.6). Since the iterative solution for the coherent field obtained in Section 4 is already known, (5.6) is the final equation needed to calculate the equivalent effective surface impedance of a rough surface. To accuracy $\sim h^2$, then, we have obtained expressions for the impedance (5.6) and dispersion equations (5.11), (5.12), and equations which determine the location of zeroes (Brewster angle) (5.4), (5.5). The derivation of the dispersion equations through the impedance concept is easier than the diagram summing technique employed in Ref. 22

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