Nonlinear rf plasma oscillations in crossed fields

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For certain beam charge density profiles, the self-magnetic field stabilizes nonlinear electron oscillations of the plasma. A new mode which exhibits a threshold-saturation effect is found.

A topic of current interest in the effort to solve several problems involved in the production and transport of intense electron beams[1-3] and in the use of these beams to excite oscillations in magnetized plasmas[4-6] is the stability of highly nonlinear rf oscillations of an axisymmetric charged column of cold plasma which is oriented along the external magnetic field. The longitudinal dimension of the column is usually much greater than the transverse dimension, and the system can be regarded as homogeneous along the direction of the magnetic field. For time-varying processes with times $\sim\omega_{pe}^{-1}$ (ω_{pe} is the electron plasma frequency), the ions are assumed to be immobile, and the stream velocity of the electrons is assumed to be much higher than the thermal velocity. The problem is thereby reduced to one of studying the nonlinear dynamics of a cold electron plasma in a radial electric field crossed with longitudinal and azimuthal magnetic fields. This dynamics has been studied thoroughly only in the case $\omega_{pe} \ll \omega_{Be}$, where ω_{Be} is the electron cyclotron frequency. The case $\omega_{pe} \sim \omega_{Be}$ is of interest in many applications. In this case, the plasma diamagnetism must be taken into consideration, and the resultant magnetic field cannot be regarded as uniform. The gradient which arises in the magnetic pressure may cause a rapid growth of oscillations and the ultimate formation of discontinuities in the distributions of the charge density and the stream velocity. Under certain conditions, however, the inverse process may also occur (an increase in the drift current reduces the pressure gradient) and stabilize the stream. These processes can be described correctly only in a self-consistent system.

In the present paper, a description of this sort is achieved on the basis of the nonrelativistic hydrodynamic equations of a cold plasma. We derive a class of exact solutions which describe charge and current profiles that are stable against nonlinear, slightly nonelectrostatic oscillations of the plasma electrons.

1. We consider the motion of the electrons in plane geometry, using the coordinate system x,y,z, where z and x are longitudinal and transverse coordinates, and the system is uniform along y and z. We assume that the charged stream with immobile ions, $N_i = \text{const}$, is oriented parallel to the external magnetic field, which is longitudinally uniform. The motion of the electrons in this case is a drift along y and z, while along x it takes the form of oscillations. Under the assumption that the electron stream velocity is substantially less than the velocity of light, c (under this assumption we can ignore the y and z components of the displacement current), we use the nonrelativistic hydrodynamic equations of a cold plasma in order to find a description of the stream:

$$\partial v_x/\partial t+v_x\partial v_x/\partial x=E_x+v_yB_z-v_zB_y, \tag{1}$$

$$\partial v_y/\partial t+v_x\partial v_y/\partial x=E_y-v_xB_z, \tag{2}$$

$$\partial v_z/\partial t+v_x\partial v_z/\partial x=E_z+v_xB_y, \tag{3}$$

$$n=1+\partial E_x/\partial x, \tag{4}$$

$$\partial n/\partial t+\partial(nv_x)/\partial x=0, \tag{5}$$

$$\partial B_y/\partial x=nv_z, \quad -\partial B_z/\partial x=nv_y,$$
$$\partial B_y/\partial t=\partial E_z/\partial x, \quad -\partial B_z/\partial t=\partial E_y/\partial x. \tag{6}$$

Here $x = x'\omega_{pe}/s$; $t = t'\omega_{pe}$; x' and t' are the coordinate and the time; $n = N_e/N_i$, where N_e is the electron density; and $\omega_{pe}^2 = 4\pi e^2 N_i/m$, where e and m are the charge and mass of the electron.

We express the components of the magnetic field and of the solenoidal electric field in terms of the vector potential $A = (0, A_y, A_z)$:

$$E_y=-\partial A_y/\partial t, \quad E_z=-\partial A_z/\partial t,$$
$$B_y=-\partial A_z/\partial x, \quad B_z=\partial A_y/\partial x.$$

Transforming to Lagrangian variables $\tau = t$, $\psi(x,t)$ in Eqs. (1)–(5) (in this case we have $v_x = \partial x/\partial\tau$, $n^{-1} = \partial x/\partial\psi$), we find

$$\frac{\partial^2 x}{\partial\tau^2}+x+\frac{1}{2}\frac{\partial}{\partial\psi}(B_y^2+B_z^2)=\psi+f''(\tau)+f(\tau),$$
$$v_y+A_y=P_y(\psi), \quad v_z+A_z=P_z(\psi),$$
$$\partial B_z/\partial\psi=A_y-P_y(\psi), \quad \partial B_y/\partial\psi=-A_z+P_z(\psi), \tag{7}$$

where the function $f(\tau)$ characterizes the coupling of the system with the source of external forces, $P_y(\psi)$ and $P_z(\psi)$ are arbitrary functions of ψ, and the prime means differentiation with respect to τ.

Differentiating the first equation in (7) with respect to ψ, and differentiating the second pair of equations with respect to x, we find

$$\frac{\partial^2 n^{-1}}{\partial\tau^2}+n^{-1}+\frac{1}{2}\frac{\partial^2}{\partial\psi^2}(B_y^2+B_z^2)=1, \tag{8}$$

$$n^{-1}=B_z^{-1}\left(\frac{\partial^2 B_z}{\partial\psi^2}+P_y'\right)=B_y^{-1}\left(\frac{\partial^2 B_y}{\partial\psi^2}-P_z'\right). \tag{9}$$

Eliminating $n^{-1}(\psi,\tau)$ from (8), (9), we find two equations for B_y and B_z. Incidentally, the system (8), (9) is also valid for a purely electron stream in vacuum. In the other case, the second term would be absent from Eq. (8).

We seek a solution of Eqs. (8), (9) in the form

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$$B_y=F^{-1}(\tau)B(\psi)\cos\Theta(\psi),\quad B_z=F^{-1}(\tau)B(\psi)\sin\Theta(\psi) \tag{10}$$

(B, Θ, and F are arbitrary functions). Substituting these expressions into (9), we find

$$P_y'\sin\Theta+P_z'\cos\Theta=0,\quad \Theta'=\mu B^{-2}. \tag{11}$$

Here and below, the prime means differentiation with respect to the corresponding independent variable, and μ is a constant of integration. We can separate variables in (8) by setting

$$F''+F+k^{-1}F^{-2}=\lambda, \tag{12}$$

where k and λ are arbitrary constants. In this case we find

$$B''+\lambda kB(1+\lambda kB^2)^{-1}B'^2=(1+\Theta'^2)B(1+\lambda kB^2)^{-1},$$
$$P_y'\cos\Theta-P_z'\sin\Theta=kB\sin 2\Theta(BB''+B'^2). \tag{13}$$

Equation (13) for B has a first integral:

$$B'^2=(C_0+B^2-\mu^2B^{-2})(1+\lambda kB^2)^{-1}, \tag{14}$$

where C_0 is a constant of integration.

The system (11), (13), (14) determines the functions P_y, P_z, B, and Θ unambiguously. The solution (10) therefore describes a class of time-dependent one-dimensional configurations which are characterized by the three parameters μ, λ, k. The parameter μ determines the relative contribution of the Lorentz forces set up by the current-density components j_y and j_z (for $\mu=0$ and $\Theta=\pi/2$, there is no longitudinal component j_z, and we have $B_y=0$). The parameter λ is a normalization parameter; the parameter k relates the spatial shape of the solutions with their evolutionary characteristics. Without any loss of generality we will restrict the analysis to the two values $\lambda=1$ and $\lambda=0$.

Let us examine solutions of (12) with $\lambda=1$. The function of $F'(F)$ has two branches, which describe solutions with positive and negative F, respectively . The shape of these branches depends on the sign and magnitude of k. The local extrema of the curve, which correspond to equilibrium points of the system, are determined by the equation

$$F_{eq}^3-F_{eq}^2+k^{-1}=0,$$

which has one and three real roots under the respective conditions

$$Q=3^{-3}k^{-2}(3^3/4-k)\gtreqless 0\,.$$

For $Q\geqslant 0$, i.e., for $k\geqslant 3^3/4$, the unique root is

$$F_{eq}=-3^{-1}(a^{1/3}-1+a^{-1/3}),$$
$$a=3^3/2k-1-[(3^3/2k-1)^2-1]^{1/2},$$

and its sign is the opposite of that of k. Solutions described by the branch containing the point F_{eq} are periodic. On the other branch we have $F(\tau\to\infty)\to\infty$. For $Q<0$, one of the roots is negative, while the two others are positive. Since we have $k^{-1}\ll 1$ in this case, we find

$$F_{1eq}\approx 1-k^{-1},\quad F_{2,3eq}\approx\mp k^{-1/2}.$$

Values $F_{ieq}\approx 1-k^{-1}$ and $F_{2eq}\approx -k^{-1/2}$ correspond to a stable equilibrium. Both branches of the $F'(F)$ curve at $k>3^3/4$ contain periodic solutions. Figure 1 shows the form of the curve $F'(F)$ for $k>0$.

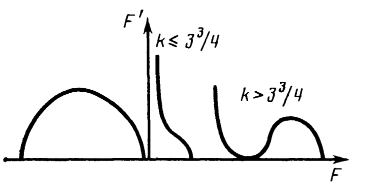

FIG. 1.

Finally, with $\lambda=0$ there exists a single root $F_{eq}=-k^{-1/3}$, regardless of the value of k.

Let us find the oscillation frequencies for small deviations of the system from equilibrium positions. Setting

$$F=F_{eq}+\delta F(\tau),\quad \delta F\ll F_{eq},$$

in (12), and linearizing this equation, we find

$$\delta F''+(1-2k^{-1}F_{eq}^{-3})\delta F=0. \tag{15}$$

With $\lambda=0$ we have $kF_{eq}^3=-1$; the oscillation frequency is $\Omega=3^{1/2}$, independent of k. With $\lambda=1$ and $k>3^3/4$, we have the following results for stable equilibrium points:

$$F_{1eq}\approx 1-k^{-1},\quad \Omega_1=1-(2k)^{-1},\quad F_{2eq}\approx -k^{-1/2},\quad \Omega_2=(2k^{1/2})^{1/2}.$$

We now consider a purely electron stream, i.e., $N_i=0$; we will compare the results in this case with those of the case discussed above. In the absence of ions the condition under which variables can be separated in (8),(9) reduces to an equation for $F(\tau)$:

$$F''+k^{-1}F^{-2}=\lambda.$$

Periodic solutions exist here only for $\lambda=1$ and $k>0$. In this case we have $F_{eq}=-k^{1/2}$ and $\Omega=(2k^{1/2})^{1/2}$, corresponding to the left branch in Fig. 1 of the solution discussed above.

We return to the analysis of the solutions with $N_i\neq 0$. In the case of three real roots, and if we choose the constant of integration to be

$$C_1=F_{eq}(3F_{eq}-4),$$

we find that the first integral of Eq. (12) can be put in the form

$$F'^2=F^{-1}[2(1-F_{eq})-F](F-F_{eq})^2.$$

This representation is correct if $F>0$, i.e., for $F_{3eq}\approx k^{1/2}$. It describes, in particular, finite solutions on the right branch of the $F'(F)$ curve in Fig. 1. Here $F(\tau)$ is expressed implicitly in terms of elementary functions:

$$\begin{aligned}\tau+\tau_0=2\arcsin[F/2(1-k^{-1/2})]^{1/2}\\ +[2(k^{1/2}-1)(2k^{1/2}-3)]\ln\{F^{1/2}(2k^{1/2}-3)^{1/2}\\ +[2(1-k^{-1/2})-F]^{1/2}\}\{F^{1/2}(2k^{1/2}-3)^{1/2}-[2(1-k^{-1/2})-F]^{1/2}\}^{-1},\end{aligned} \tag{16}$$

where the range of $F(\tau)$ is $k^{-1/2}\leqslant F\leqslant 2(1-k^{-1/2})$.

Since in a real plasma there will be a time scale which limits the applicability of our original equations, (1)–(6), we can interpret configurations with $k>0$ and $F(\tau)$ as quasisteady configurations.

Analysis shows that when we take account of the self-consistent influence exerted on the stream by the magnetic field set up by the drift currents of the system, there can be solutions whose evolutionary characteristics take the form of stable nonlinear oscillations. In contrast with purely electrostatic oscillations in a longitudinal external magnetic field, these oscillations are anharmonic. Furthermore, for certain relations among the parameters of the system, the spectrum acquires a new branch, which exhibits a threshold saturation. When the threshold is exceeded, the system loses its stability on this branch.

We turn now to the spatial distributions of the function of the system which correspond to solutions of (12). From (1), (7)–(9), (13), and (14) we find the following expressions for the charge density, x, Θ, the velocity components, and the components of the electric field:

$$n^{-1}=1+k(F-\lambda)(\lambda kB^4+2B^2+\lambda k\mu^2+C_0)(1+\lambda kB^2)^{-2},$$
$$x=kF(B^4+C_0B^2-\mu^2)^{1/2}(1+\lambda kB^2)^{-1/2}+(1-\lambda kC_0-\lambda^2k^2\mu^2)\cdot\int_{B(x=0)}^{B}(1+\lambda k\tilde{B}^2)^{-3/2}(\tilde{B}^4+C_0\tilde{B}^2-\mu^2)^{-1/2}\tilde{B}\,d\tilde{B}+f(\tau),$$
$$\Theta=\Theta_0+\mu\int_{B(x=0)}^{B}(1+\lambda k\tilde{B}^2)^{1/2}(\tilde{B}^4+C_0\tilde{B}^2-\mu^2)^{-1/2}\tilde{B}^{-1}\,d\tilde{B},$$
$$v_x=kF'(B^4+C_0B^2-\mu^2)^{1/2}(1+\lambda kB^2)^{-1/2}+f'(\tau),\qquad(17)$$
$$v_y=-B^{-1}F^{-1}[(B^4+C_0B^2-\mu^2)^{1/2}(1+\lambda kB^2)^{-1/2}\sin\Theta+\mu\cos\Theta],$$
$$v_z=B^{-1}F^{-1}[(B^4+C_0B^2-\mu^2)^{1/2}(1+\lambda kB^2)^{-1/2}\cos\Theta-\mu\sin\Theta],$$
$$E_x=k(\lambda-F)(B^4+C_0B^2-\mu^2)^{1/2}(1+\lambda kB^2)^{-1/2}-f''(\tau),$$
$$E_y=F'B^{-1}F^{-2}[(B^4+C_0B^2-\mu^2)^{1/2}(1+\lambda kB^2)^{-3/2}(1+kFB^2)\sin\Theta+\mu\cos\Theta]+f'BF^{-1}\sin\Theta$$
$$E_z=-F'B^{-1}F^{-2}[(B^4+C_0B^2-\mu^2)^{1/2}(1+\lambda kB^2)^{-3/2}(1+kFB^2)\cos\Theta-\mu\sin\Theta]-f'BF^{-1}\cos\Theta.$$

Here Θ_0 is a constant of integration, and we have chosen $dB/d\psi\geqslant0$ in (14). This choice corresponds to a specification of the solutions in one of the regions $x\geqslant0$, $x\leqslant0$. Expressions (17) along with (10), (12) determine an implicit dependence of the functions of the system on x and t. At equilibrium we have $F'=f'=0$ and $F=F_{eq}$, and the equation $E_x+v_yB_z-v_zB_y=0$ holds. Let us assume that at $F=F_{eq}$ the distribution of the charge density n_0 is uniform. It follows from (17) that an equilibrium of this sort is realized only for $\lambda=0$ and $kC_0=1-k^2\mu^2$. In this case we have $B^2_{eq}=k\mu^2+x^2/kF^2_{eq}$ and the parameters k and F_{eq} can be expressed in terms of n_0:

$$F_{eq}=n_0^{-1},\quad k=n_0^3(n_0-1)^{-1}.$$

Since we have $k>0$ here, we also have $n_0>1$. With $n_0>1$, the functional dependence $k(n_0)$ is double-valued and has a minimum $k_{min}(n_0=3/2)=3^3/4$.. The value k_{min} corresponds to the value at which the potential well disappears from the right branch of $F'(F)$ in Fig. 1, and the system loses its stability. For $n_0>3/2$ we have $F_{eq}\gtrsim k^{-1/2}$, and for $n_0<3/2$ we have $F_{eq}\approx1-k^{-1}$; i.e., positions of respectively unstable and stable equilibria are realized.

This solution can be used to describe oscillations in a ribbon-shaped diamagnetic beam with a uniform charge-density profile which has sharp inner and outer boundaries in the transverse direction. The equilibrium of a stream of this sort is characterized by a drift with velocities which are constant over the cross section:

$$v_{yeq}=-J_y(J_y^2+J_z^2)^{-1/2}(1-n_0^{-1})^{1/2},$$
$$v_{zeq}=J_z(J_y^2+J_z^2)^{-1/2}(1-n_0^{-1})^{1/2}$$

(J_y and J_z are the total drift current and the total longitudinal current) in linearly increasing electric and magnetic fields. The equilibrium transverse dimension of the beam, θ_0, can be expressed in terms of n_0, J_y, and J_z:

$$l_0=(J_y^2+J_z^2)^{1/2}n_0^{-1/2}(n_0-1)^{1/2}.$$

In the time-dependent case, the condition $n_0>1$ means that we have only the right branch of the $F'(F)$ curve in Fig. 1, and the oscillation amplitude is limited by the condition $F(\tau)<1$, which clearly holds near equilibrium points. The functions E_x, B_y, B_z, v_y, v_z, and n oscillate around their equilibrium positions, while remaining constant in sign. The laws of motion of the outer and inner boundaries of the stream are determined from the conditions for matching with the external magnetic field and from the condition that the total current in the system is conserved.

We turn now to a study of configurations with a nonuniform charge-density profile. The basic properties of these configurations are determined by the values of the parameter k. In addition to the case discussed above, the integrals in (17) can be calculated in terms of elementary functions only for $\lambda=0$ and $\lambda=1$, $C_0=0$, and $\mu=0$ (the latter case is of no practical interest). They can be estimated approximately, however, in the limiting cases $k\gg3^3/4$ and $k\sim0$. We express the parameters k, C_0, and μ^2 in terms of the characteristics of the system in equilibrium. For definiteness we consider a solid beam with sharp transverse boundaries, $-x_b\leqslant x\leqslant x_b$, with a charge-density distribution which is symmetrical with respect to x. In this case, with $\partial/\partial t\neq0$, the components of the velocity and of the electric and magnetic fields in (10) and (17) must satisfy the conditions

$$v_x(-x)=-v_x(x),\quad v_y(-x)=-v_y(x),$$
$$v_z(-x)=v_z(x),\ E_x(-x)=-E_x(x),$$
$$E_y(-x)=-E_y(x),\quad E_z(-x)=E_z(x),$$
$$B_y(-x)=-B_y(x),\quad B_z(-x)=B_z(x),$$
$$B_z(x_b)=B_0,$$

where B_0 is the given external longitudinal magnetic field. In general, the elliptic functions in (10) and (17) are double-valued. Solutions with $dB/d\psi\geqslant0$ and $dB/d\psi\leqslant0$, which correspond to the regions $x\geqslant0$ and $x\leqslant0$ and which can be joined in the $x=0$ plane, satisfy these conditions if

$$x[B^2=-C_0/2\pm(C_0^2/4+\mu^2)^{1/2}]=\Theta[\tilde{B}^2=-C_0/2\pm(C_0^2/4+\mu^2)^{1/2}]=0$$

and $\Theta_0=(2l+1)\pi/2$ (the upper and lower signs correspond to $\mu\neq0$ and $\mu=0$; l is an integer). If $k=-|k|$ and $B^2\to|k|^{-1}$, expressions (10) and (17) do not apply.

We assume that in equilibrium the following are given: the total drift current and the total longitudinal current, J_y and J_z, $v_z(z=0)=v_{z0}$ and

$$v_y^2(x_b)+v_z^2(x_b)=v_0^2.$$

From (10) and (17) we then find

$$\mu^2=F_{eq}^4v_{z0}^2(B_0-J_y)^2,$$
$$C_0=F_{eq}^2[v_{z0}^2-(B_0-J_y)^2],$$
$$\lambda kF_{eq}^2[v_0^2(B_0^2+J_z^2)-v_{z0}^2(B_0-J_y)^2]=\mathscr{E},\qquad(18)$$

where $\mathscr{E} = B_0^2 + J_z^2 - (B_0 - J_y)^2 + v_{z0}^2 - v_0^2$. Solution (18) with $\lambda k = 0$ leads to the relations $\mathscr{E} = 0$, which corresponds to a generalization of the condition for a Brillouin equilibrium to the case of a diamagnetic planar beam with a nonuniform charge-density profile. With $\lambda = 1$ and $k \neq 0$ we have $\mathscr{E} \neq 0$, and the values of k and F_{eq} are determined unambiguously from (12) and (18):

$$k = \mathscr{E}^3[v_0^2(B_0^2+J_z^2) - v_{z0}^2(B_0-J_y)^2]^{-1}$$

$$\times\ [\mathscr{E} - v_0^2(B_0^2+J_z^2) + v_{z0}^2(B_0-J_y)^2]^{-2},$$

$$F_{eq} = 1 - \mathscr{E}^{-1}[v_0^2(B_0^2+J_z^2) - v_{z0}^2(B_0-J_y)^2]. \tag{19}$$

For definiteness we assume $v_0^2 > v_{z0}^2$ at this point. Expressions (18) and (19) can be used to evaluate the parameters k, C_0, μ, and F_{eq} in various regions of values of the equilibrium characteristics of the system. If the self-magnetic fields of the beam are negligible in comparison with B_0, i.e., if $J_y \ll B_0$ and $J_z \ll B_0$ (the given-field approximation), we find from (18) and (19)

$$\mathscr{E} \approx -v_0^2 + v_{z0}^2, \quad k \approx -B_0^2(1+B_0^2)^{-2}, \quad F_{eq} \approx 1 + B_0^2.$$

At the same accuracy level it follows from (17) that the equilibrium charge density is $n_{eq} \approx$ const, but the squares of the equilibrium velocities are negative; i.e., an equilibrium of this type cannot be realized with $k < 0$.

Incorporating the self-magnetic fields of the beam leads to a decrease in the absolute value of $\mathscr{E}$. It follows from (19) that k, while remaining negative, also decreases. In the limit $\mathscr{E} \to 0$ we have $k \to 0$ and $F_{eq} \to \infty$, and expressions (19) are formally inapplicable. In solution (17) and Eq. (12) in the limit $k \to 0$ we should take the limit $\lambda = 0$; as a result, k becomes a scale factor. Let us examine solution (17) with $\lambda = 0$. The integrals in (17) can be calculated in terms of elementary functions in the case $\lambda = 0$. Assuming $k = 1$ for definiteness, we find the following expressions for the charge density, x, and Θ from (17) and (18):

$$n^{-1} = 1 + F(B_0-J_y)^2(2\eta^2 - 1 + \varepsilon),$$

$$\Theta = \pi/2 + 2^{-1}\arccos(2\varepsilon\eta^{-1} + 1 - \varepsilon)(1+\varepsilon)^{-1},$$

$$x = F(B_0-J_y)^2[\eta^4 + \eta^2(\varepsilon-1) - \varepsilon]^{1/2} +$$

$$+ 2^{-1}\ln(1+\varepsilon)^{-1}\{2\eta^2 - 1 + \varepsilon + 2[\eta^4 + \eta^2(\varepsilon-1) - \varepsilon]^{1/2}\}. \tag{20}$$

Here $F(\tau) < 0$, $F_{eq} = -1$, $\eta = B(B_0 - J_y)^{-1}$, $\eta(x=0) = 1$, and $\varepsilon = v_{z0}^2(B_0 - J_y)^2$. In equilibrium, the charge density attains a minimum at $x = 0$, and it increases monotonically as $x \to x_b$. The condition that n_{eq}^{-1} not be negative imposes restrictions on the values of the equilibrium characteristics of the system:

$$(B_0 - J_y)^2 \leqslant 1 - v_{z0}^2, \quad B_0^2 \leqslant 1 - v_{z0}^2/2.$$

The equilibrium dimension x_b is a function of B_0^2 and J_y/B_0. For a diamagnetic beam with $J_y/B_0 \ll 1$ we find, ignoring v_{z0}^2/B_0^2 in comparison with J_y/B_0,

$$x_b \approx 2^{1/2}(J_y/B_0)^{1/2}(1 - B_0^2).$$

The quantity Θ_{eq} (x_b) is determined under the same assumptions by the product of the parameters v_{z0}^2/B_0^2 and J_y/B_0:

$$\Theta_{eq} \approx \pi/2 + v_{z0}^2 J_y/B_0^3.$$

In the course of the oscillations, the charge-density profile retains its shape (Fig. 2), and the amplitude is limited by the condition $n > 0$. The velocity components v_y and v_z and the field components E_x, B_y, and B_z oscillate around their equilibrium values, while remaining constant in sign. The dimensional frequency of the small oscillations is

$$\Omega = (\omega_{pe}^2 + \omega_{Be}^2)^{1/2} = 3^{1/2}\omega_{pe},$$

where ω_{pe} and ω_{Be} are related by the condition for a Brillouin equilibrium.

Finally, we consider the limiting case $k \gg 3^3/4$. We restrict the analysis to a stream with $v_z = J_z = 0$. In this case, F_{eq} and k in (19) are expressed in terms of the ratio $v_0^2 B_0^2/\mathscr{E}$:

$$F_{eq} = 1 - v_0^2 B_0^2/\mathscr{E}, \quad k = (v_0^2 B_0^2/\mathscr{E})^{-1}(1 - v_0^2 B_0^2/\mathscr{E})^{-2}.$$

The value of k is large under the conditions $v_0^2 B_0^2/\mathscr{E} \ll 1$ and $v_0^2 B_0^2/\mathscr{E} \sim 1$. With $v_0^2 B_0^2/\mathscr{E} \ll 1$ we have $F_{eq} \approx 1 - k^{-1}$, and a solution is realized in a region with a stable-equilibrium point on the right branch of the curve of $F'(F)$. With $v_0^2 B_0^2 \sim \mathscr{E}$ we have $F_{eq} \approx \mp (k)^{-1/2}$, and the solutions with $v_0^2 B_0^2 \gtrsim \mathscr{E}$ and $v_0^2 B_0^2 \lesssim \mathscr{E}$ correspond to the left branch and to a region with an unstable equilibrium point on the right branch of the $F'(F)$ curve. Let us find the charge-density profiles and the nature of the oscillations as functions of the values of $v_0^2 B_0^2/\mathscr{E}$, J_y, and B_0. With $k \gg 3^3/4$ and $B_0 - J_y \gtrsim 1$ in the $x(B)$ integral, we have $kB^2 > 1$, and it can be evaluated approximately. From (17) we have

$$n^{-1} = F + (1-F)k^{-1}F_{eq}^{-2}(B_0-J_y)^{-2}\xi^4,$$

$$x = k^{1/2}F_{eq}(B_0-J_y)F\xi^{-1}(1-\xi^2)^{1/2}$$

$$+ 2^{-1}k^{-1/2}F_{eq}^{-1}(B_0-J_y)^{-1}[\xi(1-\xi^2)^{1/2} + \arccos\xi], \tag{21}$$

where $\xi = F_{eq}(B_0 - J_y)B^{-1}$ and $\xi(x=0) = 1$. The charge-density distribution is monotonic for $F_{eq}(B_0 - J_y) \leqslant B \leqslant F_{eq}B_0$. From (21) we find the equilibrium values $n_{eq}^{-1}(x=0)$ and $n_{eq}^{-1}(x = x_b)$ to be

$$n_{eq}^{-1}(x=0) = 1 - v_0^2 B_0^2/\mathscr{E} + (v_0^2 B_0^2/\mathscr{E})^2(B_0-J_y)^{-2},$$

$$n_{eq}^{-1}(x=x_b) = 1 - v_0^2 B_0^2/\mathscr{E} + (v_0^2 B_0^2/\mathscr{E})^2(B_0-J_y)^2 B_0^{-4}.$$

For the left branch of the $F'(F)$ curve, the condition $v_0^2 B_0^2 > \mathscr{E}$ with $n_{eq} > 0$ and $v_0^2 \ll 1$ leads to $J_y < B_0 v_0^2/2$. In this case we have n_{eq} $(x = x_b) > n_{eq}$ $(x = 0)$, but in contrast with the

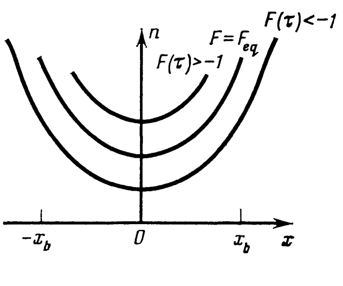

FIG. 2.

case of a Brillouin stream, discussed above, the quantity n_{eq} $(x=0)$ may be either greater than or less than unity. For $v_0^2B_0^2<\mathscr{E}$ the equilibrium transverse dimension x_{beq} is determined primarily by the second term in expression (21) for $x(B)$:

$$x_{beq}\approx B_0^{-1}(J_y/2B_0)^{1/2}.$$

The nature of the oscillations is completely analogous to that in the case discussed above. In other words, all quantities oscillate around their equilibrium positions; the charge-density profile retains its shape; and the oscillation amplitude is limited by the applicability condition for solution (17).

On the right branch of the $F'(F)$ curve the condition $v_0^2B_0^2<\mathscr{E}$ with $v_0^2\ll 1$ leads to

$$B_0^2v_0^2/2<J_y<2B_0(1-v_0^2/4).$$

The degree of nonuniformity of the charge density is determined by the ratio J_y/B_0, and the deviation from charge neutralization is determined by the ratio $v_0^2B_0^2/\mathscr{E}$. The first and second quantities are large if $J_y\ll B_0$ and $v_0^2B_0^2\sim\mathscr{E}$ (a point of unstable equilibrium), while they are small if $J_y\ll B_0$ and $v_0^2B_0^2\ll\mathscr{E}$ (a point of stable equilibrium). As before, we have $n_{eq}(x=x_b)>n_{eq}(x=0)$, and $n_{eq}(x=0)$ can be either greater than or less than unity. The transverse length scale of the beam under the condition $v_0^2B_0^2\ll\mathscr{E}$ is determined by the first term in expression (21) for $x(B)$:

$$x_b\approx B_0(\mathscr{E}/v_0^2B_0^2)^{1/2}[1-(1-J_y/B_0)^2]^{1/2}.$$

In the case $v_0^2B_0^2\lesssim\mathscr{E}$ it is instead determined by the second term:

$$x_b\approx 2^{-1}B_0^{-1}[1-(1-J_y/B_0)^2]^{1/2}.$$

The condition that J_y is bounded from above rules out the possibility of a complete reversal of the longitudinal magnetic field. In a time-varying situation, the behavior of all the functions except E_x and n is similar to that discussed above. The density profiles retain their shape only if $F(\tau)<1$; this condition corresponds to small oscillations around the stable equilibrium point. With $F(\tau)=1$ in (17) and (21), we have $n=1$ and $E_x=0$; at $F(\tau)>1$, the maximum of the density profile shifts away from the boundary toward the median plane of the stream (Fig. 3). The difference $F(\tau)-1$ reaches its highest positive values in the course of oscillations around an unstable equilibrium point. It follows from expression (16) in this case that as the oscillation amplitude increases, and as the system approaches the position of an unstable equilibrium, the time spent in the state with $F(\tau)<1$ increases without bound.

This analysis shows that in a study of slightly nonelectrostatic, nonlinear, rf oscillations of a plasma in crossed fields a self-consistent consideration of the drift currents implies the existence of solutions with stable evolutionary characteristics. In contrast with purely electrostatic harmonic

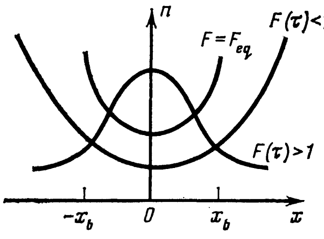

FIG. 3.

oscillations in a longitudinal external magnetic field, for which the spatial distributions of the functions do not influence the temporal characteristics of the system, the stable nonequilibrium oscillations in this case correspond to completely definite profiles of the densities of the charge and the current. In equilibrium, they have a clearly defined cylindrical nature. In the case of a slight diamagnetism of the stream, the incorporation of the drift current leads simply to an anharmonicity of the oscillations. The presence of an ion background, combined with a strong diamagnetism of the stream, gives rise (for certain relations among the parameters) to a new oscillation branch, which exhibits a threshold-saturation effect. The shape of this branch near an unstable equilibrium point and the evolutionary characteristic of the charge-density profile are evidence that weakly nonlinear helical waves of the envelope soliton type can exist in this region.

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