Stochastic nature of streamlines in steady-state flows

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Steady-state three-dimensional flows of an ideal incompressible fluid of a new class, which satisfies the Beltrami condition, are analyzed. A Hamiltonian formalism is constructed for flows with quasicrystalline and helical symmetries. The problem of the onset of chaos in the streamlines is formulated as a problem of the disruption of invariant tori. The boundary of the region of turbulence of the streamlines is calculated. It is shown that a stochastic web can form. The effect of the chaos of the streamlines on the diffusion of passive particles in the liquid is discussed. The effect of this chaos on the generation of a magnetic field in a conducting medium is also discussed.

1. INTRODUCTION

Divergence-free fields have a remarkable property: the number of field lines inside a force tube is conserved. This conservation property means that the course of the field lines can be determined by methods similar to those of Hamiltonian mechanics.[1] This circumstance in turn means that rapid progress can be made in analyzing a variety of physical problems. A clear example is the theory of the stability and destruction of magnetic surfaces.[2,3] The equations

$$\frac{dx}{B_x}=\frac{dy}{B_y}=\frac{dz}{B_z} \tag{1.1}$$

determine the course of the field lines of the magnetic field $B(x,y,z)$. If we rewrite system (1.1) in a more convenient form, e.g.,

$$\frac{dx}{dz}=\frac{B_x}{B_z},\quad \frac{dy}{dz}=\frac{B_y}{B_z} \tag{1.2}$$

we see that we are dealing with a "nonstationary" problem for a dynamic system with a two-dimensional phase space (x,y). The variable z plays the role of the time. For certain types of fields, it is sometimes possible to put the system (1.2) in Hamiltonian form. In such a case, the well-developed apparatus of the theory of dynamic systems begins to operate in its full glory. The field lines of a magnetic field may wind around invariant surfaces $z = z(x,y)$, or they may exhibit a spatial behavior which is irregular (stochastic). This comment by itself demonstrates the importance of the information obtained in this manner, since (for example) a random walk of a magnetic field line in space would also imply diffusion of magnetized charged particles.[4,5] In particular, the field lines in the cases of two-dimensional fields (for which B has no z dependence) are always regular, while in the three-dimensional case, generally speaking, some of the lines will always behave stochastically if the dependence on z is periodic.

Our purpose in the present paper is to analyze the streamlines which are formed by steady-state three-dimensional flows. For an incompressible fluid, as for a magnetic field, we have div $\mathbf{v} = 0$, and the equations of the streamlines are similar to (1.1),

$$\frac{dx}{v_x}=\frac{dy}{v_y}=\frac{dz}{v_z}, \tag{1.3}$$

or (1.2),

$$\frac{dx}{dz}=\frac{v_x}{v_z},\quad \frac{dy}{dz}=\frac{v_y}{v_z}. \tag{1.4}$$

The difference between (1.1) and (1.3) or between (1.2) and (1.4) results from the particular magnetic fields and the particular velocity fields which are of interest.

Steady-state fields $\mathbf{v}$ which satisfy Euler's equation in the absence of external forces,

$$\frac{\partial \mathbf{v}}{\partial t}+[\mathrm{rot}\,\mathbf{v},\mathbf{v}]=-\nabla\left(\frac{P}{\rho}+\frac{\mathbf{v}^2}{2}\right), \tag{1.5}$$

play a special role here. The nonlinear term on the left side of (1.5) vanishes identically for an incompressible fluid, provided that Beltrami's property holds:

$$\mathrm{rot}\,\mathbf{v}=g\mathbf{v}, \tag{1.6}$$

where g is some scalar function of the coordinates. Everywhere below we assume $g = \pm 1$ for simplicity.

An example of a flow which has property (1.6) is an Arnol'd-Beltrami-Childress (*ABC*) flow[1,6–8]:

$$\begin{aligned} v_x&=A\sin z+C\cos y,\\ v_y&=B\sin x+A\cos z,\\ v_z&=C\sin y+B\cos x. \end{aligned} \tag{1.7}$$

Arnol'd-Beltrami-Childress flows have two important properties.

First, a flow of this type is a solution of the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\nabla\frac{P}{\rho}+\mathbf{F}+\nu\Delta\mathbf{v} \tag{1.8}$$

with a suitably chosen force **F**:

$$\mathbf{F}=\nu\mathbf{v}. \tag{1.9}$$

Equation (1.9) follows from the circumstance that the *ABC* flow (1.7) satisfies the condition $\Delta v = -v$. Furthermore, there is the assertion that an *ABC* flow is stable if the viscosity is sufficiently high.[9]

Second, the streamlines of an *ABC* flow have the property of being stochastic in a region of (x,y,z) space of finite measure. This property has been studied numerically and analytically in a series of papers.[10–12] It is completely analogous to the stochastic spatial "arrangement" of the field lines of a magnetic field, as noted above. A chaos of streamlines has a variety of consequences. It imparts a diffusive nature to the dynamics of passive particles in a liquid.[13] An *ABC* flow leads to the generation of a magnetic field in a conducting medium.[14–16] If the magnetic field is frozen in,

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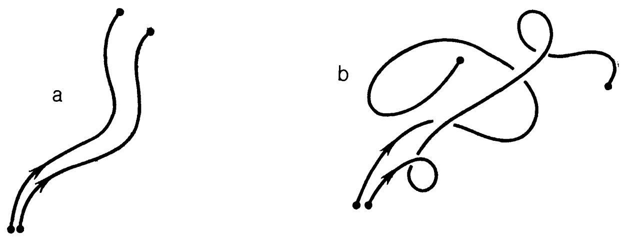

FIG. 1. Stable (a) and unstable (b) streamlines.

the diffusion of streamlines involves diffusion of magnetic field lines. This effect in turn means that magnetized particles will exhibit stochastic dynamics. We are thus essentially dealing with a new and as yet little-studied manifestation of chaos—a manifestation which stems from the nonlinear nature of the dynamic equations: A regular (for example, periodic) vector field has a stochastic tangent space. This property can be defined in a variety of ways. Let us assume, for example, that a field line of the field v emerges from the point r_1 and goes to the point r_2 (Fig. 1). These points are related by the relation

$$\mathbf{r}_2=\hat{\mathscr{L}}\mathbf{r}_1, \qquad (1.10)$$

where the operator $\hat{\mathscr{L}}$ specifies the "motion" of the streamline (or field line). Since the number of streamlines is conserved within a current tube, the onset of chaos means that these streamlines will become highly tangled in space (Fig. 1b) by comparison with regular behavior (Fig. 1a). Mixing of streamlines (or of the lines of a force field) occurs in coordinate space, as occurs for the trajectories of a particle in the phase space of Hamiltonian systems.[17]

Below we will formalize this phenomenon, which might naturally be called "turbulence of streamlines" (or of field lines of a field). This phenomenon is sometimes referred to as "Lagrangian turbulence" in contrast with ordinary "Eulerian turbulence." Let us outline the analysis below of the turbulence of streamlines.

In Sec. 2 we introduce a new class of three-dimensional steady-state flows which satisfy the Beltrami's condition (1.6). These flows might be called "flows with a quasicrystalline symmetry" or, more simply, "flows with quasisymmetry." *ABC* flows and also two-dimensional Kolmogorov-Sinaĭ flows[18] are particular cases of this class. For flows with quasisymmetry we construct a Hamiltonian formalism (Sec. 3), which we then use to formulate a problem of the appearance of turbulence of streamlines as a problem of the destruction of invariant tori.

In Sec. 4 we calculate the boundary of the turbulence region as the width of the stochastic layer near certain special streamlines which pass through steady-state points of the velocity field.

In Sec. 5 we present yet another example of steady-state flows with helical symmetry, and we establish an analogy between a flow of this sort and the magnetic field in a toroidal system.

In essence, the results show that a steady-state flow is constructed in a largely nontrivial way, although the formal expression for the velocity field may look totally harmless [see, e.g. (1.7)]. The streamlines partition the entire coordinate space with the flow into a system of cells. This system may be either simple (e.g., cubic in the case of an *ABC* flow or a hexagonal two-dimensional flow) or a very complex system, as in the case of a quasisymmetry of higher order. These cells are separated by layers of turbulent streamlines, whose thickness is determined by the parameters of the flow. This situation appears to be a fairly general one, so the well-known structures in the case of thermal convection or in other steady-state liquid flows have analogous properties. These questions will be discussed in the conclusion to this paper.

2. QUASISYMMETRIC FLOWS

Steady-state flows in a sufficiently large volume may have a regular structure of the crystalline type. The best-known examples here are convection rollers and convection cells of rectangular and hexagonal shapes. Rectangular structures may also arise in the case of electrodynamic convection and in two-dimensional flows of the Kolmogorov-Sinaĭ type.[18,19]

The circumstance that we presently observe only structures with crystalline symmetry (i.e., with rectangular and hexagonal lattices) is a reflection of their greater stability. Formally, the spatial symmetry and the hydrodynamic equations allow a considerably wider class of steady-state flows with a symmetry like that of quasicrystals (for brevity, "quasisymmetric flows").[20-22]

In two-dimensional hydrodynamics, with $\mathbf{v} = (v_x(x,y), v_y(xy))$, the Euler equation (1.5) can be put in the form

$$\frac{\partial}{\partial t}\Delta_\perp\psi-\frac{\partial(\psi,\Delta_\perp\psi)}{\partial(x,y)}=0 \qquad (2.1)$$

by means of the stream function $\psi=\psi(x,y)$, where

$$v_x=\frac{\partial\psi}{\partial y},\quad v_y=-\frac{\partial\psi}{\partial x}. \qquad (2.2)$$

Here $\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2$. Accordingly, if (for example)

$$\Delta_\perp\psi=f(\psi), \qquad (2.3)$$

where f is an arbitrary function, the nonlinear term vanishes identically, making the solution a steady-state solution.

Periodic structures and structures of a quasicrystalline type can be found by means of the function[21]

$$H_0=\sum_{j=1}^{q}\cos(\mathbf{R}\mathbf{e}_j), \qquad (2.4)$$

where $\mathbf{R}=(x,y)$, and the "hedgehog" $\mathbf{e}_j$ is formed by q unit vectors:

$$\mathbf{e}_j=\left(\cos\frac{2\pi j}{q},\ \sin\frac{2\pi j}{q}\right). \qquad (2.5)$$

In the case $q=2$, the contour lines of the function H_0 correspond to rollers (parallel straight lines); in the case $q=4$ they correspond to a square lattice; and at $q=3$ and 6 they correspond to a "kagome" hexagonal lattice. These are cases of crystalline symmetry. With $q\neq 2, 3, 4, 6$, the contour lines of function (2.4) specify a quasisymmetry.

Since H_0 satisfies the equation

$$\Delta H_0+H_0=0, \qquad (2.6)$$

it is clear that $\psi=H_0$ is one of the possible solutions of two-dimensional hydrodynamics. Generally speaking, therefore, a quasicrystalline symmetry may be realized in two-dimensional hydrodynamics if a region of stability of flows of this sort is created by means of a source and viscous terms.

In three-dimensional steady-state flows, quasisymmetry can again arise. Let us consider the simplified case in which the velocity field is determined by the expressions

$$v_x = -\frac{\partial H_0}{\partial y} + \varepsilon \sin z, \quad v_y = \frac{\partial H_0}{\partial x} - \varepsilon \cos z, \quad v_z = H_0, \tag{2.7}$$

where ε is a perturbation parameter. We will call this a "quasisymmetric flow." It is periodic in z and quasisymmetric in the (x,y) plane. By virtue of property (2.6), the Beltrami's condition (1.6) holds in this case. A solution of this sort is accordingly a steady-state solution of Eq. (1.5). At $q = 4$ we have $H_0 = 2(\cos x + \cos y)$, so (2.7) takes the form

$$\begin{aligned} v_x &= 2 \sin y + \varepsilon \sin z, \\ v_y &= -2 \sin x - \varepsilon \cos z, \\ v_z &= 2 \cos x + 2 \cos y. \end{aligned} \tag{2.8}$$

Comparison of (2.8) with (1.7) shows that flow (2.8) is an *ABC* flow (in the resulting expressions we need to make the replacements $x \to -x$, $y \to \pi/2 - y$, and $z \to \pi - z$) in this case).

The symmetry of flow (2.7) with $q \neq 2, 3, 4, 6$ is lower than that with $q = 2, 3, 4, 6$ (Ref. 23). The Fourier spectrum of the velocity components v_x, v_y, v_z has a finite number of harmonics, because the function H_0 defined by (2.4) is nearly periodic. We thus meet a new type of "less organized" order, one example of which is the structure of schechtmanite quasicrystals.[24]

3. HAMILTONIAN DESCRIPTION OF STREAMLINES

Equations (1.4) determine a curve, i.e., a streamline, in the space (x,y,z). It can be described in the following way. We denote by (x_0, y_0, z_0) a fixed ("initial") point through which the streamline passes. Its coordinates for an arbitrary value of z are then determined in the form of certain functions

$$x = x(z;\, x_0,\, y_0), \quad y = y(z;\, x_0,\, y_0), \tag{3.1}$$

which constitute the solution of system (1.4) under the initial conditions

$$x(z = z_0;\, x_0,\, y_0) = x_0, \quad y(z = z_0;\, x_0,\, y_0) = y_0.$$

Parametrization (3.1) by means of one of the variables z is convenient for introducing a canonical Hamiltonian description of the pattern of streamlines.

We introduce the new variable[3]

$$\tau = \tau(z,\, x(z;\, x_0,\, y_0),\, y(z;\, x_0,\, y_0)), \tag{3.2}$$

which depends strongly on the choice of one streamline or another. For this purpose we set

$$d\tau/dz = 1/v_z. \tag{3.3}$$

In (3.3) it is assumed that v_z is a function of z alone, if we write expressions for x and y with the help of (3.1). The variable

$$\tau = \int \frac{dz}{v_z} + \text{const} \tag{3.4}$$

will play the role of a "time."

System (1.4) for a quasisymmetric flow, (2.7), takes the form

$$\begin{aligned} \dot{x} &= -\frac{\partial H_0}{\partial y} + \varepsilon \sin z(\tau), \\ \dot{y} &= \frac{\partial H_0}{\partial x} - \varepsilon \cos z(\tau), \end{aligned} \tag{3.5}$$

where the dot means differentiation with respect to τ, and it is assumed that z is expressed in terms of τ through inversion of Eq. (3.4). It is now a straightforward matter to see that the expression

$$H = H_0 + \varepsilon V, \quad V = -x \cos z(\tau) - y \sin z(\tau) \tag{3.6}$$

is a Hamiltonian, which determines the canonical equations

$$\dot{x} = -\partial H/\partial y, \quad \dot{y} = \partial H/\partial x, \tag{3.7}$$

which are the same as (3.5). The change of variables in (3.3) takes the following form, according to (2.7):

$$\dot{z} = H_0. \tag{3.8}$$

At first glance it appears that the system (3.5), (3.8) is identical to the equations ordinarily used in advection theory[13] or in an analysis of the motion of passive particles, if suitable right sides are chosen for the equations in the latter cases. Actually, these equations are not identical. The time τ is not the ordinary time, and in general it is totally unrelated to the ordinary time. It is determined by Eq. (3.8) and is of a purely formal nature.

The introduction of the Hamiltonian formalism (3.6), (3.7) makes it possible to quickly find the basic results, regarding how the streamlines are constructed. In those cases in which Eqs. (1.2) have some additional symmetry property it is possible to construct a Hamiltonian by making use of this property. An example of such a construction of Hamiltonian equations for toroidal magnetic confinement systems is given in Refs. 25 and 26.

4. STOCHASTIC LAYERS AND WEBS

Let us consider the streamline Hamiltonian (3.6). With $\varepsilon = 0$ (the two-dimensional case) it determines a family of cylindrical surfaces which correspond to various values of the "energy" integral $H_0 = \text{const} = E$. This integral is simultaneously a stream function. A perturbation V for which ε is nonzero but very small has the consequence that a significant fraction of these surfaces, while undergoing a slight distortion in shape, remain invariant under a change in the time τ, in accordance with the Kolmogorov-Arnol'd-Moser theory.[1] At $\varepsilon = 0$, however, there are singular separatrix surfaces on which even a small perturbation has a pronounced effect. The intersection of these surfaces with the $z = \text{const}$ plane produces a quadratic ($q = 4$) or hexagonal ($q = 6$) grid (Fig. 2) with crystalline symmetry. The cases of the $q = 5$ and 8 quasisymmetry are shown in Figs. 3 and 4. Different distributions of loops are produced in different $H_0 = \text{const}$ planes. A special pattern is obtained with $H_0 = 1$ (Fig. 3a). Near this value of H_0, there is a maximum number

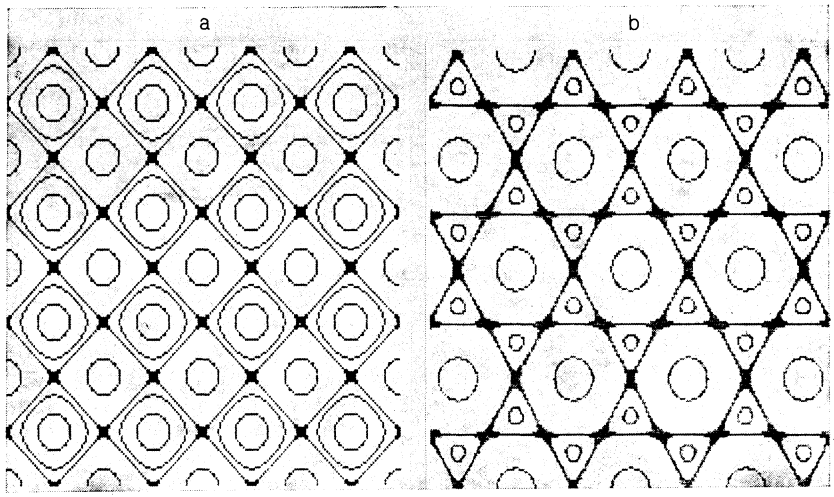

FIG. 2. Streamlines which form lattice structures in the cases of fourfold and sixfold symmetry. a—The surface $h = H_0(x,y)$ with $q = 4$ is a checkerboard alternation of hills $H_- > 0$ and valleys $H_0 < 0$. The contour lines (the cross sections of the surface at $H_0 = \text{const}$) form a square lattice in the case $H_0 = 0$. In the cases $H_0 = 1, -1$, and -3 the contour lines are closed curves; b—the contour lines for $q = 6$ [cross sections of the surface $h = H_0(x,y) = \text{const}$] form a "kagome" lattice. The closed streamlines inside the hexagons ($H_0 = 4$) are cross sections of hills, while those inside the triangles ($H_0 = -3$) are the cross sections of valleys.

of hyperbolic points.[23] The phase trajectories, however, probably do not form a single network or web. There are many discontinuities, as can be seen in Fig. 3, a and b. The distance between trajectories near a discontinuity is very small. Accordingly, even small perturbations can easily "mend" regions where the loops are not connected in the $H_0 = 1$ plane. As a result, within a thin layer ΔH near $H_0 = 1$ a common network (web) of finite thickness forms. In this sense, the value $H_0 = 1$ is a special one, since at other values of H_0 (Fig. 3c) the phase trajectories have very few hyperbolic points, and the formation of a common network through slight broadening of the separatrices or of nearby trajectories

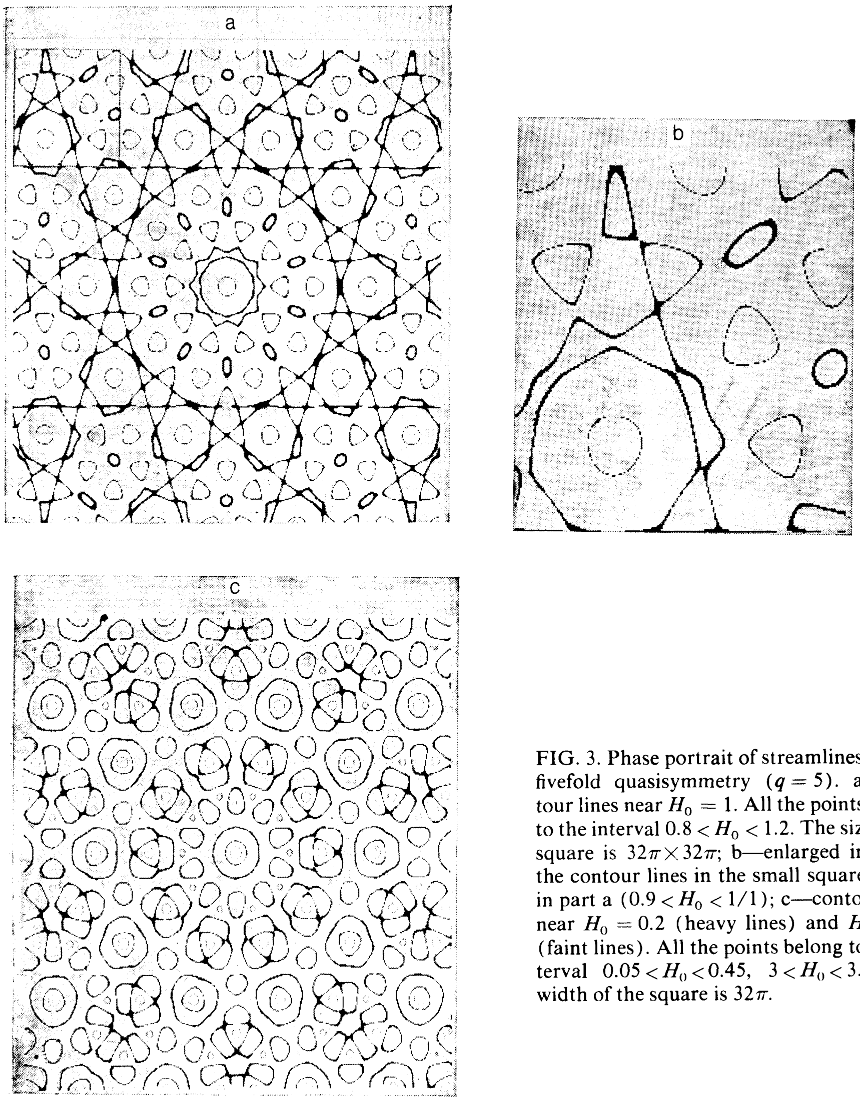

FIG. 3. Phase portrait of streamlines having fivefold quasisymmetry ($q = 5$). a—Contour lines near $H_0 = 1$. All the points belong to the interval $0.8 < H_0 < 1.2$. The size of the square is $32\pi \times 32\pi$; b—enlarged image of the contour lines in the small square drawn in part a ($0.9 < H_0 < 1/1$); c—contour lines near $H_0 = 0.2$ (heavy lines) and $H_0 = 3.2$ (faint lines). All the points belong to the interval $0.05 < H_0 < 0.45$, $3 < H_0 < 3.4$. The width of the square is 32π.

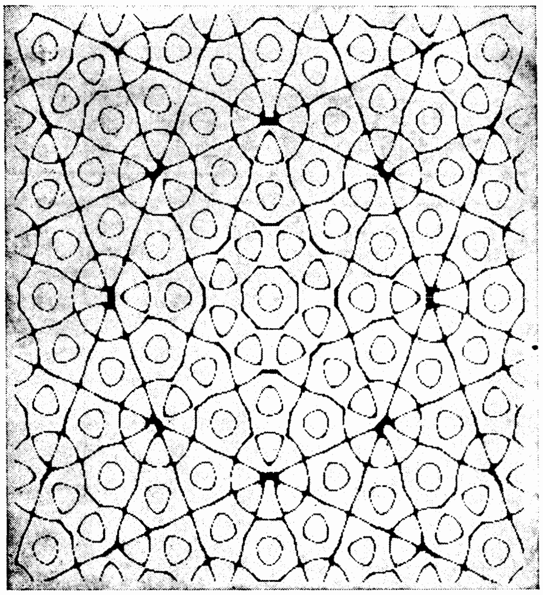

FIG. 4. Contour lines in the case of eightfold quasisymmetry ($q = 8$) near $H_0 = 0$. The width along H_0 of the window is $-0.4 < H_0 < 0.4$. The size of the square is $32\pi \times 32\pi$.

is difficult. This pattern explains the mechanism for the formation of a web under the influence of a perturbation. At the same time we see why there is quasisymmetry at $q = 5, 7, 8$,

Figure 5 shows the surface[1] $\zeta = H_0(x,y)$ for $q = 5$. We clearly see here how the saddle regions form structures of lines which are approximately straight. These lines produce the web network in Fig. 3a.

A system similar to (3.5) was studied in Refs. 20–23. It was shown there that at $z(\tau) = |\tau(| \gg 1)$ there is an unbounded network of channels (a stochastic web) in the (x,y) plane within which the particle dynamics is stochastic. The (x,y) plane is the phase plane of the particle (x is the momentum, and y the coordinate). The web has the corresponding symmetry of a periodic lattice ($q = 2, 3, 4, 6$) or a quasisymmetry. Its thickness is exponentially small $[\sim\exp(-\text{const}\,\nu)]$ in the case $\varepsilon = 1$. In this case the situation is different, since $z(\tau)$ is not a fixed function of τ, and it depends on the energy of the particle by virtue of Eq. (3.8).

Let us examine Eqs. (3.5) and (3.8) in the case $\varepsilon \ll 1$. The change in the stream function of two-dimensional structures is found with the help of the equations

$$\dot{H}_0 = \frac{\partial H_0}{\partial x}\dot{x} + \frac{\partial H_0}{\partial y}\dot{y} = \varepsilon\left(\frac{\partial H_0}{\partial x}\sin z - \frac{\partial H_0}{\partial y}\cos z\right), \quad \dot{z} = H_0. \tag{4.1}$$

For definiteness we will discuss the case

$$H_0 = \cos x + \cos y, \tag{4.2}$$

which corresponds to $q = 4$ in Eq. (2.4), i.e., to an *ABC* flow [for simplicity we are omitting the factor of 2 in (2.4)]. The unperturbed ($\varepsilon = 0$) course of the streamlines in the case $H_0 = E$ (Ref. 20) is

$$\begin{aligned} \cos x &= E/2 + (1 - E/2)\,\mathrm{cd}[(1 + E/2)\tau, \varkappa], \\ \cos y &= E/2 - (1 - E/2)\,\mathrm{cd}[(1 + E/2)\tau, \varkappa], \end{aligned} \tag{4.3}$$

where cd = cn/dn (cn and dn are elliptic functions), and $\varkappa$ is the modulus of the elliptic function, given by

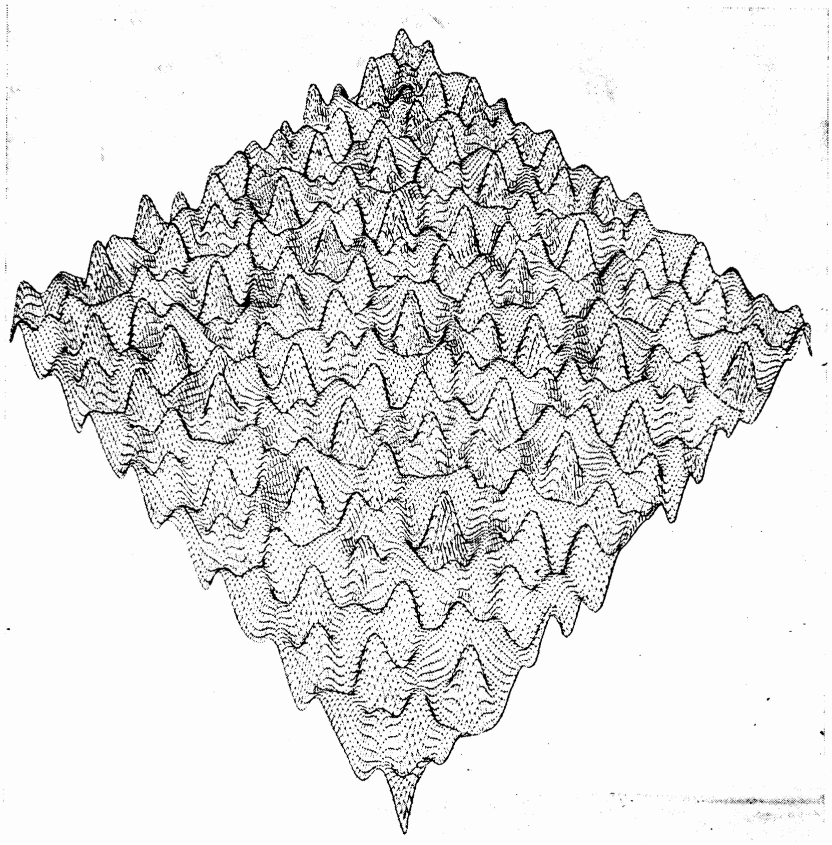

FIG. 5. Surface mapping the stream function $\zeta - H_0(x,y)$ in the case of fivefold quasisymmetry ($q = 5$).

$$\varkappa=(2-E)/(2+E). \quad (4.4)$$

Furthermore, the nonlinear "frequency" of the unperturbed motion is

$$\omega(E)=\pi/2K(\varkappa), \quad (4.5)$$

where $K(\varkappa)$ is the complete elliptic integral.

The perturbation in (4.1) has its greatest effect near the separatrix which corresponds to the value $E=0$ and to a square lattice in the phase plane (Fig. 2). Any point in the (x,y) plane near the lattice lies on a trajectory with a small value of E. It may fall in the region of the stochastic layer of the web, which forms at the position of the separatrix network under the influence of perturbation V. Near the separatrix ($E\to 0$) we find from (4.5)

$$\omega(E)\sim\pi/\ln(8|E|^{-1}). \quad (4.6)$$

The usual picture of the destruction of the separatrix and of the formation of a stochastic layer is determined by the overlap of resonances on the right side of expression (4.1) for H_0:

$$\dot{H}_0=-\varepsilon(\sin x\sin z-\sin y\cos z), \quad \dot{z}=H_0. \quad (4.7)$$

The resonance condition

$$n\omega(E)=\dot{z}=E, \quad (4.8)$$

where n is an odd number, actually cannot be satisfied as $E\to 0$ at any value $n>1$. That this is true can be seen immediately by substituting (4.6) into (4.8). Since the perturbation frequency $\dot{z}=E$ is itself determined by the value of the unperturbed Hamilton H_0, we have an adiabatic situation.

Nevertheless, the effect of the perturbation is quite strong, although it develops slowly in time. It is related to repeated, and nearly periodic, crossings of the separatrix of the system (H_0 crosses zero). Let us examine this process in more detail.

We rewrite system (4.8) in the form

$$\ddot{z}+\varepsilon(\sin z\sin x-\cos z\sin y)=0, \quad (4.9)$$

where for $\sin x$ and $\sin y$ we should substitute in explicit functions of the time τ of the unperturbed motion (4.3). Near the separatrix we have $E\to 0$ and $\varkappa\to 1$ [see (4.4)]. Using the relations

$$\sin y\approx(2|E|)^{1/2}\,\mathrm{sd}(\tau,1-E),\ \mathrm{sd}=\mathrm{sn}/\mathrm{dn},$$
$$\mathrm{sd}(\tau,\varkappa)=(1-\varkappa^2)^{-1/2}\,\mathrm{cn}(\tau-K,\varkappa),$$

and the change of variables ($z-\pi/4\to z$, $\tau-K\to\tau$), we find

$$\ddot{z}+2^{1/2}\varepsilon\,\mathrm{cn}(\tau,1-E)\sin z=0. \quad (4.10)$$

In the limit $E\to 0$, the function cn has a periodic sequence of pulses (solitons) which are narrow (in comparison with the period $2\pi/\omega$), so we can transform from Eq. (4.10) to a discrete mapping. From the exact expansion in a Fourier series,

$$\mathrm{cn}(\tau,\varkappa)=\frac{2\pi}{\varkappa^{1/2}K(\varkappa)}\sum_{n=0}^{\infty}a_n\cos\left[(2n+1)\frac{\pi\tau}{2K(\varkappa)}\right],$$
$$a_n=b^n/(1+b^{2n+1}),\quad b=\exp[-\pi K(\varkappa')/K(\varkappa)],$$
$$\varkappa'=(1-\varkappa^2)^{1/2} \quad (4.11)$$

we find, in the limit $E\to 0$,

$$\mathrm{cn}(\tau,1-E)\sim 4\omega\sum_{n=0}^{N}\cos[(2n+1)\pi\omega\tau], \quad (4.12)$$

where we have used the asymptotic expression (4.6) for ω and where the number

$$N=1/\pi\omega \quad (4.13)$$

determines the effective number of harmonics in expansion (4.11) for which the amplitude satisfies $\alpha_n\sim 1$. Specifically, as $E\to 0$ we find $b_n\propto e^{-\pi n\omega}$, and the definition (4.13) of N follows. In the limit $E\to 0$ we have $N\to\infty$, and the expression on the right side of (4.12) can be approximated by

$$\lim_{N\to\infty}\sum_{n=0}^{N}\cos[(2n+1)\pi\omega\tau]=\frac{1}{2}\sum_{n=-\infty}^{+\infty}(-1)^n\delta(\omega\tau-n). \quad (4.14)$$

Introducing the dimensionless time

$$s=\omega\tau \quad (4.15)$$

and substituting in (4.14) and (4.15), we finally find approximate equations of motion near the separatrix:

$$\frac{d^2z}{ds^2}+\frac{8^{1/2}\varepsilon}{\omega}\sin z\sum_{n=-\infty}^{+\infty}(-1)^n\delta(s-n)=0. \quad (4.16)$$

The idea of using the approximation (4.16) instead of (4.10) is based on the circumstance that the width of the pulses (solitons) which follow periodically, representing function (4.12), is $\sim 1/N\omega$ and therefore smaller by a factor of N than the period at which these pulses repeat, which is $\sim 1/\omega$.

It is now a simple matter to transform from (4.16) to a discrete mapping in the (p,z) phase space: $dz/ds=p$. Relating the variables (p_n,z_n) and (p_{n+1},z_{n+1}) in front of the two sequences of δ-function pulses in (4.16), we find

$$p_{n+1}=p_n+(-1)^nK\sin z_n,\quad z_{n+1}=z_n+p_{n+1}, \quad (4.17)$$

where the parameter K is given by

$$K=8^{1/2}\varepsilon/\omega=(8^{1/2}\varepsilon/\pi)\ln(8/|E|). \quad (4.18)$$

We have obtained a so-called standard mapping,[27] in which the condition $K\gtrsim 1$ determines the boundary of a strong stochastic behavior. In the case at hand, however, we have a different situation.

It can be seen from (4.1) that over a time $\tau\sim 1$ equal to the width of the soliton the change in H_0, i.e., E, is at most ε. Accordingly, if

$$\varepsilon>E, \quad (4.19)$$

there will be an intersection of the separatrix as z changes. Such intersections repeat quasiperiodically, causing a stochastic dynamics over the entire energy region (4.19) (Refs. 28 and 29). The inequality (4.19) begins to hold before the condition $\varepsilon>\omega$, which follows from (4.18), begins to hold. Accordingly, it is this inequality which determines the thickness of the stochastic layer, $E\sim\varepsilon$, near the separatrix, where we have $E=0$. The distance along z over which the chaos develops turns out to be quite small in this case. Specifically, the chaos onset time is given in order of magnitude by

$$d\tau \sim 1/\omega(E=\varepsilon) \sim \ln(8/\varepsilon),$$

from which we find

$$\delta z \sim E\delta\tau \sim \varepsilon \ln(8/\varepsilon).$$

This result shows that there is a finite region—a web—in the space (x,y,z) within which the streamlines are arranged stochastically. Although this conclusion is reached here only for the particular case of an *ABC* flow, it can be generalized in a fairly simple way to the case of an arbitrary symmetry or quasisymmetry q. This assertion means that steady-state flows, while producing a definite spatial structure, simultaneously produce regions of chaos in the arrangement of streamlines in the vicinity of this structure. A three-dimensional structure is a necessary condition for the formation of a structural chaos.

5. HELICAL STEADY-STATE FLOWS

In this section we introduce yet another type of three-dimensional steady-state flow: a flow with a helical symmetry of order n. Such flows are symmetric with respect to rotation through an angle of π/n around the z axis. Their existence is again based on a representation of Eqs. (2.7) for the velocity in cylindrical coordinates:

$$v_r = -\frac{1}{r}\frac{\partial H_0}{\partial \varphi} + \frac{\varepsilon}{r}\sin z,$$
$$v_\varphi = \frac{\partial H_0}{\partial r} - \frac{\varepsilon}{r}\cos z, \quad v_z = H_0. \tag{5.1}$$

The stream function $H_0(r,\varphi)$ of two-dimensional hydrodynamics satisfies the equation

$$\Delta_\perp H_0 + H_0 = 0 \tag{5.2}$$

here, and H_0 is a solution with azimuthal symmetry.

It is easy to verify that we have

$$\text{rot}\, \mathbf{v} = -\mathbf{v},$$

so the Beltrami condition holds. A solution of (5.2) is

$$H_0 = \sum_n C_n J_n(r) \cos n\varphi, \tag{5.3}$$

where C_n are coefficients, and J_n are Bessel functions.

Certain particular cases of (5.3) are of special interest. If we retain only the one term with $n = N$ in (5.3), we find a helical flow which is symmetric with respect to rotation through an angle of $2\pi/N$ in the $z = \text{const}$ plane:

$$v_r = \frac{N}{r} J_N(r) \sin N\varphi + \frac{\varepsilon}{r}\sin z,$$

$$v_\varphi = J_N'(r) \cos N\varphi - \frac{\varepsilon}{r}\cos z,$$

$$v_z = J_N(r) \cos N\varphi. \tag{5.4}$$

Equations similar to (5.4) arise for the magnetic field lines of closed confinement systems.[2,3] In the case $\varepsilon = 0$, the flow has the invariant

$$H_0 = J_N(r) \cos N\varphi, \tag{5.5}$$

which is a Hamiltonian for the field lines:

$$\frac{dr}{d\tau} = -\frac{1}{r}\frac{\partial H_0}{\partial \varphi}, \quad r\frac{d\varphi}{d\tau} = \frac{\partial H_0}{\partial r}. \tag{5.6}$$

The principle for introducing the parameter τ is the same as in (3.5):

$$dz/d\tau = H_0(r, \varphi). \tag{5.7}$$

The separatrices of the system (5.5) form a regular web which has $2N$ cells over the angle φ (Fig. 6). In two cells which are positioned symmetrically with respect to the center, the liquid flows in opposite directions. Any perturbation will disrupt the separatrix network, forming a stochastic layer in its place.[30] A perturbation along z under the condition $\varepsilon \neq 0$ in Eqs. (5.6) and (5.7) has precisely the same effect:

$$\dot r = -\frac{1}{r}\frac{\partial H_0}{\partial \varphi} + \frac{\varepsilon}{r}\sin z = v_r, \quad r\dot\varphi = \frac{\partial H_0}{\partial r} - \frac{\varepsilon}{r}\cos z = v_\varphi, \tag{5.8}$$

where the dot means differentiation with respect to τ. These equations can also be written in Hamiltonian form:

$$\dot r = -\frac{1}{r}\frac{\partial H}{\partial \varphi}, \quad r\dot\varphi = \frac{\partial H}{\partial r},$$
$$H = H_0 - \varepsilon(\varphi \sin z + \ln r \cos z). \tag{5.9}$$

For these expressions, as in the preceding section of this paper, the existence of a stochastic web is established. A particular case of helical flow (5.8) is cylindrical flow with $N = 0$. In this case, we find $H_0 = J_0(r)$ from (5.5), and the entire three-dimensional problem is integrable. Specifically, we find from (5.4)

$$v_r = \frac{\varepsilon}{r}\sin z, \quad v_\varphi = J_0'(r) - \frac{\varepsilon}{r}\cos z, \quad v_z = J_0(r).$$

Hence

$$\frac{dr}{dz} = \frac{\varepsilon}{rJ_0(r)}\sin z.$$

This equation can be integrated immediately:

$$\varepsilon \cos z + \int_0^r dr\, rJ_0(r) = \text{const}.$$

In this case there is no chaotic field line dynamics.

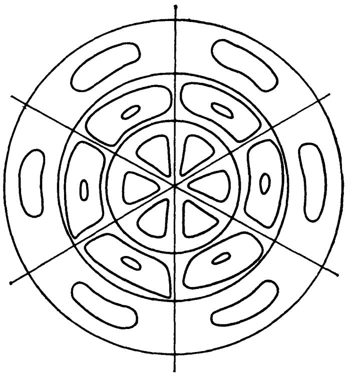

FIG. 6. Phase portrait and regular web in a cross section of a helical field with threefold symmetry (i.e., symmetry under rotation through an angle of $2\pi/3$).

6. CONCLUSION

The role played by steady-state flows has not yet been studied adequately. This comment applies primarily to the stability of these flows. On the other hand, it is clear at this point that there may exist conditions in the form of an external field (a pump) and a dissipation under which such flows can be realized. Under these conditions, quasisymmetric flows substantially extend our conception of possible regular structures in a liquid. That these structures have stochastic structures is of fundamental importance. It is apparently this property of these structure which allows the appearance of a fast dynamo.[7,14-16] Numerical analysis shows that the maximum field-generation amplitude arises in regions near saddles, i.e., in specifically those regions where the width of the stochastic layer reaches a maximum. The generation of a macroscopic vortex was also found in Ref. 31, where the Beltrami property held on the average for the small-scale random velocity field.

Furthermore, we do not rule out the possibility that quasisymmetric structures may turn out to be intermediate states in a sequence of bifurcations in the transition from the laminar regime to turbulence. In this case, much depends on the choice of boundary conditions. The use of homogeneous or periodic conditions will hinder the appearance of quasisymmetric structures.

Yet another property of structures with quasisymmetry appears important for the analysis of the development of spatial turbulence. As was shown above, these structures are generated by a separatrix grid. The destruction of this grid and the formation of a stochastic web in its place can occur not only for streamlines but also for a velocity field itself. As was shown in Refs. 20–22, an arbitrarily small time-varying perturbation disrupts the separatrices in dynamic systems and gives rise to a stochastic web of finite thickness. The role of such a perturbation could be played in actual hydrodynamics by essentially any deviation of the velocity field from a steady state. The Beltrami property, which is a property of steady-state flows, causes the nonlinear term to vanish. Accordingly, if this property does not hold to some extent, then there will be a proportionate interaction of the structures with the additional velocity caused by the deviation of the actual field from a steady state. Although the Fourier spectrum of quasisymmetric structures is a point spectrum, the existence of a finite width for a stochastic web gives rise to insignificant broadening of the points on the spectrogram. This broadening increases with increasing deviation of the structure from quasisymmetry, since the web broadens and undergoes a simultaneous distortion in shape. This may be a path by which a spatial chaos arises on the path to turbulence.

[1)] This figure was obtained by D. Chaĭkovskiĭ.

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