Conformally invariant models of two-dimensional quantum field theory with Z_n-symmetry

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A number of exact solutions of a two-dimensional conformal quantum field theory with a global Z_n symmetry is constructed. The solutions obtained can describe the critical behavior of Z_n-invariant statistical systems.

1. INTRODUCTION

One of the fundamental problems of the theory of second-order phase transitions is the description of all possible types of universal critical behavior. If one adopts the hypothesis of conformal invariance of large-scale critical fluctuations,[1,2] the problem can be reduced to that of constructing all conformally invariant solutions of Euclidean quantum field theory.

At the present time a number of exact solutions of the two-dimensional conformal quantum field theory are known (Refs. 3–18) describing the critical (or multicritical) behavior of two-dimensional statistical systems, including the Ising model[3] and its Z_n generalization,[4] the Ashkin-Teller model,[6] the RSOS models,[16,17] etc. At the basis of the study of such exactly solvable models is following general approach first applied in Ref. 3 to the Virasoro algebra.

1. The starting object for the construction of the solution is the current algebra of the conformally invariant model—a closed associative algebra containing the Virasoro algebra:

$$[L_n, L_m]=(n-m)L_{n+m}+{}^1/_{12}c(n^3-n)\delta_{n+m,0}. \qquad (1.1)$$

as a subalgebra. As current algebras were considered, e.g., the superconformal symmetry algebra (Refs. 7, 8), its $N=2$ extension (Refs. 5, 9, 11), the parafermion current algebra (Refs. 4, 5, 6), and the Kac-Moody algebras (Refs. 12, 15).

2. After the construction of the current algebra one can classify the field space of two-dimensional conformal quantum field theory in terms of its irreducible representations. Among these representations a particularly important role is played by the so-called strongly degenerate representations. Their exceptional feature in the construction of the exactly solvable models is related to the fact that, firstly, starting only from strongly degenerate representations one can construct closed operator algebras, and secondly, the correlation functions of fields belonging to strongly degenerate representations satisfy ordinary linear differential equations and can be expressed in terms of multiple contour integrals generalizing those introduced in Refs. 14 and 21.

3. The field space of conformal quantum field theory contains, in general, an infinite number of irreducible representations of the current algebra; however, for certain values of the central charge c in Eq. (1.1) it becomes possible to construct an operator algebra on the basis of a finite number of strongly degenerate representations. Models with such operator algebras are called minimal and they lend themselves to a most complete investigation.

In Ref. 13 the described scheme for constructing exactly solvable models was applied to a current algebra which, in addition to the energy-momentum tensor, contained a current W_3 of spin 3. In the present paper we construct an infinite set of exactly solvable models of two-dimensional quantum field theory, for which the current algebra together with the energy-momentum tensor contains local currents $W_k(z)$ ($k=3,4,\ldots,n$). The structure of this algebra is discussed in Sec. 2. In particular, it turns out that the W_n-algebra under investigation is an algebra with quadratic defining relations. The following two sections of the paper are devoted to a study of the irreducible representations of a W_n-invariant quantum field theory. One also finds there a derivation of the formula for the spectrum of the strongly degenerate representations, generalizing the Kac formula[3,21] for the Virasoro algebra. In Sec. 5 we study the qualitative structure of the operator algebra constructed from the strongly degenerate representations of the W_n-algebra. We note that the structure coefficients in such an operator algebra, as well as all the correlation functions, can be expressed in terms of multiple contour integrals. The simplest examples of calculations of four-point functions are listed in the Appendix. In Sec. 6 we discuss the minimal models satisfying the positivity condition. They exhibit explicit Z_n-symmetry, and are characterized by two integers n and p ($n=2, 3,\ldots,p=n+1, n+2,\ldots$), therefore we denote them by $[Z_n^{(p)}]$. The central charge c of the Virasoro algebra (1.1) in these models takes the values

$$c=(n-1)[1-n(n+1)/p(p+1)], \qquad (1.2)$$

and the spectrum of anomalous dimensions of the W_n-invariant (principal) fields is determined by the relation

$$\Delta(l|l')\equiv\Delta(l_1,\ldots,l_{n-1}|l_1',\ldots,l_{n-1}')$$
$$=\left\{12\left[\sum_{i=1}^{n-1}(pl_i\omega_i-(p+1)l_i'\omega_i)\right]^2-n(n^2-1)\right\}\Big/24p(p+1), \qquad (1.3a)$$

where the positive integers l_i, l_i' ($i=1,\ldots,n-1$) are subject to the conditions

$$\sum_{i=1}^{n-1}l_i\leqslant p,\quad \sum_{i=1}^{n-1}l_i'\leqslant p-1, \qquad (1.3b)$$

and the vectors ω_i are the fundamental weights of the Lie algebra sl(n), satisfying the relations

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$$\omega_k\omega_m = k(n-m)/n \text{ при } k\leqslant m. \qquad (1.4)$$

The $[Z_n^{(p)}]$ models include a number of known solutions. Thus, the $[M_p]$ models related to the strongly degenerate representations of the Virasoro algebra are denoted here by $[Z_2^{(p)}]$, and the $[Z_n]$ models constructed in Ref. 4 coincide with our models $[Z_n^{(n+1)}]$. We think that for $p > n+1$ the models $[Z_n^{(p)}]$ describe the multicritical behavior of two-dimensional statistical systems with global Z_n symmetry. In particular, the anomalous dimensions (1.3) agree exactly with the critical exponents characterizing the critical points of the models constructed in Ref. 18.

2. THE STRUCTURE OF THE W_n-ALGEBRAS

We consider a two-dimensional conformally invariant quantum field theory which, in addition to the energy-momentum tensor $T(z)$ ($\overline{T}(\bar z)$), which generates the conformal transformations, contains a set of conserved local currents $W_k(z)$ ($\overline{W}_k(\bar z)$), $k = 3, 4,..., n$ with spins $s_k = k(\bar s_k = -k)$. The fields $W_k(z)$ depend only on the variables $z = x_1 + ix_2$ ($\bar z = x_1 - ix_2$), i.e., have conformal dimensions $(k,0)$ ($(0,k)$). It will be convenient in some cases to denote the field by $T(z)\equiv W_2(z)$ ($\overline{T}(\bar z)\equiv \overline{W}_2(\bar z)$), and the unit operator by $I = W_0$. In the sequel we shall essentially consider the right-handed components of the fields $W_k(z)$, considering that all the conclusions extend to the left-handed components $\overline{W}_k(\bar z)$.

Let (A) denote the field space of our conformal quantum field theory, fields which are local with respect to the currents $W_k(z)$. We introduce in this space the operators $W_k(s)$ ($s = 0, \pm 1, \pm 2,...$) by means of the operator expansions:

$$A(z,\bar z)\in\{A\},$$

$$W_k(s)A(0,0) = \oint \frac{dz}{2\pi i} z^{k+s-1} W_k(z) A(0,0). \qquad (2.1)$$

We shall assume that the operators $W_k(s)$ form an associative algebra (the W_n-algebra) with nonlinear commutation relations of the following form:

$$[W_{j_1}(s_1), W_{j_2}(s_2)] = \sum_{i_1,\ldots,i_p}\sum_{u_1,\ldots,u_p} b_{j_1,j_2}^{i_1,\ldots,i_p}(s_1, s_2|u_1,\ldots,u_p)\, W_{i_1}(u_1)\ldots W_{i_p}(u_p), \qquad (2.2a)$$

where the numbers $u_1,...,u_p$ are subject to the conditions

$$\sum_{i=1}^{p} u_i = s_1+s_2, \quad u_1\leqslant u_2\leqslant\ldots\leqslant u_p, \qquad (2.2b)$$

and are such that if $u_l = u_{l+1}$, then $i_l \geqslant i_{l+1}$. (Products of this type will be called normal-ordered and in the sequel will be placed between pairs of colons.) There are the following restrictions on the structure of the coefficients b_p of this algebra: 1) the operators $W_2(s)\equiv L_s$ form a subalgebra of (1.1), 2) the coefficients $b_{j_1,j_2}^{i_1,\ldots,i_p}$ are nonzero only if

$$\sum_{k=1}^{p} i_k \leqslant j_1+j_2-2 \quad (i_1,\ldots,i_p;\ j_1,j_2\leqslant n), \qquad (2.3)$$

3) the coefficients

$$b_{j_1,j_2}^{i_1,\ldots,i_p}(0,0|0,\ldots,0) = 0, \qquad (2.4)$$

vanish.

We note that the formulated requirements are generalizations of the properties of the W_3-algebra constructed in Ref. 19 by means of an explicit solution of the associativity equations (Jacobi identities). For $n > 3$ such a method of obtaining a nontrivial example of a W_n-algebra becomes extraordinarily difficult, and therefore we guarantee that the Jacobi identity is satisfied by constructing a faithful representation of a nontrivial W_n-algebra within the universal enveloping algebra of the Heisenberg algebra.

For this purpose we consider an $(n-1)$-component free massless scalar field $\boldsymbol{\varphi}(z,\bar z) = \boldsymbol{\varphi}(z) + \bar{\boldsymbol{\varphi}}(\bar z)$, $\boldsymbol{\varphi} = (\varphi_1,...,\varphi_{n-1})$, which is determined by its two-point correlators

$$\langle\varphi_i(z)\varphi_j(0)\rangle = -2\delta_{ij}\ln z, \quad \langle\bar\varphi_i(\bar z)\bar\varphi_j(0)\rangle = -2\delta_{ij}\ln \bar z,$$
$$\langle\varphi_i(z)\bar\varphi_j(0)\rangle = 0. \qquad (2.5)$$

The fields $\varphi_j(z)$ define the operators

$$a_j(s) = \oint \frac{dz}{2\pi i} z^s \frac{\partial\varphi_j(z)}{\partial z} \quad (s=0,\pm1,\ldots), \qquad (2.6a)$$

which generate the Heisenberg algebra:

$$[a_i(s), a_j(u)] = \delta_{ij} s\delta_{s+u,0}. \qquad (2.6b)$$

In the universal enveloping algebra r_{n-1} of the algebra (2.6) we construct a set of fields $W_k(z)$ ($k = 0,2,...,n$) in the following manner. We consider the formal differential operator of the form

$$R_n = :\prod_{m=1}^{n}\left(2^{1/2} i\alpha_0\frac{\partial}{\partial z} + 2^{-1/2}\mathbf{h}_m\frac{\partial\boldsymbol{\varphi}}{\partial z}\right):, \qquad (2.7)$$

where the symbol :....: denotes normal ordering of the fields $\boldsymbol{\varphi}(z)$, and the n vectors $\mathbf{h}_i$ ($i = 1,...,n$); $\sum_{i=1}^{n}\mathbf{h}_i = 0$) form an overcomplete set in the $(n-1)$-dimensional Euclidean space, subject to the conditions

$$\mathbf{h}_i\mathbf{h}_j = \delta_{ij} - 1/n. \qquad (2.8)$$

By means of the usual commutation relations of the operator $\partial/\partial z$ the product R_n can be reduced to the form

$$R_n = \sum_{k=0}^{n} W_k(z)\left(2^{1/2} i\alpha_0\frac{\partial}{\partial z}\right)^{n-k}, \qquad (2.9)$$

which determines uniquely the system of fields $W_k(z)$. In particular

$$W_0 = I, \quad W_1 = \sum_{k=1}^{n} 2^{-1/2}\mathbf{h}_k\frac{\partial\boldsymbol{\varphi}}{\partial z} = 0, \quad \text{since} \sum_{k=1}^{n}\mathbf{h}_k = 0,$$

$$W_2 = \frac{1}{2}\sum_{k<j} :\left(\mathbf{h}_k\frac{\partial\boldsymbol{\varphi}}{\partial z}\right)\left(\mathbf{h}_j\frac{\partial\boldsymbol{\varphi}}{\partial z}\right): + i\alpha_0\sum_{k=1}^{n}(n-k)\,\mathbf{h}_k\frac{\partial^2\boldsymbol{\varphi}}{\partial z^2}$$
$$= -{}^1/_4 :\left(\frac{\partial\boldsymbol{\varphi}}{\partial z}\right)^2: + i\alpha_0\boldsymbol{\rho}\frac{\partial^2\boldsymbol{\varphi}}{\partial z^2}, \qquad (2.10)$$

where the vector

$$\boldsymbol{\rho} = \frac{1}{2}\sum_{k=1}^{n}(n+1-2k)\,\mathbf{h}_k, \quad \boldsymbol{\rho}^2 = \frac{n(n^2-1)}{12}. \qquad (2.11)$$

It is easy to see that the field $W_2(z) = T(z)$ defined by the relation (2.10) generates the Virasoro algebra (1.1) with central charge

$$c=(n-1)-24\alpha_0^2\rho^2=(n-1)[1-2n(n+1)\alpha_0^2]. \tag{2.12}$$

The asymptotic condition $T(z)\propto z^{-4}$ for $z\to\infty$ (Ref. 3) requires that the field $\varphi(z)$ should behave asymptotically, for $z\to\infty$ like[14]

$$\varphi\infty 2i\alpha_0 \ln z, \tag{2.13}$$

where

$$\alpha_0=\alpha_0\rho.$$

In the classical case the transition from the variables $u_k = \partial\varphi_k/\partial z$ to the variables $W_k(z)$ is called the Miura transformation. The properties of this transformation are discussed in the review.[22] In particular, it follows from the results exposed in this review that the Poisson brackets of the variables

$$W_j(s)=\oint \frac{dz}{2\pi i} z^{j+s-1} W_j(z)$$

have a quadratic representation in terms of these same variables,[1)] i.e., the following relation holds

$$\{W_{j_1}(s_1), W_{j_2}(s_2)\} \tag{2.14}$$
$$=\sum_{i_1,i_2}\sum_{u_1,u_2} \tilde{b}_{j_1,j_2}^{i_1,i_2}(s_1,s_2|u_1,u_2) W_{i_1}(u_1) W_{i_2}(u_2),$$

where

$$u_1+u_2=s_1+s_2, \quad i_1+i_2\leqslant j_1+j_2-2, \quad \tilde{b}_{j_1,j_2}^{i_1,i_2}(0,0|0,0)=0$$

(here the brackets are calculated with the help of the relations (2.6), where in (2.6b) one must replace the commutator with the classical Poisson bracket).

The transition to the quantum case leads to a deformation of the coefficients $\tilde{b}$ in Eq. (2.14) and a replacement of the ordinary product by the normal ordered product of the operators $W_j(s)$. We list here only the commutators of the operators $W_j(s)$ with the operators $L_p\equiv W_2(p)$ and $W_3(0)$, which allow us to prove that the fields $W_j(s)$ generate a quantum W_n-algebra, and to determine the coefficients b_p in the relation (2.2) from its associativity condition:

$$[L_p, W_j(k)]= W_j(p+k)[p(j+1)-k] \tag{2.15a}$$

$$-\sum_{q=1}^{j}(2^{1/2}i\alpha_0)^q\frac{(n-j+q)!}{(n-j)!}A_{p,q}^{j}W_{j-q}(p+k), \tag{2.15b}$$

$$[W_3(0), W_j(k)]$$
$$= 2k(\delta_{j,n}-1)W_{j+1}(k)+2^{1/2}i\alpha_0(2j-3+k)kW_j(k)$$
$$-\frac{2}{n}\sum_{q=1}^{j}(2^{1/2}i\alpha_0)^{q-1}\frac{(n-j+q)!}{(n-j)!}$$
$$\times\Big\{\sum_{p=-\infty}^{\infty} C_q^{p-k}{:}W_{j-q}(p)L_{k-p}{:} + B_{k,q}^{j}W_{j-q}(k)\Big\}.$$

Here the coefficients $A_{p,q}^j$, $B_{k,q}^j$, C_q^p are defined by the relations

$$C_q{}^p=\frac{p(p-1)\dots(p-q+1)}{q!},$$

$$A_{p,q}^j=C_{1+q}^{1+p}\left[\frac{q-1}{4n\alpha_0^2}-j+1-\frac{(n-1)(q-1)}{2}\right],$$

$$B_{k,q}^j=(-1)^q C_{q+1}^{q+1-i}k+[C_{q+2}^{q-1+i}(-1)^q-C_{q+2}^{i+1}] \times\left[\frac{q-1}{4n\alpha_0^2}-j+1-\frac{(n-1)(q-1)}{2}\right], \tag{2.15c}$$

where in the last equation the variable i equals $k/2$ for even k and $(k+1)/2$ for odd k.

The corresponding coefficients $\tilde{b}$ in Eq. (2.14) can be obtained from the relations (2.15) if one sets $\alpha_0=\infty$.

3. THE FIELD SPACE OF A W_n-INVARIANT FIELD THEORY AND ITS REPRESENTATION IN Γ_{n-1}

The field space of a W_n-invariant field theory can be classified in terms of the irreducible representations $[\Phi(\beta)]$ of the W_n-algebra, i.e., represented in the form

$$\{A\}=\bigoplus_{\beta}[\Phi(\beta)]. \tag{3.1}$$

Each of the subspace $[\Phi(\beta)]$ is generated by a W_n-invariant (principal) field $\Phi(\beta)$ satisfying the equations

$$W_k(s)\Phi(\beta)=0, \quad W_k(0)\Phi(\beta)=w_k(\beta)\Phi(\beta),$$
$$(k=2,\dots,n;\ s>0) \tag{3.2}$$

with some numerical parameters $w_k(\beta)$. We note that the last equations are compatible on account of the condition (2.4). The set of parameters $[w(\beta)]\equiv[w_2(\beta),\dots,w_n(\beta)]$ completely determines the representation $[\Phi(\beta)]$. The whole space $[\Phi(\beta)]$ consists of fields which are obtained from the field $\Phi(\beta)$ by successive application of the operators $W_k(s)$ with $s<0$ and $k=2,\dots,n$.

For the W_n-algebra defined by the commutation relations (2.15) the space (A) can be represented in Γ_{n-1}. For this we define in Γ_{n-1} the fields $V_\beta = {:}\exp[i\beta\varphi(z)]{:}$. We note that on account of the asymptotic condition (2.14) the correlation functions of the fields V_β have the form

$$\langle V_{\beta_1}(z_1)\dots V_{\beta_N}(z_N)\rangle=\prod_{i<j}^{N}(z_i-z_j)^{2\beta_i\beta_j}G(\beta_1,\dots,\beta_N), \tag{3.3a}$$

where

$$G(\beta_1,\dots,\beta_N)=\begin{cases}1, & \sum_{i=1}^{N}\beta_i=2\alpha_0\\ 0, & \sum_{i=1}^{N}\beta_i\neq 2\alpha_0\end{cases}. \tag{3.3b}$$

One can verify that the fields $V_\beta(z)$ satisfy the equations (3.2), i.e., are W_n-invariant fields ($\Phi([w])=V_\beta$), where the eigenvalues $[w(\beta)]$ are determined by the most singular term of the expansion operator

$$W_k(z)V_\beta(0)=w_k(\beta)z^{-k}V_\beta(0)+O(z^{-k+1}). \tag{3.4}$$

It follows from Eqs. (2.7) and (2.9) for the fields $W_k(z)$ that the number $w_k(\beta)$ can be obtained by means of the relation

$$\prod_{m=1}^{n}\left(2^{1/2}i\alpha_0\frac{\partial}{\partial z}-2^{1/2}iz^{-1}\mathbf{h}_m\beta\right)=\sum_{k=1}^{n}w_k(\beta)z^{-k}\left(2^{1/2}i\alpha_0\frac{\partial}{\partial z}\right)^{n-k}. \tag{3.5}$$

Applying the relation (3.5) to the functions z^j ($j=0,$

$1,\ldots,n-2$), we obtain a system of linear equations for the parameters $w_k(\beta)$

$$(2^{1/2}i)^n\prod_{m=0}^{n}[(n-m+j)\alpha_0-\mathbf{h}_m\beta] \tag{3.6}$$

$$=\sum_{k=0}^{j}\frac{j!}{(j-k)!}(2^{1/2}i\alpha_0)^k w_{n-k}(\beta).$$

The solution of this system has the form

$$w_k(\beta)=(-2^{1/2}i)^k\sum_{1\leq i_1<i_2\ldots<i_k\leq n}\prod_{m=1}^{k}[\alpha_0(k-m)+\mathbf{h}_{i_m}\beta]. \tag{3.7a}$$

In particular,

$$w_2(\beta)\equiv\Delta(\beta)=\beta^2-2\alpha_0\beta. \tag{3.7b}$$

We note that the left-hand side of the equation (3.6) is invariant to the discrete transformation of the parameters $\beta\to\beta'$:

$$\mathbf{h}_m\beta+m\alpha_0=\mathbf{h}_{m'}\beta'+\alpha_0 m', \tag{3.8}$$

where the numbers m' are obtained from the numbers $m=1,\ldots,n$ by some permutation P. Thus, to a given set of numbers $[w(\beta)]$ correspond $n!$ different values of the parameters $\boldsymbol{\beta}$. In particular, to the permutation $(1,\ldots,n)\to(n,n-1,\ldots,1)$ corresponds the transformation $\beta\to 2\alpha_0-\beta$: where the vector $\boldsymbol{\beta}^*$ is defined by

$$\beta^*\mathbf{h}_i=\beta\mathbf{h}_{n-i}. \tag{3.9}$$

The representation $[\Phi([w])]=[\Phi(\beta)]=[V_\beta]$ of the W_n-algebra in Γ_{n-1} can be constructed by applying to the invariant field V_β the operators $W_k(s)$ with $s<0$. Then one must identify the representations $[V_\beta]$ and $[V_{\beta'}]$, where the numbers β and β' are related by (3.8); in particular, $[V_\beta]=[V_{2\alpha_0-\beta^*}]$.

4. STRONGLY DEGENERATE REPRESENTATIONS AND THEIR SPECTRUM

The degenerate representations of the algebra of supplemental symmetries play an important role in the construction of exact solution of conformal quantum field theory. Almost all the known solutions are related to representations of this kind (see Refs. 3, 8, 10, 13).

A representation $[\Phi(\beta)]$ of a W_n-algebra will be called degenerate if the space of fields obtained from the field $\Phi(\beta)$ by applying the operators $W_k(s)$ with $s<0$ contains the null-vector χ_N satisfying the equations

$$W_k(s)\chi_N=0\quad(k=2,\ldots,n;\ s>0),$$
$$L_0\chi_N=(\Delta+N)\chi_N \tag{4.1}$$

for some integer N (called the level of degeneracy). In this case our representation becomes reducible. In order to obtain irreducible representations we must set the vector χ_N together with the whole subspace $[\chi_N]$ it generates equal to zero. A representation $[\Phi(\beta)]$ may, in general, contain several independent null-vectors χ_{N_i}. This requires factoring the space $[\Phi(\beta)]$ with respect to the whole subspace $[\chi_{N_i}]$. We shall call the representation $[\Phi(\beta)]$ strongly degenerate if it contains no fewer than $n-1$ independent null-vectors.

A representation of a W_n-algebra in Γ_{n-1} allows for an explicit construction of the null-vectors in $[V(\beta)]$ and to obtain restrictions on the parameters β corresponding to strongly degenerate representations. With this in mind we consider the following operators in Γ_{n-1}:

$$Q_j^{(\pm)}=\oint\frac{dz}{2\pi i}V_j^{(\pm)}(z)\equiv\oint\frac{dz}{2\pi i}:\exp[i\alpha_\pm\mathbf{e}_j\boldsymbol{\varphi}(z)]:, \tag{4.2}$$

where the vectors e_j ($j=1,\ldots,n-1$) form a basis in the space of positive roots of the algebra sl(n), and are determined by the relations

$$\mathbf{e}_i=\mathbf{h}_i-\mathbf{h}_{i+1},\quad \mathbf{e}_i\mathbf{e}_j=2\delta_{ij}-\delta_{i+1,j}-\delta_{i,j+1},$$
$$\mathbf{e}_i\mathbf{h}_j=\delta_{ij}-\delta_{i+1,j},\quad \mathbf{e}_i\boldsymbol{\rho}=1. \tag{4.3}$$

If the parameters $\alpha_\pm$ satisfy the equations

$$\alpha_++\alpha_-=\alpha_0,\quad \alpha_+\alpha_-=-{}^1/_2, \tag{4.4}$$

then the operators $Q_j^{(\pm)}$ commute with all the operators $W_k(s)$ ($k=2,\ldots,n;\ s=0,\pm1,\ldots$). To prove this assertion it suffices to prove that the singular part of the operator product of the fields $W_k(\eta)$ and $V_j^{(\pm)}(z)=:\exp[i\alpha_\pm\mathbf{e}_j\boldsymbol{\varphi}(z)]:$ can be represented in the form

$$W_k(\eta)V_j^{(\pm)}(z)=\frac{\partial}{\partial z}X_k^j(z,\eta)+O(1). \tag{4.5}$$

If one considers the differential operators $\partial^m/\partial\eta^m$ ($m=0,1,\ldots,n$) as elements of an independent basis in an ($n+1$)-dimensional linear space, and takes into account that they act only on functions which depend on the variable η, then the last relation can be rewritten in the form

$$R_n(\eta)V_j^{(\pm)}(z)=\frac{\partial}{\partial z}\sum_{m=2}^{n}X_m^j(z,\eta)\frac{\partial^{n-m}}{\partial\eta^{n-m}}+O(1). \tag{4.6}$$

In order to expand the operator product $R_n(\eta)V_j^{(\pm)}(z)$ we make use of Wick's theorem. It is necessary to consider that owing to the relations (4.3) the fields $e_j\varphi(z)$, which enter into the operator $V_j^{(\pm)}$ have nonvanishing contractions only with two successive factors in the product $R_n(\eta)$:

$$:\left[2^{1/2}i\alpha_0\frac{\partial}{\partial\eta}+2^{-1/2}\mathbf{h}_j\frac{\partial\boldsymbol{\varphi}}{\partial\eta}\right]\left[2^{1/2}i\alpha_0\frac{\partial}{\partial\eta}+2^{-1/2}\mathbf{h}_{j+1}\frac{\partial\boldsymbol{\varphi}}{\partial\eta}\right]:.$$

which implies that the operator product $R_n(\eta)V_j^{(\pm)}(z)$ has the form (4.6).

$$:\left(2^{1/2}i\alpha_0\frac{\partial}{\partial\eta}+2^{-1/2}\mathbf{h}_j\frac{\partial\boldsymbol{\varphi}}{\partial\eta}\right)$$
$$\times\left(2^{1/2}i\alpha_0\frac{\partial}{\partial\eta}+2^{-1/2}\mathbf{h}_{j+1}\frac{\partial\boldsymbol{\varphi}}{\partial\eta}\right):V_j^{(\pm)}(z)$$
$$=\frac{\partial}{\partial z}[(z-\eta)^{-1}V_j^{(\pm)}(z)]+O(1), \tag{4.7}$$

The operators $Q_j^{(\pm)}$ which commute with all the operators $W_k(s)$ allow one to construct null vectors χ_{N_i} by the standard method and to obtain the restrictions on the parameters β. This procedure has been described in detail in Ref. 13 for the W_3-algebra. The transition to the case $n>3$

does not lead to any complications. Therefore we list only the equations for the vectors β, which determine the strongly degenerate representations of the W_n-algebra. The corresponding equations have the form

$$e_i\beta=\alpha_+(1-l_i)+\alpha_-(1-l_i'),\quad i=1,\dots,n-1, \tag{4.8}$$

where l_i, l_i' are positive integers. In the case when the relations (4.8) are satisfied, the representation $[V_\beta]$ will contain $n-1$ null-vectors χ_{N_i} with degeneracy level $N_i = l_i l_i'$. The solution of the equations (4.8) can be represented in the form

$$\beta(l|l')\equiv\beta(l_1,\dots,l_{n-1}|l_1',\dots,l_{n-1}')$$

$$=\sum_{i=1}^{n-1}[\alpha_+(1-l_i)+\alpha_-(1-l_i')]\omega_i, \tag{4.9}$$

where the vectors ω_i define the fundamental weights of the Lie algebra sl(n), which satisfy the conditions

$$\omega_i e_j=\delta_{ij}. \tag{4.10}$$

Substituting the vectors $\beta(l_i,\dots,l_{n-i}|l_i',\dots,l_{n-1}')$ into Eq. (3.7), we obtain an array of numbers $\omega_k(l_1,\dots,l_{n-1}|l_1',\dots,l_{n-1})$ which determines the spectrum of the strongly degenerate representations of the W_n-algebra. In particular, the anomalous dimensions of the strongly degenerate fields is determined by the formula

$$\Delta(l|l')\equiv w_2(l|l')=\Big[\sum_{i=1}^{n-1}(l_i\alpha_+ + l_i'\alpha_-)\omega_i\Big]^2-\alpha_0^2. \tag{4.11}$$

5. THE CORRELATION FUNCTIONS AND THE STRUCTURE OF THE OPERATOR ALGEBRA OF STRONGLY DEGENERATE FIELDS

We denote by $\Phi(l|l')$ the strongly degenerate principal fields $\Phi([w(l|l')])$ and consider the L-point correlation functions of the form

$$\langle\Phi(l_1|l_1',z_1)\dots\Phi(l_L|l_L',z_L)\rangle, \tag{5.1a}$$

where we have temporarily omitted the dependence of the correlation functions on the variables $\bar z_i$. The representation of the W_n-algebra and invariant fields

$$\Phi(l|l')\to V_{\beta(l|l')}\ (V_{2\alpha_0-\beta^*(l|l')})$$

described in the previous sections, in terms of free fields, allows one to construct in the standard manner an integral representation for the correlation functions (5.1a) (see Refs. 13 and 14). Indeed, up to normalization, one can associate to the correlator (5.1a) the expression

$$\Big\langle\prod_{i=1}^{n-1}\Big[\prod_{a=1}^{M_i}\oint d\xi_a^{(i)}V_i^{(+)}(\xi_a^{(i)})$$

$$\times\prod_{b=1}^{M_i'}\oint d\eta_b^{(i)}V_i^{(-)}(\eta_b^{(i)})\Big]V_{\beta_1}(z_1)\dots V_{\beta_L}(z_L)\Big\rangle, \tag{5.1b}$$

where the expectation value in (5.1b) is taken according to Eq. (3.3), β_i denote either $\beta(l_i|l_i')$ or $2\alpha_0-\beta^*(l_i|l_i')$, and the integration is over noncontractible closed curves on the Riemann surface of the integrand. The correlation functions (5.1) are nonzero if one can choose integers M_i, M_i' ($i=1,\dots,n$) such that the condition (3.3b) should be satisfied

$$\sum_{i=1}^{L}\beta(i)+\alpha_+\sum_{j=1}^{n-1}M_je_j+\alpha_-\sum_{j=1}^{n-1}M_j'e_j=2\alpha_0. \tag{5.2}$$

To calculate the three-point correlators (5.1) in the integral (5.1b) one can choose

$$\beta(1)=\beta[l(1)|l'(1)],\quad \beta(2)=\beta[l(2)|l'(2)],$$

$$\beta(3)=2\alpha_0-\beta^*[l(3)|l'(3)].$$

Then the equation (5.2) takes on the form

$$\alpha_+\sum_{i=1}^{n-1}\{[1-l_i(1)]+[1-l_i(2)]-[1-l_{n-i}(3)]\}\omega_i$$

$$+\alpha_-\sum_{i=1}^{n-1}\{[1-l_i'(1)]+[1-l_i'(2)]-[1-l_{n-i}'(3)]\}\omega_i$$

$$+\alpha_+\sum_{i=1}^{n-1}M_ie_i+\alpha_-\sum_{i=1}^{n-1}M_i'e_i=2\alpha_0. \tag{5.3}$$

We associate to each field $\Phi(l|l')$ two Z_n-charges, $q_+(l)$ and $q_-(l')$:

$$q_+(l)=\sum_{i=1}^{n-1}(l_i-1)i,\quad q_-(l')=\sum_{i=1}^{n-1}(l_i'-1)i. \tag{5.4}$$

If the numbers α_+ and α_- are incommensurable, then multiplying each side of Eq. (5.3) by ω_{n-1} and making use of the relations (1.4) and (4.10), it is easy to see that the condition for M_i (M_i') to be integers requires conservation of the Z_n-charges q_+ and q_-, i.e.,

$$q_\pm(1)+q_\pm(2)+q_\pm(3)=0\ (\mathrm{mod}\ n). \tag{5.5}$$

This conservation law is valid for any correlation function of the fields $\Phi(l|l')$. Some four-point correlation functions for the fields $\Phi(l|l';z,z)$ are listed in the Appendix.

The analysis of the correlation functions shows that the fields $\Phi(l|l')$ form a closed operator algebra, i.e., that the operator expansion of the product of the fields $\Phi(l|l')$ and $\Phi(m|m')$ at nearby points admits a symbolic representation of the form

$$\Phi(l|l')\Phi(m|m')=\sum_{(s|s')}C^{(s|s')}_{(l|l')(m|m')}[\Phi(s|s')], \tag{5.6}$$

where the expression in square brackets contains the contribution of the fields that pertain to the representation $[\Phi(s|s')]$. The coefficiens **C** in front of the principal fields $[\Phi(s|s')]$ in the expansion (5.6) are called the structure constants of the operator algebra. The possible sets of numbers $(s|s')$ for fixed sets $(I|l')$ and $(m|m')$ for which the coefficients **C** are not zero define the selection rules or the qualitative structure of the operator algebra of the fields $\Phi(l|l')$.

The qualitative structure of the operator algebra of the fields $\Phi(l|l')$ can be described by means of a Clebsch-Gordan expansion of the product of finite-dimensional representations of the Lie algebra sl(n) with highest weight determined by the arrays of numbers (l) $((l'))$ and $(m)((m'))$. The coefficients $\mathbf{C}$ are nonzero only for those arrays of numbers (s) $((s'))$ which define representations occurring in the Clebsch-Gordon expansion of the product of the representation $(l)\otimes(m)$ $((l')\otimes(m'))$. For the cases $n=2, 3$, such selection rules have been established in Refs. 3 and 13.

6. MINIMAL MODELS

If the numbers α_+ and α_- which satisfy (4.4) are incommensurable then the operator algebra (5.6) contains the fields $\Phi(l|l')$ with all possible sets of numbers (l) and (l'). However, if the parameters α_+ and α_- satisfy the relation

$$2\alpha_+^2=-\alpha_+/\alpha_-\equiv\rho=p/p', \tag{6.1}$$

where p and p' are mutually prime integers, then one can construct a field theory containing a finite number of principal fields $\Phi(l|l')$ (a similar situation arises also for other known solutions of two-dimensional quantum field theory, see Refs. 3, 10, and 13). In particular, if the condition (6.1) is satisfied, it is necessary to make the following identification of the fields Φ:

$$\Phi(l|l')=\Phi(\tilde{l}_1|\tilde{l}_1')=\Phi(\tilde{l}_2|\tilde{l}_2')=\Phi(\tilde{l}_{n-1}|\tilde{l}'_{n-1}), \tag{6.2}$$

where

$$(\tilde{l}_k)=\Big(l_{n-k+1}, l_{n-k+2},\dots,l_{n-1}, p'-\sum_{i=1}^{n-1} l_i, l_1,\dots,l_{n-k-1}\Big),$$

$$(\tilde{l}_k')=\Big(l'_{n-k+1}, l'_{n-k+2},\dots,l'_{n-1}, p-\sum_{i=1}^{n-1} l_i', l_1',\dots,l'_{n-k-1}\Big),$$

i.e., the given fields are characterized by identical values of the numbers w_m:

$$[w(l|l')]=[w(\tilde{l}_s|\tilde{l}_{s'})],\ s=1,\dots,n-1.$$

The relations (6.2) lead to additional selection rules, an analysis of which shows that the set of fields

$$\{A\}=\bigoplus_{(l),(l')}[\Phi(l|l')], \tag{6.3}$$

where the arrays of integers (l) and (l') are subject to the conditions

$$\sum_{i=1}^{n-1} l_i\leqslant p'-1,\quad \sum_{i=1}^{n-1} l_i'\leqslant p-1,$$

form a closed operator algebra (a minimal model).

Of greatest interest, from a physical point of view, is the "principal series" of minimal models, which corresponds to the choice $p'=p+1$. In this case the parameter

$$\alpha_0^2=(\alpha_++\alpha_-)^2=1/2p(p+1)$$

and the central charge c in Eq. (2.12) take on the values (1.2). Making use of a method developed in Ref. 23 one can show that the minimal models of the principal series with $p=n+1, n+2,\dots$ satisfy the positivity condition (see Ref. 20). We denote such models by $[Z_n^{(p)}]$. After identification of (6.2) with $p'=p+1$ the $[Z_n^{(p)}]$ models contain

$$p!(p-1)!/n!(n-1)!(p-n)!(p-n+1)!$$

spinless local fields $\Phi(l|l')$ with dimensionalities $\Delta(l|l')$, defined by the equations (1.3). All values of the numbers $w_k(l|l')$ with $k>2$ corresponding to these fields can be obtained from the formulas (6.1), (4.9), and (3.7).

We note that on account of the commensurability of the parameters α_+ and α_- and the identifications (6.2) the conservation law of the two Z_n-charges q_+ and q_- is violated in the minimal models, and only one Z_n charge q is conserved, which has for the $[Z_n^{(p)}]$ models the value

$$q(l|l')=kq_+(l)-(1+k)q_-(l'), \tag{6.4a}$$

where $k\equiv p(\mathrm{mod}\ n)$; the operator algebra of these models is invariant with respect to the transformation

$$\Phi(l|l')\to\exp[2\pi iq(l|l')/N]\Phi(l|l'). \tag{6.4b}$$

7. CONCLUSION

In the preceding sections we have constructed an infinite series of exactly solvable models in two-dimensional conformal quantum field theory, exhibiting explicit Z_n symmetry (the $[Z_n^{(p)}]$ models). An important problem which has remained essentially unresolved, consist in the description of a class of statistical systems which exhibit the critical behavior of $[Z_n^{(p)}]$. We note that (for a given n) the simplest models, namely $[Z_n^{(n+1)}]$, are characterized by the value of the central charge $c=2(n-1)/(n+2)$ and agree with the models $[Z_n]$ constructed earlier in Ref. 4. These models describe, in particular, the Z_n-generalizations of the ising model. In this case there is the following correspondence between the spin fields σ_k, the parafermions ψ_k, and the thermal operators $\varepsilon^{(j)}$ of the $[Z_n]$ model and the fields $\Phi(l|l')$ of the model $[Z_n^{(n+1)}]$ (up to the identifications (6.2)):

$$\sigma_k\to\Phi(l|l')\begin{cases} l_i=1, & i\neq k,\quad l_k=2\\ l_i'=1, & i=1,\dots,n-1\end{cases} \tag{7.1a}$$

$$\psi_k\to\Phi(l|l')\begin{cases} l_i=1, & i=1,\dots,n-1\\ l_i'=1, & i\neq k,\quad l_k=2\end{cases}, \tag{7.1b}$$

$$\varepsilon^{(j)}\to\Phi(l|l')\begin{cases} l_i=1, & i\neq j,\quad n-j,\quad l_i=l_{n-j}=2\\ l_i'=1, & i=1,\dots,n-1\end{cases} \tag{7.1c}$$

The fields σ_k, ψ_k, and $\varepsilon^{(j)}$ have, respectively, the dimensions

$$d_k=k(n-k)/2n(n+2),$$

$$\Delta_k=k(n-k)/n,\quad D_j=j(j+1)/(n+2).$$

We also note that the integral representation for the $[Z_n^{(p)}]$ models, described in Sec. 5, may be useful for the investigation of the many-point correlation functions in an $\mathrm{su}(2)\times\mathrm{su}(2)$ invariant Wess-Zumino model,[12,20] since the correlation functions of this theory are simply related to the correlators in the $[Z_b]=[Z_n^{(n+1)}]$ models.[4]

In conclusion the authors use this occasion to express their gratitude to A. A. Belavin, V. G. Drinfel'd, A. B. Zamolodchikov, B. L. Feĭgin, and T. G. Khovanova for useful remarks and discussions.

APPENDIX

In this Appendix we list some of the simplest four-point functions in the $[Z_n^{(p)}]$ models. We denote by $\sigma_k(z,\bar{z})$ and $\psi_k(z,\bar{z})$ the fields $\Phi(l|l')$ where the arrays of numbers $(l|l')$ are given by Eqs. (7.1a,b). It follows from Eq. (1.3) that the fields σ_k and ψ_k have the dimensions

$$d_k=(p-n)k(n-k)/2n(p+1),\quad \Delta_k=(p+n+1)k(n-k)/2np.$$

Making use of the projective invariance of the theory we set

$$z_1=\bar{z}_1=0;\quad z_2=z,\ z_2=\bar{z};\quad z_3=\bar{z}_3=1;\quad z_4=\bar{z}_4=\infty$$

and consider the correlation function

$$G_l(z,\bar{z})=\langle\sigma_1(0,0)\sigma_1^+(z,\bar{z})\sigma_l(1,1)\sigma_l^+(\infty,\infty)\rangle, \qquad \text{(A.1)}$$

where $\sigma_l^+\equiv\sigma_{n-l}$. To such a set of fields one can associate the following value of the vectors $\beta(i)$ in Eq. (5.1b):

$$\beta(1)=-\alpha_+\omega_1,\quad \beta(2)=-\alpha_+\omega_{n-1},$$
$$\beta(3)=-\alpha_+\omega_l,\quad \beta(4)=2\alpha_0+\alpha_+\omega_l. \qquad \text{(A.2)}$$

Then the equation (5.2) takes on the form

$$\alpha_+(\omega_1+\omega_{n-1})=\alpha_+\sum_{i=1}^{n-1}M_i\mathbf{e}_i+\alpha_-\sum_{i=1}^{n-1}M_i'\mathbf{e}_i', \qquad \text{(A.3)}$$

and its simplest solutions can be defined by the formulas

$$M_i'=0,\quad M_i=1,\quad i=1,\dots,n-1.$$

In order to obtain the correlation function for the local fields $\Phi(l\,|\,l';z,\bar{z})$, one must follow Refs. 14, replace the integrand in Eq. (5.1b) by the square of its absolute value, and replace the contour integrals in (5.1b) by integrals over the whole two-dimensional plane:

$$\oint d\xi\to\int\frac{d\xi\,d\bar{\xi}}{2\pi i}.$$

Then the function $G_l(z,\bar{z})$ is represented by the integral

$$G_l(z,\bar{z})=C|z|^{2\rho/n}|1-z|^{2l\rho/n}\int\prod_{i=1}^{n-1}\frac{d\xi_i\,d\bar{\xi}_i}{2\pi i}|\xi_1|^{-2\rho}|\xi_l-1|^{-2\rho}$$
$$\times|\xi_{n-1}-z|^{-2\rho}\prod_{i=1}^{n-2}|\xi_i-\xi_{i+1}|^{-2\rho}, \qquad \text{(A.4)}$$

where $\rho=2\alpha_+^2=p/(p+1)$.

With the help of the relation

$$\int\frac{d\eta\,d\bar{\eta}}{2\pi i}|\xi_1-\eta|^c|\xi_2-\eta|^d=\text{const}\,|\xi_1-\xi_2|^{2+c+d}$$

the integral (A.4) can be reduced to the form

$$G_l(z,\bar{z})=C'|z|^{2\rho/n}|1-z|^{2l\rho/n}$$
$$\times\int\frac{d\xi\,d\bar{\xi}}{2\pi i}|\xi|^{-2+2l(1-\rho)}|\xi-1|^{-2\rho}|\xi-z|^{-2+2(n-l)(1-\rho)}$$
$$=|1-z|^{2l\rho/n}\{|z|^{-4d_1}|f_1(z)|^2+h_l|z|^{2\rho/n}|f_2(z)|^2\}, \qquad \text{(A.5a)}$$

where f_1 and f_2 are hypergeometric functions:

$$f_1=F\Big(\frac{p}{p+1},\frac{l}{p+1},\frac{n}{p+1},z\Big),$$
$$f_2=F\Big(1-\frac{n-l}{p+1},2-\frac{n+1}{p+1},2-\frac{n}{p+1},z\Big), \qquad \text{(A.5b)}$$
$$h_l=\frac{(p-n)^2}{(p+1-n)^2}\Gamma\Big(\frac{1}{p+1}\Big)\Gamma^2\Big(\frac{n}{p+1}\Big)$$
$$\times\Gamma\Big(1-\frac{n+1}{p+1}\Big)\Gamma\Big(1-\frac{n-l}{p+1}\Big)$$
$$\times\Gamma\Big(1-\frac{l}{p+1}\Big)\Big/\Gamma\Big(\frac{p}{p+1}\Big)\Gamma^2\Big(1-\frac{n}{p+1}\Big)\Gamma\Big(\frac{n+1}{p+1}\Big)$$
$$\times\Gamma\Big(\frac{n-l}{p+1}\Big)\Gamma\Big(\frac{l}{p+1}\Big).$$

The constant C' in (A.5a) is chosen in such a way that the fields should be normalized by the condition

$$\langle\sigma_k(z,\bar{z})\sigma_l^+(0,0)\rangle=\delta_{k,l}|z|^{-4d_k}. \qquad \text{(A.6)}$$

The corresponding correlators for the fields ψ_k can be obtained from Eqs. (A.5) by means of the substitution $p\to-p-1$, $d_k\to\Delta_k$.

The correlation function

$$G_l^k(z,\bar{z})=\langle\sigma_k(0,0)\sigma_l(z,z')$$
$$\times\sigma_k^+(1,1)\sigma_l(\infty,\infty)\rangle,\quad k\leqslant l,\ k+l\leqslant n \qquad \text{(A.7)}$$

for $k>1$ cannot be expressed in terms of hypergeometric functions, however it can be expressed in terms of multiple contour integrals introduced and investigated in Ref. 14. In this case we can express the parameters β_i in Eq. (5.1b) in the form

$$\beta(1)=-\alpha_+\omega_k,\quad \beta(2)=-\alpha_+\omega_l,$$
$$\beta(3)=-\alpha_+\omega_{n-k},\quad \beta(4)=2\alpha_0+\alpha_+\omega_l. \qquad \text{(A.8)}$$

The equation (5.2) can then be written as follows:

$$\alpha_+(\omega_k+\omega_{n-k})=\sum_{i=1}^{n-1}(\alpha_+M_i+\alpha_-M_i')\mathbf{e}_i. \qquad \text{(A.9)}$$

The simplest solution of Eq. (A.9) has the form

$$M_i'=0,\ i=1,2,\dots,n-1;\quad M_i=M_{n-i}=i,\ i\leqslant k;$$
$$M_k=M_{k+1}=\dots=M_{n-k}=k. \qquad \text{(A.10)}$$

We define the functions

$$D_M(\xi)=\prod_{a<b}^{M}|\xi_a-\xi_b|^2,\quad A_{M,N}(\xi,\eta)=\prod_{a=1}^{M}\prod_{b=1}^{N}|\xi_a-\eta_b|^2. \qquad \text{(A.11)}$$

Then the correlator $G_l^k(z,\bar{z})$ can be represented by the following integral:

$$G_l^k(z,\bar{z})=|z|^{2k(n-l)\rho/n}|1-z|^{2kl\rho/n}$$
$$\times\int\prod_{i=1}^{n-1}\prod_{a=1}^{M_i}d\xi_a^{(i)}\frac{d\bar{\xi}_a^{(i)}}{2\pi i}\prod_{a=1}^{k}|\xi_a^{(k)}|^{-2\rho}$$
$$\times|\xi_a^{(l)}-z|^{-2\rho}|\xi^{(n-k)}-1|^{-2\rho}\prod_{i=1}^{n-2}D_{M_i}^{2\rho}(\xi^{(i)})A_{M_i,M_{i+1}}(\xi^{(i)},\xi^{(i+1)}). \qquad \text{(A.12)}$$

By means of the relation

$$\int\prod_{a=1}^{M}\frac{d\xi_a\,d\bar{\xi}_a}{2\pi i}D_M(\xi)A_{M,M}^{c}(\xi,\eta)A_{M,1}^{d}(\xi,\sigma)$$

$$=\left[\frac{\Gamma(1+c)}{\Gamma(-c)}\right]^{M}\frac{\Gamma(1+d)\Gamma(-M-Mc-d)}{\Gamma(-d)\Gamma(M+1+Mc+d)}D_M^{1+2c}(\eta)$$

$$\times A_{M,1}^{1+c+d}(\eta,\sigma) \tag{A.13}$$

and the change of variables $\xi\to z(\xi-1)/(\xi-z)$ the integral (A.12) can be reduced to the form

$$G_l^k(z,\bar z)=C'(|z|^l|1-z|^{(n-l)})^{2k(p-n)/n(p+1)}$$

$$\times\int\prod_{a=1}^{k}\frac{d\xi_a\,d\bar\xi_a}{2\pi i}|\xi_a|^{2\gamma_1}|\xi_a-1|^{2\gamma_2}|\xi_a-z|^{2(p-n)/(p+1)}D_k^{/(p+1)}(\xi), \tag{A.14}$$

where

$$\gamma_1=(p+k+l-n)/(p+1),\quad \gamma_2=-(p+k+l)/(p+1).$$

Integrals of the type (A.14) have been investigated in detail in Ref. 14. The results of these papers allow, in particular, to determine all the structure constants $C_{k,l}^{(3)}$ where the superscript denotes an arbitrary principal field which appears in the operator expansion of the product of the fields σ_k and σ_l. Thus, e.g.,

$$(C_{k,l}^{k+l})^2=P(n-k+l)P(l)P(k)P(n)/P(k+l)P(n-l)P(n-k), \tag{A.15}$$

where

$$P(s)=\prod_{i=1}^{s}\Gamma\left(\frac{i}{p+1}\right)\Gamma^{-1}\left(1-\frac{i}{p+1}\right)=P(p-s).$$

We note that in the limit when the numbers $p, n, k, l\to\infty$ such that $p-n=\text{const}$, $k/n=\mu$, $l/n=\nu$, the correlation function $G_l^k(z,\bar z)$ is expressed in terms of the density $\rho(z,\bar z)$ in a system of k particles interacting according to the law $-\ln|\xi_i-\xi_j|^2$ and being situated at a temperature $(kT)^{-1}=2/(p+1)$; there are additional sources placed at the points 0, 1, and ∞ in the system. In the limit $k\to\infty$, $p\to\infty$ the density of particles in this system can be expressed by means of the mean field method in terms of solutions of the Liouville equation with fixed singularities (determined by the exponents γ_1 and γ_2 in (A.14) at the points 0, 1, and ∞). The solutions of the Liouville equation with such singularities can be expressed in terms of hypergeometric functions (see Ref. 24). Simple calculations lead to the following result:

$$G_l^k(z,\bar z)=\text{const}\{|z|^{2\mu\nu}|1-z|^{2\mu(1-\nu)}[\rho(z,\bar z)]^{1/2}\}^{(p-n)}+O(n^{-1}), \tag{A.16}$$

where

$$[\rho(z,\bar z)]^{-1/2}=\int\frac{d\xi\,d\bar\xi}{2\pi i}|\xi|^{-2\mu}|\xi-z|^{-2\nu}|\xi-1|^{-2+2\mu}$$

$$=\frac{\Gamma(\mu)\Gamma(\nu)\Gamma(1-\mu-\nu)}{\Gamma(1-\mu)\Gamma(1-\nu)\Gamma(\mu+\nu)}|F(\nu,\nu,\mu+\nu,z)|^2$$

$$+\frac{\Gamma(1-\mu)\Gamma(1-\nu)\Gamma(-1+\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)\Gamma(2-\mu-\nu)}$$

$$\times|z|^{2-2\mu-2\nu}|F(1-\mu,1-\mu,2-\mu-\nu,z)|^2.$$

[1)]This important fact was called to our attention by T. C. Khovanova and V. G. Drinfel'd.

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