Nonperturbative measurement in systems with spectrally selective squeezing of quantum fluctuations

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We consider a Weber detector with an optical readout system from the standpoint of the quantum theory of measurement. We take explicit account of the effect on the antenna of the measurement device, which consists of three circulators, a homodyne detector, and two degenerate parametric amplifiers, which determine the state of the field in the backward wave. Relative to external effects on the mechanical oscillator, the sensivity of such a system is limited, within the scope of this model, by the squeezing factor in the parametric amplifiers. There is no need in our approach to postulate any reduction in the state of the system due to the act of measurement.

1. INTRODUCTION

The high-Q-solid-state Weber detector with an electromagnetic parametric detection system for mechanical vibrations is one type of existing ground-based gravitational antenna.[1] The mechanical resonator (detector) is usually no more than 2–3 m in size, and the fundamental longitudinal mode typically has a frequency of $\sim 10^4$ rad/sec. Such a detector would produce an extremely weak response to a burst of gravitational radiation that conforms to astrophysical predictions.[2] In particular, the amplitude of the end-face vibrations can be comparable to, or even less than, the quantum certainty in the state of the oscillator in coordinate space. In this respect, future progress in gravitational-wave detection experiments is tied to a new measurement technique, that of so-called quantum nonperturbative measurement (QNM).[3–5] The problem dealt with by QNM involves the search for measurement techniques and specific devices which make it possible, using the response of a test oscillator, to detect forces weaker than the quantum sensitivity threshold or "standard quantum limit":

$$F_q=\zeta\hat{\tau}^{-1}(\hbar m\omega_\mu)^{1/2}. \tag{1.1}$$

Here $\zeta\sim 1$ is a statistical factor, m and ω_μ are the mass and eigenfrequency of the test oscillator, and $\hat{\tau}$ is the duration of the external signal.

Various gravitational antenna models geared toward the construction of a "quantum-consistent" QNM system have already been discussed in the literature (see Refs. 4–7, for example). Usually, the first two sections of the antenna are studied from the standpoint of quantum mechanics, namely the mechanical oscillator (gravitational detector) which responds directly to the external signal, and the electromagnetic oscillator connected to it (the so-called detection element, or mechanical-to-electromagnetic energy transducer). One quite specific goal of these analyses has been to find a QNM variable, that is, an observable (at the transducer output) whose measurement enables one to determine the amplitude of the external force to any desired accuracy. The actual instrument which is to measure the QNM variable, however, in the form of a real device, has itself been left unspecified. Instead, a "thought measurement" has been introduced, accompanied by the collapse of the wave function induced by measurement of the QNM variable. This affects the subsequent evolution of the state of the measurement system as a whole, as well as the final accuracy of the external force estimate.[6–8] Obviously, such a partially reasoned approach is only justified as a first approximation to a means of circumventing the quantum sensitivity limits (1.1).

A real experiment requires readings obtained from a complete measurement system with a final stage (or classical sensor) that produces a direct indication of the measurement information and "dequantization" of the measured variable (that is, quantum noise can be neglected following the final stage). The problem of the final stage is therefore nontrivial. A simple adaptation of classical measurement methods to situations involving quantum noise will not succeed, as the effect of the measurement instrument back on the system being measured will generally not allow the quantum limit (1.1) to be circumvented.

In the present paper, we attempt to solve the problem of a complete QNM system in principle by modeling the gravitational antenna as a resonant mechanical detector with a Fabry-Perot optical interferometer as a transducer. In this model, quantum noise plays a fundamental role at sufficiently low temperatures, leading to the limit (1.1) when a simple photodetector is used to measure the interferometer output.

The key physical concept which we wish to exploit in designing a complete QNM system is the idea of producing a specific back reaction of the final stage (that is, choosing its structure) so as to maintain the possibility of circumventing the limit (1.1). Examples of such "artificial manipulation" of the backward radiation in an instrument which measures the quadrature component of the electromagnetic field in a nonperturbative fashion may be found in Refs. 9–12; there the back reaction of the instrument consists of augmented noise in the associated quadrature component. In our case, as we shall shortly see, the form of the back reaction of the final stage is more complicated, as it depends on the structure of the leading sections of the receiving system.

Our calculations support the notion that it is possible to surpass the quantum sensitivity threshold (1.1), as predicted previously.[3–7] Moreover, if we use a device whose back reaction squeezes the noise in a specially chosen spectrally selective manner, there is in principle no limit to the available improvement in sensitivity (within the confines of the specific model).

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2. MODEL OF THE MEASUREMENT SYSTEM

The measurement system which we analyze is shown in Fig. 1. It consists of a mechanical test oscillator of mass m and frequency ω_μ acted upon by the measured force $F(t)$. To this mass we affix the movable mirror 6 of a Fabry-Perot interferometer. The pump beam for this interferometer comes from the external laser 4 (frequency ω_p) through the fixed mirror 5. The output radiation from the resonator, $E_{\rm out}$, passes through the movable mirror to a device which measures the quadrature component of the electromagnetic field. Following the ideas put forth in Refs. 9–11, this device includes the three-input circulator 7. The interferometer output, falling upon channel 1, is directed to the measurement channel 2, which contains the optical local oscillator 13, 14, providing a reference field, and the photodetector 15. The backward wave from the local oscillator through the circulator enters the auxiliary channel 3, consisting of a pair of degenerate parametric amplifiers 9, 11 and the blackbody absorber 12. Thus, the only wave returning to the interferometer comes from channel 3, and is derived from the blackbody background (with $kT\ll\hbar\omega_0$) of the parametric amplifier system.

In the linear single-mode approximation, the abbreviated equations of motion for this system take the form (see Appendix 1)

$$\ddot{x}+\omega_\mu^2x=-2\gamma[E_1\sin(\Delta t+\varphi)+E_2\cos(\Delta t+\varphi)]+f_g(t),$$
$$\dot{E}_1+\delta_eE_1+2\beta x\cos(\Delta t+\varphi)=2\delta_eE_{b1},$$
$$\dot{E}_2+\delta_eE_2-2\beta x\sin(\Delta t+\varphi)=2\delta_eE_{b2},$$
$$\beta=\frac{\omega_0E_0}{2l_0},\quad \gamma=\frac{SE_0}{4\pi mt_0^2},\quad \Delta=\omega_p-\omega_0,\quad f_g(t)=\frac{F(t)}{m},\tag{2.1}$$

where E_0 is the amplitude of the forced field at the stationary mirror [see (A12)] due to the pump laser, E_{b1} and E_{b2} are the field components of the back reaction from the device, so that

$$E_{ba}=E_{b1}\cos\omega_0t+E_{b2}\sin\omega_0t,\tag{2.2}$$

while the field leaving the interferometer is

$$E_{out}=E_1\cos\omega_0t+E_2\sin\omega_0t,\tag{2.3}$$

and $F(t)$ is the external force being measured.

Equations (2.1) are typical equations for a parametric measurement system consisting of a test oscillator plus parametric sensor, and have been thoroughly studied in both the classical and quantum limits for the case in which $E_{\rm ba}$ is stationary white noise.[17,18] One feature of our analysis will be our specific choice of nonstationary noise $E_{\rm ba}$, which in fact reflects the chosen device design.

3. CALCULATION OF THE SENSITIVITY

In estimating the background noise it must be borne in mind that the measurement channel collects not only the interferometer output $E_{\rm out}$, but also the device return noise $E_{\rm ba}$ reflected from the output mirror [see (A11)]. The overall field at the measurement device is then

$$E_m=(E_1-E_{b1})\cos\omega_0t+(E_2-E_{b2})\sin\omega_0t$$
$$=E_{1m}\cos\omega_0t+E_{2m}\sin\omega_0t.\tag{3.1}$$

For the sake of definiteness, let there be a phase-sensitive device measuring only the quadrature component E_{2m} in a narrow filter passband $\Delta\omega_f\ll\omega_\mu$. We can easily quantize (2.1); that is, we replace the variables by the appropriate operators, which satisfy the canonical commutation relations.[19] Proceeding in the usual way to a Fourier representation

$$\hat{f}(t)=\int_0^\infty[\hat{f}(\omega)e^{-i\omega t}+\hat{f}^+(\omega)e^{i\omega t}]d\omega,$$

$$(\hat{f}(\omega))^+=\hat{f}^+(\omega)=\hat{f}(-\omega),$$

$$\hat{f}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(t)e^{i\omega t}dt,\qquad \hat{f}^+(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(t)e^{-i\omega t}dt,$$

we obtain for the spectrum of noise fluctuations in $\hat{E}_{2m}$ ($\alpha=2\delta_e\psi,\psi=\beta\gamma$)

$$\hat{E}_{2m}(\omega)=\frac{\hat{E}_s(\omega)}{\delta_e-i\omega}\left[\delta_e+i\omega+\frac{i\alpha}{\delta_e-i\omega}(d_+^{-1}-d_-^{-1})\right]$$
$$-\frac{\alpha\hat{E}_c(\omega)}{(\delta_e-i\omega)^2}(d_+^{-1}+d_-^{-1})$$
$$+\frac{\alpha\hat{E}_{2c}(\omega)}{\delta_e-i\omega}\left\{\frac{\exp(i2\varphi)}{[\delta_e-i(\omega+2\Delta)]d_+}+\frac{\exp(-i2\varphi)}{[\delta_e-i(\omega-2\Delta)]d_-}\right\}$$
$$+\frac{i\alpha\hat{E}_{2s}(\omega)}{\delta_e-i\omega}\left\{\frac{\exp(i2\varphi)}{[\delta_e-i(\omega+2\Delta)]d_+}-\frac{\exp(-i2\varphi)}{[\delta_e-i(\omega-2\Delta)]d_-}\right\},\tag{3.2}$$

where

$$d=d(\omega)=\omega_\mu^2-\omega^2-\frac{4\psi\Delta}{(\delta_e-i\omega)^2+\Delta^2},\quad d_\pm=d(\omega\pm\Delta).\tag{3.3}$$

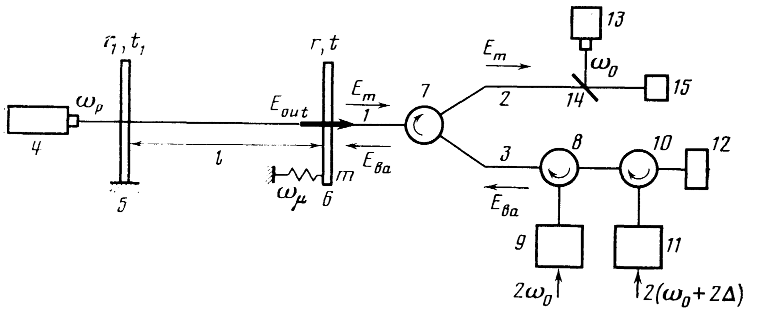

FIG. 1. The gravitational antenna with its measurement equipment. 1) Communications channel between the antenna and the measurement equipment; 2) measurement channel; 3) return noise channel; 4) pump laser (frequency ω_p); 5) "perfect" mirror, with $|r_1|\to1$, $|t_1|\to0$; 6) partially transmitting mirror with reflection and transmission coefficients $r, t(|t_1|\ll|t|)$; 7, 8, 10) circulators; 9, 11) narrow-band degenerate amplifiers (four-wave mixers) operating in reflecting mode, with pump frequencies $2\omega_0$ and $2(\omega_0+2\Delta)$; 12) absorber; 13) homodyne detector reference laser at frequency ω_0; 14) homodyne mirror ($|r|\approx0$); 15) photodetector (quantum efficiency $\eta\approx1$).

In Eq. (3.2), we have explicitly introduced the quadrature noise components at frequencies ω_0 and $\omega_0 + 2\Delta$, for which $\hat{E}_{\rm ba}$ must be written in the form

$$\hat{E}_{ba}=\hat{E}_c \cos \omega_0 t+\hat{E}_s \sin \omega_0 t+\hat{E}_{2c} \cos(\omega_0 t+2\Delta t)$$

$$+ \hat{E}_{2s} \sin(\omega_0 + 2\Delta)t. \tag{3.4}$$

$\hat{E}_c$, $\hat{E}_s$ and $\hat{E}_{2c}$, $\hat{E}_{2s}$ are the quadrature components of the noise in (overlapping) narrow spectral bands near the lower ($\omega_p - \Delta = \omega_0$) and upper ($\omega_p + \Delta = \omega_0 + 2\Delta$) sidebands of the pump frequency ω_p (with an offset of Δ). These are just the frequencies which affect measurements, in accordance with (2.1). This situation is typical of parametric receiving systems: the desired information is carried by one of the IF product frequencies (the so-called signal frequency, here given by $\omega_s = \omega_p - \Delta$), while noise is introduced at both the signal frequency and the idler frequency ($\omega_i = \omega_p + \Delta$).

From (2.1), we obtain

$$(E_{2m}(\omega))_{\rm sig} = \frac{f_0 \hat{\tau} \beta i}{4(\delta_e - i\omega)}(d_-^{-1} - d_+^{-1}) \tag{3.5}$$

for the spectral amplitude of the signal being measured, where we take $f_g(t) = f_0 \sin\omega_\mu t$ for $0 < t < \hat{\tau}$ and $f_g(t) = 0$ for 0 for $t < 0$ and $t > \hat{\tau}$, with $\omega_\mu \hat{\tau} \gtrsim 2\pi$. Measurement of the signal (3.5) in channel 2 (Fig. 1) should therefore take place in a quantum noise background with the spectrum (3.2).

According to the theory of QNM,[4–6] quantum mechanics imposes no limitations on the attainable measurement accuracy for one quadrature component of a single (isolated) oscillator. In the setup shown in Fig. 1, however, the field oscillator is not isolated: it is coupled to the mechanical oscillator through the external classical field. The ultimate measurement accuracy of a quadrature component $\hat{E}_{2m}$ is therefore not obvious; one needs to guarantee both a special type of interaction between the oscillators,[4,5] and a special type of back reaction of the device ($\hat{E}_{\rm ba}$ in Fig. 1). This turns out to be possible if one selects the device design on the basis of channel 3, and controls the free parameters of the problem, namely the classical pump frequency and phase. From an experimental standpoint, this means precisely controlling a measurement system incorporating a device with suitable performance.

An analysis of Eq. (3.2) makes it clear what is needed for this control. To measure $\hat{E}_{2m}$ to an accuracy exceeding the quantum limit, noise attributable to one of the conjugate quadrature components must be eliminated from the spectrum at frequencies $\omega \sim 0$ and $\omega \sim 2\Delta$ (that is, in the vicinity of ω_0 and $\omega_0 + 2\Delta$). "Squeezing" of the noise in the remaining components will then result in a nonperturbative measurement.

An appropriate choice of offset Δ can eliminate the $\hat{E}_c(\omega)$ term at $\omega = 0$, and the right choice of pump phase φ can take out the $\hat{E}_{2c}(\omega)$ term at $\omega = 0$. The first of these offsets is the solution of the equation

$$d_+^{-1}(0)+d_-^{-1}(0)=0, \qquad (\delta_e^2+4\Delta^2)(\omega_\mu^2-\Delta^2)-4\psi\Delta=0. \tag{3.6}$$

The noise contribution of $\hat{E}_{2c}(\omega)$ goes to zero at $\omega = 0$ if the phase φ satisfies

$$\frac{\exp(i2\varphi)}{(\delta_e-i2\Delta)\,d_+(0)}+\frac{\exp(-i2\varphi)}{(\delta_e+i2\Delta)\,d_-(0)}=0,$$

$$\exp(i2\varphi)=\frac{\delta_e-i2\Delta}{(\delta_e^2+4\Delta^2)^{1/2}}. \tag{3.7}$$

If (3.6) and (3.7) are satisfied, the only noise left in the system at $\omega = 0$ will be in one of the two quadrature components $\hat{E}_s$ or $\hat{E}_{2s}$ (or in $\hat{E}_{2c}$). If the design of the measurement device makes it possible to squeeze the noise in these components, it should then be possible in principle to measure signal variations $\hat{E}_{2m}$ below the quantum limit. Physically speaking, condition (3.6) provides for frequency balance in the parametric system, and (3.7) provides for phase balance.

The signal-to-noise ratio is obtained from (3.2) and (3.5) in the usual way[20]:

$$\mu=\pi^{-1}\int_0^\infty \frac{E_s(\omega)E_s^*(\omega)\,d\omega}{\langle \hat{E}_{2m}(\omega)\hat{E}_{2m}^+(\omega)+\hat{E}_{2m}^+(\omega)\hat{E}_{2m}(\omega)\rangle/2}. \tag{3.8}$$

To calculate the noise spectral density

$$N_2(\omega)=\langle \hat{E}_{2m}(\omega)\hat{E}_{2m}^+(\omega)\rangle_{sym}$$

we need to specify the spectral densities

$$\langle \hat{E}_c(\omega)\hat{E}_c^+(\omega)\rangle_{sym},\ \langle \hat{E}_s(\omega)\hat{E}_s^+(\omega)\rangle_{sym},\ \langle \hat{E}_{2c}(\omega)\hat{E}_{2c}^+(\omega)\rangle_{sym},$$
$$\langle \hat{E}_{2s}(\omega)\hat{E}_{2s}^+(\omega)\rangle_{sym},$$

and their correlations as well.[19,21] According to (3.4) (see also Fig. 1), we can take the spectral components at $\omega \sim \omega_0$ and $\omega \sim \omega_0 + 2\Delta$ to be uncorrelated (to zeroth order in $2\Delta/\omega_0$ (Ref. 19)), and furthermore, we have for the squeezed noise in channel 3

$$\langle \hat{E}_s(\omega)\hat{E}_s^+(\omega)+\hat{E}_s^+(\omega)\hat{E}_s(\omega)\rangle/2=\langle \hat{E}_s(\omega)\hat{E}_s^+(\omega)\rangle_{sym}=N_0/g_0,$$
$$\langle \hat{E}_c(\omega)\hat{E}_c^+(\omega)\rangle_{sym}=N_0 g_0,\ \langle \hat{E}_s(\omega)\hat{E}_c^+(\omega)\rangle_{sym}\approx 0,$$
$$\langle \hat{E}_{2s}(\omega)\hat{E}_{2s}^+(\omega)\rangle_{sym}=N_0/g_2,\ \langle \hat{E}_{2c}(\omega)\hat{E}_{2c}^+(\omega)\rangle_{sym}=N_0 g_2,$$
$$\langle \hat{E}_{2s}(\omega)\hat{E}_{2c}^+(\omega)\rangle\approx 0, \tag{3.9}$$

where N_0 is the spectral density of the vacuum fluctuations, and g_0 and g_2 are the squeezing parameters at ω_0 and $\omega_0 + 2\Delta$. Since in general the expression for the signal-to-noise ratio is quite unwieldy, it makes the most sense to consider several special cases.

1. Large damping, $\delta_e \gg \omega_\mu$. Here the expression for μ simplifies. Retaining only the principal terms (in ω_μ/δ_e) and bearing in mind that the filter bandwidth satisfies $\Delta\omega_f \ll \omega_\mu$, we obtain

$$\mu\approx f_0^2\hat{\tau}^2\beta\omega_\mu\delta_e^3\{4N_0\gamma[(\delta_e^6(g_0^{-1}+g_2^{-1})$$
$$+16\psi^2(g_0+g_2))(\psi^2(g_0+g_2)+\omega_\mu^4\delta_e^2/g_0)]^{1/2}\}^{-1} \tag{3.10}$$

from (3.2), (3.5), (3.8), and (3.9), whereupon with $\psi \ll \omega_\mu^3$ we find

$$(g_0)_{opt}\approx(g_2)_{opt}=g\approx\omega_\mu^2\delta_e/2^{1/2}\psi. \tag{3.11}$$

Finally,

$$\mu=\mu_q g=\mu_q\omega_\mu/\Delta\omega_\Phi, \quad \mu_q\approx f_0{}^2\hat{\tau}^2\beta/8N_0\gamma\omega_\mu\delta_e, \tag{3.12}$$

in which μ_q is the conventional quantum limit for the signal-to-noise ratio. The filter bandwidth (that is, the bandwidth of an optimal filter) given by the integrand in (3.8) is

$$\Delta\omega_f=2^{1/2}\psi/\delta_e\omega_\mu=\omega_\mu/g. \tag{3.13}$$

Notice that the magnitude of the coupling can be changed quite easily (by changing the pump laser power $W_L \sim E_L^2$, for example), while at the same time the maximum attainable squeezing g is dictated by the technical specifications of the parametric amplifiers. Equation (3.11) should be looked upon, then, as the condition which determines the optimal coupling in a system with given squeezing g. On the one hand, the signal-to-noise ratio is maximized for a given g, and on the other, the filter bandwidth is also maximized (the discrimination time is minimized, reducing the role of thermal noise in the antenna). If for a low value of the squeezing (3.11) is not true (the coupling ψ is too strong), the first relation in (3.12) will still hold, but the filter bandwidth will be less than the maximum possible for given μ (3.13).

2. Small damping, $\delta_e \ll \omega_\mu$. Here the expression for μ depends strongly on the relation between the system parameters ψ and $\omega_\mu\delta_e^2/2$. The coupling coefficient between the optical resonator and the mechanical system is $\varkappa=(2\psi/\omega_\mu)^{1/2}$, so in going from $\varkappa>\delta_e\,(\psi>\omega_\mu\delta_e^2/2)$ to $\varkappa<\delta_e\,(\psi<\omega_\mu\delta_e^2/2)$, the nature of the coupling between the oscillators changes. In general, the integral for μ can still not be evaluated exactly, so we make use of approximate series expansions (which are accurate to within a factor of order unity). For $\psi\lesssim\omega_\mu\delta_e^2/2(\varkappa<\delta_e)$ we obtain

$$\mu\approx\frac{\beta^2 f_0{}^2\hat{\tau}^2\omega_\mu{}^4}{4N_0\pi}\int_0^\infty\frac{(\psi^2+\omega_\mu{}^2\delta_e{}^2\omega^2)\,d\omega}{A+B\omega^2+C\omega^4}, \tag{3.14}$$

where

$$A=4\omega_\mu{}^4\psi^4\left(g_0{}^{-1}+\frac{\delta_e{}^2}{\omega_\mu{}^2 g_2}\right),$$

$$B=\delta_e{}^2\psi^2\left(\frac{g_0\psi^2\delta_e{}^2}{\omega_\mu{}^2}+\frac{8\omega_\mu{}^6}{g_0}+\frac{4\delta_e{}^2\omega_\mu{}^4}{g_2}+\psi^2 g_2\right), \tag{3.15}$$

$$C=4\omega_\mu{}^2\delta_e{}^2(\psi^2\omega_\mu{}^2 g_0+\omega_\mu{}^6\delta_e{}^2/g_0+\delta_e{}^2\psi^2 g_2).$$

Equation (3.14) is maximized when

$$g_0\approx\omega_\mu{}^2 g_2/\delta_e{}^2, \tag{3.16}$$

which corresponds to noise reduction at the frequency $\omega_0+2\Delta$ by the narrow-band resonant response of a tuned circuit with $\delta_e\ll\omega_\mu$ (the squeezing g_2 is rather less than for $\delta_e\gg\omega_\mu$, when the measurement system transforms noise at ω_0 and $\omega_0+2\Delta$ identically). Calculating (3.14) when (3.16) holds, we obtain the signal-to-noise ratio

$$\mu=\mu_q g_0. \tag{3.17}$$

This equation holds for optimal coupling

$$\psi_{opt}=\omega_\mu{}^2\delta_e/g_0 \tag{3.18}$$

and filter bandwidth

$$\Delta\omega_f=\omega_\mu/g_0. \tag{3.19}$$

If the coupling is less than the optimum value (3.18), the signal-to-noise ratio (3.17) will remain the same, but the filter bandwidth will be less than that given by (3.19):

$$\Delta\omega_f=\psi/\omega_\mu\delta_e. \tag{3.20}$$

A calculation of the signal-to-noise ratio for $\psi>\omega_\mu\delta_e^2/2$ $(\varkappa>\delta_e)$ and $g_0<2\omega_\mu/\delta_e$ shows that the coupling ψ must be reduced until $\psi\approx\omega_\mu\delta_e^2/2$, at which point the limiting value (3.17) of μ is obtained and the filter bandwidth is $\Delta\varphi_f\approx\delta_e/2$.

To sum up, we have found that for small damping $\delta_e<\omega_\mu$ and $g_0>2\omega_\mu/\delta_e$, the optimum coupling is given by (3.18), while the signal-to-noise ratio and filter bandwidth are given by (3.17) and (3.19) [if $\psi<\psi_{\rm opt}$, the filter bandwidth corresponds to (3.20)]; for $g_0<2\omega_\mu/\delta_e$, the optimum coupling is $\psi_{\rm opt}=\omega_\mu\delta_e^2/2$ and (3.17) and (3.20) give the signal-to-noise ratio and bandwidth.

When the coupling ψ is optimized, therefore, μ grows linearly with g_0 for either high or low damping (although in the latter instance, g_2 can be smaller $(\omega_\mu/\delta_e)^2$ than in the former). The actual experimental value is $\psi\approx10^5$ $\sec^{-3}\ll\omega_\mu^3\approx10^{12}\sec^{-3}$, and it is therefore impossible to satisfy (3.11) for moderately small values of g_0 and large damping $\delta_e>\omega_\mu$. In other words, the filter bandwidth (3.13) will not have been optimized (it will be less than the maximum possible for the given signal-to-noise ratio), which will then have a negative impact on the system's immunity to thermal noise from the mechanical oscillator. At the same time, the filter bandwidth (3.20) is much larger than (3.13) when the damping is small, thereby reducing the relative importance of oscillator noise.

4. DISCUSSION

In the preceding sections, we discussed a self-contained model of a parametric measurement system which differs from its predecessors most notably by its incorporation of a third signal path, a device with a specified structure which makes a nonperturbative measurement of a quadrature component of the electromagnetic field. The main result obtained with this model is its indication of a special operating mode in which the sensitivity to an external force exceeds the standard quantum limit (1.1). In fact, according to Eqs. (3.12) and (3.17), the minimum detectable external force is

$$(F_0)_{min}=\zeta\hat{\tau}^{-1}(\hbar m\omega_\mu)^{1/2}g^{-1/2}=F_q g^{-1/2},$$

so that the improvement in sensitivity is determined by the return-noise squeezing factor g obtained with this instrument. With the problem stated in this way, the traditional quantum mechanical question of the physical realizability of the measurement process[3-8] is no longer an issue, since the device which provides the measurement and produces the backward noise wave (the dequantization noise) is indicated explicitly.

To conclude, let us discuss the limitations introduced by imperfections of the individual design elements (Fig. 1). We will be most interested in those that result primarily in losses in the measurement system, leading to spreading of the squeezed state.

We assume finite damping δ_μ (or finite Q_μ) of the mechanical oscillator. The signal-to-noise ratio in the thermal noise background of a mechanical oscillator at temperature T_μ is

$$\mu_T=\frac{f_0{}^2\hat{\tau}^2}{16}\Delta\omega_f\left[\frac{2}{\pi}\frac{\hbar\omega_\mu{}^2}{mQ_\mu}\left(\frac{1}{2}+\left(\exp\left(\frac{\hbar\omega_\mu}{kT_\mu}\right)-1\right)^{-1}\right)\right]^{-1}. \tag{4.1}$$

Comparing (4.1) with (3.12) and (3.17), we obtain a lower limit on the filter bandwidth:

$$\Delta\omega_f \gtrsim \omega_\mu Q_\mu^{-1/2} \text{ for } kT_\mu \ll \hbar\omega_\mu,$$

$$\Delta\omega_f \gtrsim \omega_\mu\left(\frac{kT_\mu}{\hbar\omega_\mu Q_\mu}\right)^{1/2} \text{ for } kT_\mu \gg \hbar\omega_\mu, \tag{4.2}$$

whereupon we have for maximum signal-to-noise ratio

$$\mu_{max}=\mu_q Q_\mu{}^{1/2} \quad \text{for } kT_\mu \ll \hbar\omega_\mu,$$

$$\mu_{\max}=\mu_q\left(\frac{\hbar\omega_\mu Q_\mu}{kT_\mu}\right)^{1/2} \quad \text{for } kT_\mu \gg \hbar\omega_\mu \tag{4.3}$$

or for the force (see Ref. 18),

$$F_{min}=F_q Q_\mu^{-1/4}, \quad kT_\mu \ll \hbar\omega_\mu, \qquad F_{min}=F_q\left(\frac{\hbar\omega_\mu Q_\mu}{kT_\mu}\right)^{-1/4},$$

$$kT_\mu \gg \hbar\omega_\mu. \tag{4.4}$$

For $T=10^{-3}$ K $\gg \hbar\omega_\mu/k \approx 10^{-7}$ K and $Q_\mu \approx 10^{10}$, we have $F_{\min}/F_q \approx 3\cdot 10^{-2}$

Losses δ in the optical resonator, due for example to the finite transmission coefficient of the "perfect" mirror, will also lead to the destruction of squeezed states of the field. The preservation of squeezed states requires that

$$\delta \lesssim \delta_e/g_m, \tag{4.5}$$

where $g_m=\max(g_0,g_2)$. Note that in general the damping δ can come from anywhere: absorption in the mirrors, diffraction, and so on. Similarly, losses in the photodetector and circulators of the setup shown in Fig. 1 will generate equilibrium noise with a spectral density[12] $N_{ab}=(1-\eta)N_0$ (where η is the quantum efficiency of the photodetector or $(1-\eta)$ is the optical loss of the circulators), so that as $\eta\to 1$, these noise sources will vanish asymptotically (the effects of a nonideal photodetector can be reduced by the use of a degenerate parametric amplifier[13]). Taking all limitations into account is only possible when one analyzes a specific hardware implementation of the design in Fig. 1, which is a problem in its own right. Here our objective has been to ascertain the basic structure of a closed-loop QNM design for detecting weak external signals with a test oscillator. Our motivation has come from the first experiments to produce a squeezed state of the electromagnetic field, even though the squeezing values achieved are still only 30–40%.[22,23]

APPENDIX

We obtain the equation of motion of the present system from the correspondence principle. For simplicity, let $r_1=-r_{01}$ and $r=-r_0$ be real, with $r_{01}>0$ and $r_0>0$; t_1 and t should then be purely imaginary[13]: $t_1=it_{01}$, $t=it_0$. Assume further that the mirrors are lossless: $r_{01}^2+t_{01}^2=1$, $r_0^2+t_0^2=1$. If we consider only one polarization, the field incident on the measurement device is

$$E_m=E_L\frac{t_1t\exp(ikl)}{1-r_1r\exp(2ikl)}+rE_{ba}+\frac{r_1rt^2\exp(2ikl)}{1-r_1r\exp(2ikl)}E_{ba}, \tag{A1}$$

where $k=\omega/c$, l is the length of the resonator, E_m is the field incident on the measure device, E_L is the field due to the pump laser, and E_{ba} is the noise field due to the back reaction from the device.

Next we consider the classical pump limit $r_{01}\to 1$, $|t_1|\to 0$, but with $t_{01}E_L=$ const. Then from the left, only the laser pump field passes into the interferometer, and

$$E_m=-r_0E_{ba}-\frac{t_0{}^2\exp(2ikl)}{1-r_0\exp(2ikl)}E_{ba}+\frac{it_0\exp(ikl)}{1-r_0\exp(2ikl)}E_Lt_1. \tag{A2}$$

The measured field E_m therefore consists of a back-reaction field E_{ba} reflected from the input mirror of the interferometer, and the field E_{out} which exists the interferometer (the forced oscillation field, the third term in (A2), contains no noise when $|t_1|\to 0$). Expanding the factors of E_{ba} and E_L in (A2) near one of the resonator eigenfrequencies $\omega_0=n\pi c/l$ and introducing the damping $\delta_e=c(1-r_0)/2lr_0$, we obtain ($|\omega-\omega_0|l/c\ll 1$, and we are in the single-mode regime)

$$E_m=(\omega_0-\omega-i\delta_e)^{-1}\left[\frac{E_Lt_1t_0(-1)^n}{2lr_0/c}-\frac{E_{ba}t_0{}^2i}{2lr_0/c}\right]-E_{ba}r_0. \tag{A3}$$

We find then from (A3) that the resonant terms satisfy

$$\ddot{E}+2\delta_e\dot{E}+\omega_0{}^2E=F, \tag{A4}$$

where $E=E_p+E_{out}$ is the output radiation from the resonator, consisting of forced oscillations plus a perturbation due to the back reaction of the measurement device, $F=F_L+F_{ba}$, F_L is the "driving" force due to the pump laser, F_{ba} is the "force" due to the back reaction from the measurement device, and

$$F_{ba+}=-4i\omega_0\delta_eE_{ba+}, \quad F_{L+}=2i\omega_0t_{01}E_{L+}(\delta_ec/lr_0)^{1/2} \tag{A5}$$

are the amplitudes of the driving forces:

$$F=F^++F^-=F_+\exp(-i\omega t)+F_-\exp(i\omega t), \quad (F^+)^*=F^-.$$

If the length of the interferometer now changes as $l=l_0+x(t)$ $(|x(t)|\ll l)$, then so will the mode frequency $\omega_0^2(t)\approx\omega_0^2(1-2x(t)/l)$. We then obtain (keeping the leading term in x/l)

$$\ddot{E}+2\delta_e\dot{E}+\omega_0{}^2(1-2x/l)E=F_{ba}+F_L. \tag{A6}$$

It should be noted that there is an additional phase shift of the driving force and the response in connection with the choice of a transmission coefficient in the form $t=it_0$ [namely the imaginary unit i in (A5)]. One can derive an equation analogous to (A6) using Slater's method.[14–16]

Equation (A6) must next be supplemented by the equation for the mechanical degree of freedom (for an oscillator of mass m and resonant frequency ω_μ:

$$\ddot{x}+\omega_\mu{}^2x=F_p(t)+f_s(t), \tag{A7}$$

where x is the displacement of the movable mirror from its equilibrium position, and $F_p(t)$ and $f_g(t)$ are the mass-normalized light-pressure force on the mirror and the external (gravitational) force (we neglect damping in the mechanical system). We find the light-pressure force from the Maxwell stress tensor (S is the area of the mirror; for simplicity, we assume that the pump beam has cross section S):

$$F_p(t)=\frac{S}{8\pi m}[(E^2(x-0,t)+H^2(x-0,t))-(E^2(x+0,t)+H^2(x+0,t))], \qquad (A8)$$

where the first and second terms describe the pressure force on the mirror due to fields coming from the left (inside), and the third and fourth describe those coming from the right (outside). In calculating the pressure force, we can assume the mirror to be stationary (taking the displacement into account gives terms of the next higher order). The fields $E(x-0,t)$, $H(x-0,t)$, $E(x+0,t)$, and $H(x+0,t)$ can be found in the same way as (A1)–(A5); substituting these into (A8) and linearizing, we have the equation for the mechanical system (with the assumption that E_{ba} does not contain the pump):

$$\ddot{x}+\omega_\mu^2 x=-\frac{S}{\pi m t_0^2}E_{out}\left(\frac{\pi}{2}\right)E_p\left(\frac{\pi}{2}\right)+f_g(t), \qquad (A9)$$

where $E_{out}(\pi/2)$ and $E_p(\pi/2)$ are the output noise and the forced oscillations, both shifted by $\pi/2$. To sum up, then, after linearization of (A6), we obtain from (A5), (A6), and (A9) a closed set of equations:

$$\begin{aligned}&\ddot{E}_p+2\delta_e\dot{E}_p+\omega_0^2E_p=F_L,\\&\ddot{E}_{out}+2\delta_e\dot{E}_{out}+\omega_0^2E_{out}-(2\omega_0^2x/l_0)E_p=-4\omega_0\delta_eE_{ba}(\pi/2),\\&\ddot{x}+\omega_\mu^2x=-(S/\pi m t_0^2)E_{out}(\pi/2)E_p(\pi/2)+f_g(t),\end{aligned} \qquad (A10)$$

$$E_m=-r_0E_{ba}+E_{out}. \qquad (A11)$$

writing out the expression for the pump explicitly as $F_L=F_0\sin(\omega_pt+\varphi_0)$, we find from the first of Eqs. (A10) that

$$E_p=\frac{F_0\sin(\omega_pt+\varphi_0+\theta)}{[(\omega_0^2-\omega_p^2)^2+4\delta_e^2\omega_p^2]^{1/2}}=E_0\sin(\omega_pt+\varphi), \qquad (A12)$$

with $\varphi=\varphi_0+\theta$ (φ is an arbitrary phase determined by θ_0, and θ is the delay of the response relative to the driving force).

Representing the output field E_{out} and the back-reaction field E_{ba} in terms of their quadrature components [see Eqs. (2.2) and (2.3)], we obtain from (A11) the abbreviated basic equations of the measurement system in the form (2.1).

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