Nonlinear stage of the three-wave parametric interaction in an inhomogeneous medium

E. Z. Gusakov and A. N. Savel'ev

A. F. Ioffe Physicotechnical Institute, Academy of Sciences of the USSR, Leningrad
(Submitted 11 August 1987)
Zh. Eksp. Teor. Fiz. **94**, 89–103 (April 1988)

An explicit solution is obtained for the relationship between the amplitudes of incident and outgoing waves in the region of the three-wave parametric decay resonance in a plane-layered medium.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

The spatial inhomogeneity of a medium has a considerable stabilizing effect on parametric decay instabilities excited in the medium by a monochromatic pump.[1,2] The convective transport of wave energy out of the narrow region of interaction, in which the decay conditions are satisfied for the projections of the wave vectors onto the direction of the gradient (the z axis), i.e.,

$$k_1(z_*,\ \omega_1)=k_2(z_*,\ \omega_2)+k_3(z_*,\ \omega_3),\qquad \omega_1=\omega_2+\omega_3, \tag{1}$$

leads not only to a higher instability threshold, but also to saturation, even in the linear approximation in the amplitudes of the excited waves 2 and 3 [we shall refer to them as *secondary* waves; the pump will be designated wave 1]. An absolute parametric instability is not excited in this process. Instead, there is convective spatial amplification of the secondary waves incident on the three-wave resonance region $z\simeq z_*$. The parametric interaction is incoherent, the wave amplification occurs in a wide frequency range, and the phases of different-frequency waves are uncorrelated.

When the convective amplification factor is large enough, the increase in the secondary-wave amplitude gives rise to a significant loss of energy by the pump wave and to its depletion. This process was examined in Ref. 3 on the assumption that the interaction was incoherent and the spectrum of secondary waves was sufficiently wide, and the problem of the depletion of the pump wave during the development of the Raman instability in laboratory plasma was solved.

We note, however, that the parametric interaction in a spatially inhomogeneous medium can also occur in the coherent state that is realized when there are favorable conditions for the return of at least some of the secondary wave energy transported out of the decay region $z\simeq z_*$ to this region. Theoretical analysis of different special cases[2,4–6] shows that, as a rule, this kind of feedback does actually occur. What happens is that, when the pump wave is introduced, convective spatial amplification gives rise to the excitation of the coherent process, i.e., absolute parametric instability. The spectrum is then discrete, the frequencies of the unstable waves are given by the condition for the quantization of the feedback loop, and the instability threshold and growth rate are determined from the balance condition for amplification in the decay region $z\simeq z_*$ and losses incurred during propagation in the feedback loop. A small excess above the absolute instability threshold gives rise to excitation of only one (the most unstable) mode of the system, but the spectrum fills up as the pump wave field increases.

We note that, when the absolute instability threshold is slightly exceeded, pump depletion effects in the region of the three-wave resonance $z\simeq z_*$ may become very significant. They then lead to instability saturation, and determine the degree of excitation of secondary waves. In contrast to Ref. 3, the significant effect is not only the loss of energy by the pump wave as it crosses the $z\simeq z_*$ layer, but also the change in the amplitudes and phases of the secondary wave.

In the determination of these quantities, we shall confine our attention to the simple case where only one pair of secondary waves with frequencies ω_2, ω_3 is excited in the inhomogeneous medium under the influence of the pump wave of frequency ω_1. We shall consider the case in which waves 2 and 3 have opposite group velocities ($v_2v_3<0$), in which the absolute instability is usually found to occur and, to be specific, we shall suppose that $v_1>v_3>0>v_2$. Moreover, we shall consider that, in the neighborhood of $z=z_*$,

$$k_1(z)-k_2(z)-k_3(z)=z/h^2.$$

The truncated equations for the amplitudes of the three interacting waves then become

$$v_1\frac{d}{dz}y_1=iy_2y_3\exp\left(-\frac{iz^2}{2h^2}\right),\qquad v_2\frac{d}{dz}y_2=iy_1y_3^*\exp\left(\frac{iz^2}{2h^2}\right),$$
$$v_3\frac{d}{dz}y_3=iy_1y_2^*\exp\left(\frac{iz^2}{2h^2}\right), \tag{2}$$

where y_m ($m=1,2,3$) are proportional to the amplitudes of the interacting waves and v_m are the corresponding group velocites. Before we formulate the boundary conditions for (2), let us agree to use the upper index $(\nu)=(\pm)$ to label quantities referring to incident $(-)$ and outgoing $(+)$ waves in the three-wave interaction region. It is readily seen that we then have

$$y_m^{(\nu)}\equiv y_m^{(\operatorname{sign}(v_mz/|v_mz|))},\qquad m=1,2,3.$$

It can be shown that the asymptotic solutions of (2) for $z/h\to\pm\infty$ are

$$\begin{aligned}
y_1^{(\nu)}(z)&\sim A_1^{(\nu)}\exp\{-i(\varkappa_2^{(\nu)}+\varkappa_3^{(\nu)})\ln|z/h|\}+O(h/z),\\
y_2^{(\nu)}(z)&\sim A_2^{(\nu)}\exp\{i(\varkappa_3^{(\nu)}-\varkappa_1^{(\nu)})\ln|z/h|\}+O(h/z),\\
y_3^{(\nu)}(z)&\sim A_3^{(\nu)}\exp\{i(\varkappa_2^{(\nu)}-\varkappa_1^{(\nu)})\ln|z/h|\}+O(h/z),
\end{aligned}\tag{3}$$

where $O(h/z)$ are rapidly oscillating functions that decay as h/z for $z\to\pm\infty$,

$$A_m^{(\nu)}=|A_m^{(\nu)}|\exp(i\varphi_m^{(\nu)}),\quad m=1,\ 2,\ 3$$

 0038-5646/88/040698-08$04.00

are complex constants, and the quantities $\varkappa_m^{(\nu)}$ $(m = 1,2,3)$ are given by

$$\varkappa_m^{(\nu)} = -h^2 |A_m^{(\nu)}|^2 / v_n v_p, \quad n \neq p \neq m. \tag{4}$$

The solutions given by (3) have an obvious physical interpretation: the waves cease to interact far from resonance in the sense that the absolute values of their amplitudes become asymptotically constant. The six complex constants $A_m^{(\nu)}$ $(m = 1,2,3)$ determine the amplitudes of the waves far from resonance, and only three of them are independent and must be specified as boundary conditions, which, in our notation, are $A_1^{(-)}, A_2^{(-)}, A_3^{(-)}$. The dependence of the amplitudes of the outgoing waves $A_1^{(+)}, A_2^{(+)}, A_3^{(+)}$ on the amplitudes of the incident waves $A_1^{(-)}, A_2^{(-)}, A_3^{(-)}$ is then given by unknown functions, and it is our aim here to determine these functions.

In the limit of the homogeneous medium $h \to \infty$, the set of equations given by (2) has an exact solution that can be written in terms of elliptic functions.[7] In the other limiting case, in which the secondary wave amplitudes are negligible in comparison with the pump, and the amplitude y_1 of the latter can be regarded as given and the first equation can be eliminated from (2), we again have an exact general solution that can be expressed in terms of the parabolic cylinder functions. This limiting case is examined in detail in Ref. 8. Since each of these two limiting cases is described by a very complicated exact solution, one can hardly expect to obtain an explicit exact general solution of (2). However, some progress can be made in the solution of the above problem of the relationship between the asymptotic constants even when the exact solution is not known. Actually, the stationary three-wave interaction problem is physically completely equivalent to the interaction of very long, smooth, flat-topped wave packets. The requirement that the packets be smooth is significant because arbitrary packets, e.g., triangular packets, can excite absolute instability, i.e., a phenomenon that does not occur in the stationary problem with an unbounded pump wave.

2. ASYMPTOTIC PROPERTIES OF THE NONSTATIONARY PROBLEM

The set of equations describing the three-wave decay interaction between smoothly-varying wave packets in a spatially inhomogeneous medium is

$$\begin{aligned}
&(\partial/\partial t + v_1 \partial/\partial z) y_1 = i y_2 y_3 \exp(-iz^2/2h^2),\\
&(\partial/\partial t + v_2 \partial/\partial z) y_2 = i y_1 y_3^* \exp(iz^2/2h^2),\\
&(\partial/\partial t + v_3 \partial/\partial z) y_3 = i y_1 y_2^* \exp(iz^2/2h^2).
\end{aligned} \tag{5}$$

In the asymptotic region, in which $|z/h| \gg 1$, the solutions of (5) can be written in the form of the sum of rapidly oscillating ($\langle \tilde{y}_m \rangle$) and smoothly varying ($\langle y_m \rangle$) functions of z:

$$y_m(z, t) = \langle y_m(z, t) \rangle + \tilde{y}_m(z, t).$$

Substituting this in (5), and separating terms with different oscillation scales, we obtain the following expressions for the rapidly-oscillating part $\tilde{y}_m$:

$$\begin{aligned}
&\tilde{y}_1 \approx -\frac{h^2 \langle y_2 \rangle \langle y_3 \rangle}{v_1 z} \exp\left(-\frac{iz^2}{2h^2}\right),\\
&\tilde{y}_2 \approx \frac{h^2 \langle y_1 \rangle \langle y_3^* \rangle}{v_2 z} \exp\left(\frac{iz^2}{2h^2}\right),\\
&\tilde{y}_3 \approx \frac{h^2 \langle y_1 \rangle \langle y_2^* \rangle}{v_3 z} \exp\left(\frac{iz}{2h^2}\right).
\end{aligned} \tag{6}$$

It has been assumed in the derivation of (6) that

$$\frac{\partial}{\partial t} \tilde{y}_m \ll v_m \frac{\partial}{\partial z} \tilde{y}_m.$$

We note that the rapidly-oscillating parts of the solution, $\tilde{y}_m$, are forced and fall as $\sim h/z$ with distance from the decay region. The smooth parts, $\langle y_m \rangle$, obey the following equations as $z/h \to \pm\infty$:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial z}\right) \langle y_1 \rangle = \frac{ih^2}{z} \left(\frac{|\langle y_3 \rangle|^2}{v_2} + \frac{|\langle y_2 \rangle|^2}{v_3}\right) \langle y_1 \rangle,\\
&\left(\frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial z}\right) \langle y_2 \rangle = \frac{ih^2}{z} \left(\frac{|\langle y_1 \rangle|^2}{v_3} - \frac{|\langle y_3 \rangle|^2}{v_1}\right) \langle y_2 \rangle,\\
&\left(\frac{\partial}{\partial t} + v_3 \frac{\partial}{\partial z}\right) \langle y_3 \rangle = \frac{ih^2}{z} \left(\frac{|\langle y_1 \rangle|^2}{v_2} - \frac{|\langle y_2 \rangle|^2}{v_1}\right) \langle y_3 \rangle
\end{aligned} \tag{7}$$

obtained using (6). The solution of (7) that is valid for $z/h \to \pm\infty$, can be written in the form

$$\langle y_m^{(\nu)} \rangle = A_m^{(\nu)}(\xi_m) \exp(i\gamma^{(\nu)}(z, \xi_m)),$$

where $\xi_m = z - v_m t$, and the phases $\gamma_m^{(\nu)}(z, \xi_m)$ are given by

$$\begin{aligned}
\gamma_1^{(\nu)} &= -\int_{z_*}^{z} [\varkappa_2^{(\nu)} + \varkappa_3^{(\nu)}] \frac{dz'}{z'}\\
&\quad - [\varkappa_2^{(\nu)}(-v_2 t) + \varkappa_3^{(\nu)}(-v_3 t)] \ln\left|\frac{z_*}{h}\right|,\\
\gamma_2^{(\nu)} &= \int_{z_*}^{z} [\varkappa_3^{(\nu)} - \varkappa_1^{(\nu)}] \frac{dz'}{z'} + [\varkappa_3^{(\nu)}(-v_3 t) - \varkappa_1^{(\nu)}(-v_1 t)] \ln\left|\frac{z_*}{h}\right|,\\
\gamma_3^{(\nu)} &= \int_{z_*}^{z} [\varkappa_2^{(\nu)} - \varkappa_1^{(\nu)}] \frac{dz'}{z'} + [\varkappa_2^{(\nu)}(-v_2 t) - \varkappa_1^{(\nu)}(-v_1 t)] \ln\left|\frac{z_*}{h}\right|,
\end{aligned} \tag{8}$$

where $z_* \to +0$ for $z/h > 0$ and $z_* \to -0$ for $z/h < 0$. The quantities $\varkappa_m^{(\nu)} \equiv \varkappa_m^{(\nu)}(\xi_m)$ are specified, as before, by (4), but, in contrast to the stationary case, they are slowly-varying functions of ξ_m and not constants. The integrals in (8) are evaluated for each phase $\gamma_m^{(\nu)}$ with constant ξ_m, i.e., along the trajectory of the corresponding wave packet. For example, the integrals in the expression for the phase $\gamma_3^{(\nu)}$ must be understood in the sense

$$\begin{aligned}
&\int^{z} [\varkappa_2^{(\nu)}(z' - v_2 t) - \varkappa_1^{(\nu)}(z' - v_1 t)] \frac{dz'}{z'}\\
&\equiv \int^{z} \left[\varkappa_2^{(\nu)}\left(z'\left(1 - \frac{v_2}{v_3}\right) + \frac{v_2}{v_3}\xi_3\right)\right.\\
&\quad \left. - \varkappa_1^{(\nu)}\left(z'\left(1 - \frac{v_1}{v_3}\right) + \frac{v_1}{v_3}\xi_3\right)\right] \frac{dz'}{z'}.
\end{aligned} \tag{9}$$

It is clear from the structure of the solutions $\langle y_m(z,t) \rangle$ that the three-wave interaction occurring far from the resonance region leads only to a change in the phases of the wave packets. The amplitude moduli $|A_m^{(v)}|$ are then freely propagating (without change in shape) real wave packets. The significant point is that a change in the phase of a packet far from resonance leads to an interaction with all the waves propagating there, both incident and outgoing. As the point of observation z approaches the region of resonance to distances much shorter than the inhomogeneity scale of the interacting wave packets ($|z| \ll H$, but $|z| \gg h$), the expressions given by (8) become much simpler, and the functions $\langle y_m(z,\xi_m)\rangle$ assume the form

$$\langle y_1^{(v)}\rangle \approx A_1^{(v)}(-v_1t) \exp\left\{-i[\varkappa_2^{(v)}(-v_2t)+\varkappa_3^{(v)}(-v_3t)]\ln\left|\frac{z}{h}\right|\right\},$$

$$\langle y_2^{(v)}\rangle \approx A_2^{(v)}(-v_2t)\exp\left\{i[\varkappa_3^{(v)}(-v_3t)-\varkappa_1^{(v)}(-v_1t)]\ln\left|\frac{z}{h}\right|\right\}, \tag{10}$$

$$\langle y_3^{(v)}\rangle \approx A_3^{(v)}(-v_3t)\exp\left\{i[\varkappa_2^{(v)}(-v_2t)-\varkappa_1^{(v)}(-v_1t)]\ln\left|\frac{z}{h}\right|\right\}.$$

We note that these relationships are similar to the asymptotic expressions (3) for the boundary conditions in the stationary problem, except that the amplitudes of the incident and outgoing waves in (10) are smooth functions of time and, as will be shown later, the amplitudes of the outgoing waves depend on time only through the time dependence of the incident waves. Moreover, when $h \ll H$, Eqs. (5) describing the interaction between the packets are identical with the equations of the stationary problem (2) for $|z| \ll H$, since

$$\frac{\partial}{\partial t}y_m \approx \frac{v_m y_m}{H} \ll v_m\frac{\partial}{\partial z}y_m \approx \frac{v_m y_m \varkappa_n}{z}.$$

The above analysis confirms the intuitive idea that the three-wave interaction between smooth wave packets is physically equivalent to the stationary problem. The required relationships between the asymptotic constants $A_m^{(v)}$ in the stationary problem can then be obtained from the corresponding relationships between the asymptotic complex profiles of wave packets $A_m^{(v)}(\xi_m)$ in the space-time problem by simply omitting the argument ξ_m of the function $A_m^{(v)}(\xi_m)$, since the formulas expressing this relationship in the problem with continuous wave packets are adiabatic. We note that the complication which arises as we pass from the stationary to the space-time problem can be completely removed by applying to the latter the inverse scattering method[9] for the interaction between finite packets. It will be shown below that this method can be used to relate the asymptotic solutions of (5) for $t \to \pm\infty$ that correspond to nonoverlapping packets of, respectively, incident and outgoing waves $y_m^{(v)}(\xi_m)$ in the far asymptotic region of the coordinate z. We note that the amplitudes of packets in the asymptotically far region, $y_m^{(v)}(\xi_m)$, differ from the required amplitudes $A_m^{(v)}(\xi_m)$ by only the phases that arise from the mutual crossing of the packets far from resonance. Actually, we find, using (8), that

$$y_m^{(v)}(\xi_m) = A_m^{(v)}(\xi_m)\exp(i\gamma_m^{(v)}(\pm\infty,\xi_m)), \tag{11}$$

where $\gamma_m^{(v)}(\pm\infty,\xi_m)$ are obtained from (8) for $z \to \pm\infty$.

3. THE LAX REPRESENTATION AND THE SOLUTION OF THE DIRECT SCATTERING PROBLEM

We now turn to the relationship between the asymptotic solutions of (5) for $t \to \pm\infty$ and note that, as shown in Refs. 10 and 11, there is a transformation that reduces (5) to equations that describe the interaction between wave packets in a homogeneous medium. This transformation is

$$u_m(z,t) = y_m(z,t)\exp\{-iq_m(z-v_mt)^2\}, \quad m=1, 2, 3;$$

$$q_1 = \frac{v_2v_3}{2h^2(v_1-v_2)(v_3-v_1)}, \quad q_2 = \frac{v_3v_1}{2h^2(v_1-v_2)(v_3-v_2)},$$

$$q_3 = \frac{v_1v_2}{2h^2(v_2-v_3)(v_1-v_3)}. \tag{12}$$

For the functions $u_m(z,t)$, we then have the set of equations

$$\left(\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial z}\right)u_1 = iu_2u_3, \quad \left(\frac{\partial}{\partial t}+v_2\frac{\partial}{\partial z}\right)u_2 = iu_1u_3^*,$$

$$\left(\frac{\partial}{\partial t}+v_3\frac{\partial}{\partial z}\right)u_3 = iu_1u_2^*. \tag{13}$$

The initial conditions for the packets $u_m(z,t)$ at times $t=\tau \to -\infty$ can be formulated as follows:

$$u_m(z,\tau) = y_m^{(-)}(z-v_m\tau)\exp\{-iq_m(z-v_m\tau)^2\}, \quad m=1, 2, 3. \tag{14}$$

If $|u_m| = |y_m^{(-)}| \to 0$ as $|z| \to \infty$ at time $t=\tau$, then, following Ref. 12, we can apply the inverse scattering method to (13). The interaction between packets is examined in great detail in Ref. 12 within the framework of (13), but most attention is devoted to solutions corresponding to the discrete spectrum of the direct scattering problem and, naturally, there are no universal recipes for solving the inverse problem. On the other hand, we are interested in the opposite situation, in which the discrete spectrum is absent and the initial conditions have the specific form given by (14). The set of equations given by (13) was examined in Ref. 13 for initial conditions in the form given by (14), but most attention was again devoted to soliton solutions corresponding to the discrete spectrum and the integral relations given in Ref. 12 were again employed. However, the inverse problem was not solved for a pure nonequilibrium spectrum.

We shall show below that the inverse scattering problem *can* be solved explicitly for the initial conditions (14), which means that the relationship between the wave packet amplitudes before and after the interaction can be found. It will be useful to reproduce now the necessary data on the inverse problem method for (13). These can be found in Ref. 12.

The Lax representation for (13) is

$$\frac{\partial L}{\partial t} = i[L,M], \quad L = iA\frac{\partial}{\partial z}+[A,Q], \quad M = iB\frac{\partial}{\partial z}+[B,Q], \tag{15}$$

where the operators A, B are constant diagonal matrices of rank 3×3 with elements $a_1 > a_2 > a_3$, b_1, b_2, b_3, respectively. In our case of the decay process, the matrix Q takes the form

$$Q=\begin{pmatrix} 0 & p(a_1-a_2)^{-1/2}u_2 & p(a_1-a_3)^{-1/2}u_1 \\ -p(a_1-a_2)^{-1/2}u_2^* & 0 & p(a_2-a_3)^{-1/2}u_3 \\ -p(a_1-a_3)^{-1/2}u_1^* & -p(a_2-a_3)^{-1/2}u_3^* & 0 \end{pmatrix}, \qquad (16)$$

where

$$p=\frac{a_3}{(v_3-v_1)}\left\{\frac{(a_1-a_2)}{(a_1-a_3)(a_2-a_3)}\right\}^{1/2},$$

and the elements of the matrices A and B must be such that

$$v_1=\frac{a_3b_1-a_1b_3}{a_1-a_3},\quad v_2=\frac{a_2b_1-a_1b_2}{a_1-a_2},\quad v_3=\frac{a_3b_2-a_2b_3}{a_2-a_3}. \qquad (17)$$

It is important to note that there is a misprint in the equation corresponding to (17) in Ref. 12 (their signs should be reversed). Eliminating the quantities b_1, b_2, b_3 from (17), we readily obtain the following useful relationships:

$$\frac{v_1-v_2}{v_3-v_2}=\frac{a_1(a_3-a_2)}{a_2(a_3-a_1)},\quad \frac{v_1-v_3}{v_2-v_3}=\frac{a_3(a_1-a_2)}{a_2(a_1-a_3)}. \qquad (18)$$

Since, in our case, $v_1>v_3>0>v_2$, one would expect that $a_1>a_2>0>a_3$.

The first step in the inverse scattering method is to solve the direct spectral problem for the operator L, given by (15), with a real spectral parameter λ:

$$L\Psi=\lambda\Psi, \qquad (19)$$

where Ψ is a vector function. Since, by hypothesis, $u_m(z)\to 0$ for $|z|\to\infty$, the asymptotic solution of (19) is

$$\Psi_m\sim\exp(-i\lambda z/a_m).$$

We shall now determine the two fundamental sets of solutions of (19) that have a particularly simple behavior for $z\to\pm\infty$, namely,

$$\Psi^{(\pm)}(z,\lambda)\to\exp(-i\lambda A^{-1}z),\quad z\to\pm\infty, \qquad (20)$$

where $\Psi^{(+)}$ are matrices with elements $\Psi_{mn}^{(+)}$, where the first index (rows) is the vector index and the second (columns) is the number of the fundamental solutions. The scattering matrix $S(\lambda)$ of the operator L is then determined by

$$\Psi^{(+)}(z,\lambda)=\Psi^{(-)}(z,\lambda)S(\lambda). \qquad (21)$$

Since the coefficients in (19) are functions of time because $u_m=u_m(z,t)$, the scattering matrix is also a function of time, i.e., $S=S(\lambda,t)$. The essential point is, however, that the function $S(t)$ is determined relatively simply with the aid of the representation (15) for $z\to\pm\infty$, even if we do not know $u(z,t)$ (Ref. 12). The expression relating the scattering matrices at times τ and t is

$$S_{mn}(\lambda,t)=S_{mn}(\lambda,\tau)\exp\left\{i\lambda\left(\frac{b_n}{a_n}-\frac{b_m}{a_m}\right)(t-\tau)\right\}, \qquad (22)$$

where S_{mn} are the elements of the matrix S.

Since we are interested only in the relationship between the solutions of (13) that are asymptotic for $t\to\pm\infty$, the direct spectral problem $L\Psi=\lambda\Psi$ becomes significantly simpler because the wave packets are separated in space $t=\tau\to-\infty$. The scattering problem is then solved successively for each packet $u_m(z,\tau)$, and the scattering matrix can be written as the product of three partial scattering matrices $S_1^{(-)}, S_2^{(-)}, S_3^{(-)}$. The partial scattering matrices are the scattering matrices of the operators L_1, L_2, L_3, obtained from L by substituting into it $u_2=u_3\equiv 0$, $u_1=u_3\equiv 0$, and $u_1=u_2\equiv 0$, respectively. The order in which the partial matrices are multiplied together at time $t=\tau$ is determined by the disposition of the wave packets in space prior to the interaction, i.e., by the ratio of their group velocities. In our case, $v_1>v_3>0>v_2$, we have

$$S(\tau)=S_1^{(-)}S_3^{(-)}S_2^{(-)}. \qquad (23)$$

The partial matrices are much simpler than the complete matrix and, in our case, they can be evaluated explicitly. We shall demonstrate this for $S_1^{(-)}$.

The corresponding partial spectral problem that can be solved in the localization region of the packet $u_1(z,\tau)$ is

$$\begin{gathered} ia_1\frac{d\Psi_1}{dz}+p(a_1-a_3)^{1/2}u_1(z,\tau)\Psi_3=\lambda\Psi_1,\\ ia_2\frac{d\Psi_2}{dz}=\lambda\Psi_2,\\ ia_3\frac{d\Psi_3}{dz}+p(a_1-a_3)^{1/2}u_1^*(z,\tau)\Psi_1=\lambda\Psi_3. \end{gathered} \qquad (24)$$

Substituting

$$\Psi_m=f_m\exp\left(-\frac{i\lambda z}{a_m}\right)$$

and recalling (14), we obtain

$$\begin{aligned} a_1\frac{df_1}{dz}&=ip(a_1-a_3)^{1/2}y_1^{(-)}(z-v_1\tau)f_3\\ &\quad\times\exp\left\{-iq_1(z-v_1\tau)^2+\frac{i\lambda z(a_3-a_1)}{a_3a_1}\right\},\\ a_3\frac{df_3}{dz}&=ip(a_1-a_3)^{1/2}y_1^{(-)*}(z-v_1\tau)f_1\\ &\quad\times\exp\left\{iq_1(z-v_1\tau)^2-\frac{i\lambda z(a_3-a_1)}{a_3a_1}\right\},\\ \frac{df_2}{dz}&=0. \end{aligned} \qquad (25)$$

We note that the scale of variation of the exponentials on the right-hand sides of these equations ($\lesssim h$) is much smaller than the characteristic size of the wave packet ($h\ll H$). The scattering matrix for (25) was investigated in Ref. 14 for these conditions. In the neighborhood of $z_1^{(-)}$, i.e., near the point of stationary phase of the exponential, for which

$$\xi_1^{(-)}\equiv z_1^{(-)}-v_1\tau=\lambda(a_3-a_1)/2q_1a_3a_1,$$

we can neglect the variation of $y_1^{(-)}(z-v_1\tau)$ and reduce (25) to the equation of a parabolic cylinder. Well away from the point of stationary phase, the approximate solution of (25) can be readily found in the asymptotic region, in which $q_1(z-z_1^{(-)})^2\gg 1$. It is found to be

$$f_1=C_1\exp\left\{-i\int_{z_0}^{z}\frac{\varkappa_1^{(-)}(z'-v_1\tau)}{z'-z_1^{(-)}}dz'\right\}$$

$$f_3=C_3\exp\left\{i\int_{z_0}^{z}\frac{\varkappa_1^{(-)}(z'-v_1\tau)}{z'-z_1^{(-)}}dz'\right\}. \tag{26}$$

When the function $y_1^{(-)}(z-v_1\tau)$ is sufficiently smooth, the resonance and asymptotic regions are found to overlap. The solutions given by (26) and those of the equation of the parabolic cylinder must be joined in the overlap region and, when this is done, we obtain the solution of the problem defined by (25) on the entire z axis. The application of this procedure yields all the partial scattering matrices.

Henceforth, it will be convenient to use the normalized scattering matrices with elements

$$s_{mn}=|a_m/a_n|^{1/2}S_{mn}.$$

It is shown in Ref. 12 that the general form of the normalized partial matrices for $v_1>v_3>v_2$ is:

$$s_1^{(-)}=\begin{pmatrix}\sigma_1^{(-)*} & 0 & \eta_1^{(-)*}\\ 0 & 1 & 0\\ \eta_1^{(-)} & 0 & \sigma_1^{(-)}\end{pmatrix},$$

$$s_2^{(-)}=\begin{pmatrix}\sigma_2^{(-)} & \eta_2^{(-)} & 0\\ -\eta_2^{(-)*} & \sigma_2^{(-)*} & 0\\ 0 & 0 & 1\end{pmatrix},\quad s_3^{(-)}=\begin{pmatrix}1 & 0 & 0\\ 0 & \sigma_3^{(-)*} & \eta_3^{(-)*}\\ 0 & \eta_3^{(-)} & \sigma_3^{(-)}\end{pmatrix}, \tag{27}$$

The elements of these matrices, evaluated by the above procedure, are as follows:

$$\sigma_1^{(-)}(\xi_1^{(-)}(\lambda))=\exp\left\{-i\int_{-\infty}^{+\infty}\frac{\varkappa_1^{(-)}(z'-v_1\tau)}{z'-z_1^{(-)}-i0}dz'\right\},$$

$$\eta_1^{(-)}(\xi_1^{(-)}(\lambda))=-\frac{(2\pi\varkappa_1^{(-)})^{1/2}}{\Gamma(1+i\varkappa_1^{(-)})}\exp\left\{\frac{\pi\varkappa_1^{(-)}}{2}-i\frac{\pi}{4}+i\Phi_1^{(-)}\right.$$
$$+i\varkappa_1^{(-)}\ln|2q_1\varepsilon^2|+i\int_{z_1^{(-)}+\varepsilon}^{+\infty}\frac{\varkappa_1^{(-)}(z'-v_1\tau)}{z'-z_1^{(-)}}dz'$$
$$\left.-i\int_{-\infty}^{z_1^{(-)}-\varepsilon}\frac{\varkappa_1^{(-)}(z'-v_1\tau)}{z'-z_1^{(-)}}dz'\right\},$$

$$\sigma_2^{(-)}(\xi_2^{(-)}(\lambda))=\exp\left\{i\int_{-\infty}^{+\infty}\frac{\varkappa_2^{(-)}(z'-v_2\tau)}{z'-z_2^{(-)}+i0}dz\right\} \tag{28}$$

$$\eta_2^{(-)}(\xi_2^{(-)}(\lambda))=\frac{(-2\pi\varkappa_2^{(-)})^{1/2}}{\Gamma(1-i\varkappa_2^{(-)})}\exp\left\{\frac{\pi\varkappa_2^{(-)}}{2}-\frac{3i\pi}{4}-i\Phi_2^{(-)}\right.$$
$$-i\varkappa_2^{(-)}\ln|2q_2\varepsilon^2|+i\int_{-\infty}^{z_2^{(-)}-\varepsilon}\frac{\varkappa_2^{(-)}(z'-v_2\tau)}{z'-z_2^{(-)}}dz'$$
$$\left.-i\int_{z_2^{(-)}+\varepsilon}^{+\infty}\frac{\varkappa_2^{(-)}(z'-v_2\tau)}{z'-z_2^{(-)}}dz'\right\}$$

where $\varepsilon\to+0$. The elements $\sigma_3^{(-)},\eta_3^{(-)}$ are not written out in (28) because, when $\varkappa_1^{(-)}$, $\Phi_1^{(-)}$, q_1 are replaced with $\varkappa_3^{(-)}$, $\Phi_3^{(-)}$, q_3 they become identical with the elements $\sigma_1^{(-)}$, $\eta_1^{(-)}$. All the functions $\varkappa_m^{(-)}$, under the integral sign have been evaluated at the corresponding points of stationary phase:

$$\varkappa_m^{(-)}=\varkappa_m^{(-)}(\xi_m^{(-)})=\varkappa_m^{(-)}(z_m^{(-)}-v_m\tau).$$

The relationship between the points of stationary phase and the spectral parameter is

$$\begin{aligned}\xi_1^{(-)}&\equiv z_1^{(-)}-v_1\tau=\lambda(a_3-a_1)/2q_1a_3a_1,\\ \xi_2^{(-)}&\equiv z_2^{(-)}-v_2\tau=\lambda(a_2-a_1)/2q_2a_2a_1,\\ \xi_3^{(-)}&\equiv z_3^{(-)}-v_3\tau=\lambda(a_3-a_2)/2q_3a_3a_2.\end{aligned} \tag{29}$$

The quantity $\Phi_m^{(-)}$ in (28) denotes the functions

$$\Phi_1^{(-)}=-\arg[y_1^{(-)}(\xi_1^{(-)})]-\frac{\lambda^2(a_3-a_1)^2}{4q_1a_3^2a_1^2}-\frac{\lambda v_1\tau(a_3-a_1)}{a_3a_1}$$

$$\Phi_2^{(-)}=-\arg[y_2^{(-)}(\xi_2^{(-)})]-\frac{\lambda^2(a_2-a_1)^2}{4q_2a_2^2a_1^2}-\frac{\lambda v_2\tau(a_2-a_1)}{a_2a_1} \tag{30}$$

$$\Phi_3^{(-)}=-\arg[y_3^{(-)}(\xi_3^{(-)})]-\frac{\lambda^2(a_3-a_2)^2}{4q_3a_3^2a_2^2}-\frac{\lambda v_3\tau(a_3-a_2)}{a_3a_2}.$$

4. SOLUTION OF THE INVERSE PROBLEM

To abbreviate the derivation, we reproduce the procedure for the solution of the inverse problem, but only in the special case $y_2^{(-)}(z,\tau)\equiv0$, i.e., when the incident packet is absent for one of the interacting waves. For the general boundary conditions, we shall reproduce only the final result

$$A_m^{(+)}=A_m^{(+)}(A_n^{(-)}).$$

It is clear that, when $y_2^{(-)}(z,\tau)\equiv0$, the partial scattering matrix $s_2^{(-)}(\lambda,\tau)$ becomes a unit matrix, and the total normalized matrix assumes the following form, according to (23):

$$s(\lambda,\tau)=s_1^{(-)}s_3^{(-)}. \tag{31}$$

For $t\to+\infty$, we find from (22) that

$$s_{mn}(\lambda,t)=(s_1^{(-)}s_3^{(-)})_{mn}\exp\left\{i\lambda\left(\frac{b_n}{a_n}-\frac{b_m}{a_m}\right)(t-\tau)\right\}. \tag{32}$$

On the other hand, it is natural to suppose that, at time $t+\infty$, the wave packets that have undergone an interaction will also be separated in space. Their mutual disposition in space will then change in accordance with the group velocity ratio, and the scattering matrix will again be factored in the form $s(\lambda,t)=s_2^{(+)}s_3^{(+)}s_1^{(+)}$, where the partial normalized matrices are calculated from the packets that have interacted at some time $t\to+\infty$. Equating these two representations of the complete normalized scattering matrix, we obtain

$$(s_2^{(+)}s_3^{(+)}s_1^{(+)})_{mn}(\lambda,t)$$
$$=(s_1^{(-)}s_3^{(-)})_{mn}(\lambda,\tau)\exp\left\{i\lambda\left(\frac{b_n}{a_n}-\frac{b_m}{a_m}\right)(t-\tau)\right\}. \tag{33}$$

We note that the form of the normalized partial scattering matrices $s_1^{(+)}$, $s_2^{(+)}$, $s_3^{(+)}$ is the same as (27) if we replace the elements $\sigma_m^{(-)}$, $\eta_m^{(-)}$ with $\sigma_m^{(+)}$, $\eta_m^{(+)}$ ($m=1,2,3$) in (27). This device together with (33) and (17) enables us to find the general relationship between the elements $\sigma_m^{(+)}$, $\eta_m^{(+)}$ and the elements $\sigma_m^{(-)}$, $\eta_m^{(-)}$:

$$
\begin{gathered}
|\sigma_3^{(+)}|^2=1+|\sigma_1^{(-)}\eta_3^{(-)}|^2, \\
\eta_3^{(+)}=\sigma_1^{(-)}\eta_3^{(-)}\exp\left\{\frac{i\lambda v_3(a_2-a_3)}{a_2a_3}(t-\tau)\right\}, \\
|\sigma_1^{(+)}|^2=\left|\frac{\sigma_1^{(-)}\sigma_3^{(-)}}{\sigma_3^{(+)}}\right|^2, \\
\eta_1^{(+)}=\frac{\eta_1^{(-)}}{\sigma_3^{(+)}}\exp\left\{\frac{i\lambda v_1(a_1-a_3)}{a_1a_3}(t-\tau)\right\}, \\
|\sigma_2^{(+)}|^2=\left|\frac{\sigma_3^{(-)}}{\sigma_3^{(+)}}\right|^2, \\
\eta_2^{(+)}=\frac{\eta_3^{(-)}\eta_1^{(-)*}}{\sigma_3^{(+)*}}\exp\left\{\frac{i\lambda v_2(a_2-a_1)}{a_2a_1}(t-\tau)\right\}.
\end{gathered}
\tag{34}
$$

To express the partial matrices $s_m^{(+)}$ in terms of the elements of the partial matrices $s_m^{(-)}$, we must reconstruct the arguments of the functions $\sigma_m^{(+)}$ from their moduli. It is well known that this can be done if we know the positions of the zeros of the functions $\sigma_m^{(+)}(\lambda)$ in the complex plane of λ. In our case of the parametric decay interaction between smooth wave packets $y_m^{(-)}(z,\tau)$ in an inhomogeneous medium, the functions $\sigma_m^{(-)}$ have no zeros, according to (28), which means that by virtue of (34) the functions $\sigma_m^{(+)}$ have no zeros either. This enables us to write down the required arguments:

$$
\arg[\sigma_m^{(+)}(\lambda)]=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\ln|\sigma_m^{(+)}|}{\lambda'-\lambda}d\lambda'. \tag{35}
$$

To relate the elements of the partial scatering matrices $\sigma_m^{(+)}$, $\eta_m^{(+)}$ to the amplitudes of the outgoing wave packets, we introduce the natural assumptions (confirmed by the ensuing analysis) that the packets $y_m^{(+)}(z-v_mt)$, like the incident packets $y_m^{(-)}(z-v_m\tau)$, are smooth functions. The quantities $\sigma_m^{(+)}$, $\eta_m^{(+)}$ can then be determined from $y_m^{(+)}(z-v_mt)$ by means of the procedure described in the last Section. Naturally, the expressions for $\sigma_m^{(+)}$, $\eta_m^{(+)}$ are found to be analogous to (28), and differ only by the replacement of $\varkappa_m^{(-)}$, $\Phi_m^{(-)}$, $\xi_m^{(-)}$, τ with $\varkappa_m^{(+)}$, $\Phi_m^{(+)}$, $\xi_m^{(+)}$, t, whereas the expressions for $\Phi_m^{(+)}$ are of the form given by (30) except that $\arg[y_m^{(-)}(\xi_m^{(-)})]$ and τ must be replaced with $\arg[y_m^{(+)}(\xi_m^{(+)})]$ and t. The formulas relating the stationary points $z_m^{(+)}$ and the spectral parameter then have the form

$$
\begin{aligned}
\xi_1^{(+)}&\equiv z_1^{(+)}-v_1t=\lambda(a_3-a_1)/2q_1a_3a_1, \\
\xi_2^{(+)}&\equiv z_2^{(+)}-v_2t=\lambda(a_2-a_1)/2q_2a_2a_1, \\
\xi_3^{(+)}&\equiv z_3^{(+)}-v_3t=\lambda(a_3-a_2)/2q_3a_3a_2.
\end{aligned}
\tag{36}
$$

Comparison of (29) with (36) shows that all the stationary points $\xi_m^{(+)}$ and $\xi_m^{(-)}$ are related to one another. This relationship can be determined by eliminating the spectral parameter λ from (29) and (36). Using (12) and (18), we obtain the relationship between $\xi_m^{(+)}$ and $\xi_m^{(-)}$, which does not contain the nonphysical parameters a_m:

$$
\frac{\xi_1^{(-)}}{v_1}=\frac{\xi_2^{(-)}}{v_2}=\frac{\xi_3^{(-)}}{v_3}=\frac{\xi_1^{(+)}}{v_1}=\frac{\xi_2^{(+)}}{v_2}=\frac{\xi_3^{(+)}}{v_3}. \tag{37}
$$

This result is a consequence of the local point interaction between the wave packets in the decay region $z=0$. Indeed, the outgoing wave packet $y_m^{(+)}$ is generated at the point $z=0$ at time T under the influence of the local values of the amplitudes of the packets of incident waves $y_m^{(-)}$, which, at the time T, are located at the point of decay. In other words, the trajectories of all the packets, both incident and outgoing, cross at the decay point $z=0$.

Equating the elements $\sigma_m^{(+)}$, $\eta_m^{(+)}$, calculated directly from the unknown packets $y_m^{(+)}(z-v_mt)$ after the interaction, and the same elements $\sigma_m^{(+)}$, $\eta_m^{(+)}$, expressed in accordance with (34) and (35) in terms of the partial normalized matrices $\sigma_m^{(-)}$, $\eta_m^{(-)}$ prior to interaction, we obtain a set of transcendental equations for the unknown functions $\varkappa_m^{(+)}$, $\arg[y_m^{(+)}]$. This set can be solved for the required amplitudes and phases of the packets. This solves the problem of the connection between the profiles of wave packets y_m^{ν} in an inhomogeneous medium before and after the interaction for $t\to\pm\infty$. We shall reproduce here only the answer for the packet $y_3^{(+)}$, bearing in mind the subsequent interpretation of the results in relation to the stationary problem:

$$
\begin{aligned}
\varkappa_3^{(+)}(\xi_3^{(+)})=\frac{1}{2\pi}\ln[1+2\,\mathrm{sh}(\pi\varkappa_3^{(-)}(\xi_3^{(-)}))\exp\{2\pi\varkappa_1^{(-)}(\xi_1^{(-)}) \\
+\pi\varkappa_3^{(-)}(\xi_3^{(-)})\}],
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\arg[y_3^{(+)}(\xi_3^{(+)})]=\arg[y_3^{(-)}(\xi_3^{(-)})] \\
+\arg\left[\frac{\Gamma(1+i\varkappa_3^{(-)}(\xi_3^{(-)}))}{\Gamma(1+i\varkappa_3^{(+)}(\xi_3^{(+)}))}\right]+\Lambda,
\end{aligned}
\tag{39}
$$

where

$$
\begin{aligned}
\Lambda=[\varkappa_3^{(+)}(\xi_3^{(+)})-\varkappa_3^{(-)}(\xi_3^{(-)})]\ln|2q_3\varepsilon^2|+\int_{\varepsilon\to+0}^{+\infty}[\varkappa_3^{(+)}(x+\xi_3^{(+)}) \\
-\varkappa_3^{(-)}(x+\xi_3^{(-)})]\frac{dx}{x}+\int_{-\infty}^{+\infty}\varkappa_1^{(-)}(x+\xi_1^{(-)})\frac{dx}{x} \\
-\int_{-\infty}^{\varepsilon\to-0}[\varkappa_3^{(+)}(x+\xi_3^{(+)})-\varkappa_3^{(-)}(x+\xi_3^{(-)})]\frac{dx}{x}.
\end{aligned}
$$

Equation (38) and (39) have the following meaning. Equation (38) is used to determine the modulus of the amplitude of the resonance-crossing packet $y_3^{(+)}(\xi_3^{(+)})$ at each point $\xi_3^{(+)}=z-v_3t$. In this procedure, the quantities $\xi_3^{(-)}$, $\xi_1^{(-)}$ on the right-hand side of (38) must be replaced with $\xi_3^{(-)}=\xi_3^{(+)}$, $\xi_1^{(-)}=(v_1/v_3)\xi_3^{(+)}$, in accordance with (37). Equation (39) gives the slowly-varying phase of the same packet for $t\to+\infty$. It is important to note that the functions $\varkappa_3^{(+)}(\xi_3^{(+)})$, $\arg[y_3^{(+)}(\xi_3^{(+)})]$, calculated from (38) and (39), are actually smooth functions because of the smoothness of the amplitude of the wave packets prior to interaction, which totally justifies the assumptions made in the derivation of (38) and (39).

As noted in Sec. 2 [formula (11)], the amplitudes $y_3^{(\pm)}$ of the wave packets in the far asymptotic regions differ from the amplitudes $A_3^{(\pm)}$ in which we are interested here only by their phases, i.e.,

$$\arg[y_3^{(+)}(\xi_3^{(+)})]=\arg[A_3^{(+)}(\xi_3^{(+)})]+\gamma_3^{(+)}(+\infty,\xi_3^{(+)}),$$
$$\arg[y_3^{(-)}(\xi_3^{(-)})]=\arg[A_3^{(-)}(\xi_3^{(-)})]+\gamma_3^{(-)}(-\infty,\xi_3^{(-)}), \tag{40}$$

where $\gamma_3^{(\pm)}(\pm\infty,\xi_3^{(\pm)})$ are given by (8) for $z\to\pm\infty$. Substituting (40) in (39), we obtain

$$\arg[A_3^{(+)}(\xi_3^{(+)})]=\arg[A_3^{(-)}(\xi_3^{(-)})]+\arg\left[\frac{\Gamma(1+i\varkappa_3^{(-)}(\xi_3^{(-)}))}{\Gamma(1+i\varkappa_3^{(+)}(\xi_3^{(+)}))}\right]+\Lambda+\gamma_3^{(-)}(-\infty,\xi_3^{(-)})-\gamma_3^{(+)}(+\infty,\xi_3^{(+)}). \tag{41}$$

To interpret the integral terms in (41), we must take into account the following relationships, essentially the Manley-Rowe relations:

$$\varkappa_1^{(+)}(\xi_1^{(+)})=\varkappa_1^{(-)}(\xi_1^{(-)})+\varkappa_3^{(-)}(\xi_3^{(-)})-\varkappa_3^{(+)}(\xi_3^{(+)}),$$
$$\varkappa_2^{(+)}(\xi_2^{(+)})=\varkappa_3^{(-)}(\xi_3^{(-)})-\varkappa_3^{(+)}(\xi_3^{(+)}). \tag{42}$$

These results are obtained from (34) with the aid of the following formula that follows from (28):

$$|\sigma_m^{(\nu)}|^2=\exp\{2\pi\varkappa_m^{(\nu)}(\xi_m^{(\nu)})\}.$$

Using the Manley-Rowe relations together with (37), it can be shown, after some cumbersome derivations, that the integral terms in (41) acutally cancel out and (41) assumes the simple form

$$\arg[A_3^{(+)}(\xi_3^{(+)})]=\arg[A_3^{(-)}(\xi_3^{(-)})]+\arg\left[\frac{\Gamma(1+i\varkappa_3^{(-)}(\xi_3^{(-)}))}{\Gamma(1+i\varkappa_3^{(+)}(\xi_3^{(+)}))}\right]. \tag{43}$$

It was noted in Sec. 2 that, to obtain the relationship between the asymptotic constants in the stationary problem from (38), (42), and (43), it is sufficient to discard the arguments $\xi_m^{(\nu)}$ in (38), (42), and (43). This gives us the relationships between the amplitude $A_3^{(+)}$ of the departing wave $y_3^{(+)}$ and the amplitudes $A_1^{(-)}, A_3^{(-)}$ of the incident waves $y_1^{(-)}, y_3^{(-)}$ (the case in which the incident wave $y_2^{(-)}$ is absent). Let us now examine the limiting cases of (38), (42), and (43).

In the case of a weak wave $y_3^{(-)}$,

$$2\pi\varkappa_3^{(-)}\ll 1,$$

and a pump $y_1^{(-)}$ of moderate intensity,

$$2\pi\varkappa_3^{(-)}\exp(2\pi\varkappa_1^{(-)})\ll 1,$$

Eqs. (38), (42), and (43) become identical with the well-known result of the linear theory[1] that describes the convective amplification of the incident wave:

$$\varkappa_3^{(+)}\approx\varkappa_3^{(-)}\exp(2\pi\varkappa_1^{(-)}),\quad \varkappa_1^{(+)}\approx\varkappa_1^{(-)},$$
$$\arg[A_3^{(+)}]\approx\arg[A_3^{(-)}]. \tag{44}$$

For a powerful pump wave, for which

$$2\pi\varkappa_3^{(-)}\exp(2\pi\varkappa_1^{(-)})\gg 1,$$

the pump becomes strongly depleted as its energy is transferred to secondary waves:

$$\varkappa_3^{(+)}\approx\varkappa_1^{(-)}+\frac{1}{2\pi}\ln(2\pi\varkappa_3^{(-)}). \tag{45}$$

This effect is even more clearly defined when both incident waves are strong, i.e., $2\pi\varkappa_1^{(-)}\gg 1$, $2\pi\varkappa_3^{(-)}\gg 1$:

$$\varkappa_3^{(+)}\approx\varkappa_1^{(-)}+\varkappa_3^{(-)}-\frac{1}{2\pi}\exp(-2\pi\varkappa_3^{(-)}). \tag{46}$$

We note that, in this case, there is also an appreciable change in the phase of the departing wave $y_3^{(+)}$:

$$\arg[A_3^{(+)}]\approx\arg[A_3^{(-)}]+\arg\left[\frac{\Gamma(1+i\varkappa_3^{(-)})}{\Gamma(1+i\varkappa_1^{(-)}+i\varkappa_3^{(-)})}\right]. \tag{47}$$

5. RELATIONSHIP BETWEEN THE ASYMPTOTIC CONSTANTS OF THE STATIONARY PROBLEM ($v_1>v_3>0>v_2$)

The above algorithm can be used to derive the formulas relating the asymptotic complex amplitudes $A_1^{(\nu)}, A_2^{(\nu)}, A_3^{(\nu)}$ of the interacting waves y_1, y_2, y_3 in the case of general boundary conditions. It is useful to recall the necessary definitions:

$$A_m^{(\nu)}=|A_m^{(\nu)}|\exp(i\varphi_m^{(\nu)}),\quad m=1,2,3;$$
$$\varkappa_1^{(\nu)}=-h^2|A_1^{(\nu)}|^2/v_2v_3>0,$$
$$\varkappa_2^{(\nu)}=-h^2|A_2^{(\nu)}|^2/v_1v_3<0,$$
$$\varkappa_3^{(\nu)}=-h^2|A_3^{(\nu)}|^2/v_1v_2>0,$$

where $(\nu)=(\pm)$ correspond to departing and incident waves at the resonance point $z=0$, respectively. The final results can be summarized as follows.

The moduli of the amplitudes are given by

$$\varkappa_3^{(+)}=\frac{1}{2\pi}\ln[1+\rho_3^2+\delta_3^2+2\rho_3\delta_3\cos(\alpha_3-\beta_3)],$$
$$\varkappa_2^{(+)}=\varkappa_2^{(-)}+\varkappa_3^{(-)}-\varkappa_3^{(+)},\quad \varkappa_1^{(+)}=\varkappa_1^{(-)}+\varkappa_3^{(-)}-\varkappa_3^{(+)}. \tag{48}$$

The phases of the departing waves have the form

$$\varphi_m^{(+)}=\arg[\Gamma(1-i\varkappa_m^{(+)})]+\operatorname{arctg}\left[\frac{\rho_m\sin\alpha_m+\delta_m\sin\beta_m}{\rho_m\cos\alpha_m+\delta_m\cos\beta_m}\right],$$
$$m=1,2,3, \tag{49}$$

where the quantities α_m are given by the simple general formula

$$\alpha_m=\varphi_m^{(-)}+\arg[\Gamma(1+i\varkappa_m^{(-)})],\quad m=1,2,3. \tag{50}$$

The remaining quantities are as follows:

$$\rho_1=2^{1/2}\mathrm{sh}^{1/2}(\pi\varkappa_1^{(-)})\exp\{\pi(\varkappa_2^{(-)}-\varkappa_3^{(-)}-\varkappa_1^{(-)}/2)\},$$
$$\delta_1=2\,\mathrm{sh}^{1/2}(\pi\varkappa_3^{(-)})\mathrm{sh}^{1/2}(-\pi\varkappa_2^{(-)})\exp\{\pi(\varkappa_2^{(-)}-\varkappa_3^{(-)})/2\}, \tag{51}$$
$$\beta_1=\varphi_3^{(-)}+\varphi_2^{(-)}+\pi/4+\arg[\Gamma(1+i\varkappa_3^{(-)})\Gamma(1+i\varkappa_2^{(-)})];$$

$$\rho_2=2^{1/2}\mathrm{sh}^{1/2}(-\pi\varkappa_2^{(-)})\exp\{\pi(\varkappa_1^{(-)}-\varkappa_3^{(-)}-\varkappa_2^{(-)}/2)\},$$
$$\delta_2=2\,\mathrm{sh}^{1/2}(\pi\varkappa_1^{(-)})\mathrm{sh}^{1/2}(\pi\varkappa_3^{(-)})\exp\{\pi(\varkappa_1^{(-)}-\varkappa_3^{(-)})/2\}, \tag{52}$$
$$\beta_2=\varphi_1^{(-)}-\varphi_3^{(-)}+3\pi/4+\arg[\Gamma(1+i\varkappa_1^{(-)})\Gamma(1-i\varkappa_3^{(-)})];$$

$$\rho_3=2^{1/2}\mathrm{sh}^{1/2}(\pi\varkappa_3^{(-)})\exp\{\pi(\varkappa_1^{(-)}+\varkappa_2^{(-)}+\varkappa_3^{(-)}/2)\},$$
$$\delta_3=2\,\mathrm{sh}^{1/2}(-\pi\varkappa_2^{(-)})\mathrm{sh}^{1/2}(\pi\varkappa_1^{(-)})\exp\{\pi(\varkappa_1^{(-)}+\varkappa_2^{(-)})/2\}, \tag{53}$$
$$\beta_3=\varphi_1^{(-)}-\varphi_2^{(-)}+3\pi/4+\arg[\Gamma(1+i\varkappa_1^{(-)})\Gamma(1-i\varkappa_2^{(-)})].$$

It is important to note that the presence of the dependence on $2\rho_3\delta_3\cos(\alpha_3-\beta_3)$ in (48) reflects the interference of the interacting waves. In particular, when

$$\rho_3=\delta_3,\quad \alpha_3-\beta_3=\pi \tag{54}$$

we have complete suppression of the departing wave $y_3^{(+)}$ and an amplification of the pump wave $y_1^{(+)}$. For weak incident secondary waves $2\pi\varkappa_3^{(-)}\ll 1$, $-2\pi\varkappa_2^{(-)}\ll 1$ and a strong pump $2\pi\varkappa_1^{(-)}>1$, the suppression conditions (54) reduce to

$$\varkappa_3^{(-)}+\varkappa_2^{(-)}=0,$$
$$\varphi_1^{(-)}-\varphi_2^{(-)}-\varphi_3^{(-)}=\frac{\pi}{4}-\arg[\Gamma(1+i\varkappa_1^{(-)})], \tag{55}$$

found in Ref. 8 in the linear theory.

6. CONCLUSION

We have examined the nonlinear stage of the three-wave decay interaction in an inhomogeneous medium in the most interesting case, in which the group velocities of the interacting waves are such that $v_1>v_3>0>v_2$ and absolute parametric instability can be excited in an inhomogeneous medium by closing the feedback loop by some suitable mechanism. It is clear that the above algorithm can be used to examine all other possible cases of group velocity ratios and to obtain in a systematic manner all the possible formulas for the relationship between the asymptotic constants in the three-wave stationary problem in an inhomogeneous medium.

[1] A. D. Piliya, Proc. Tenth Conf. on Phenomena in Ionized Gases, Oxford, 1971, p. 320.
[2] M. N. Rosenbluth, Phys. Rev. Lett. **29**, 565 (1972).
[3] L. M. Gorbunov, V. M. Domrin, and R. R. Ramazashvili, Zh. Eksp. Teor. Fiz. **70**, 2161 (1976) [Sov. Phys. JETP **43**, 1128 (1976)].
[4] A. D. Piliya, Pis'ma Zh. Eksp. Teor. Fiz. **17**, 374 (1973) [JETP Lett. **17**, 266 (1973)].
[5] D. Pesme, G. Laval, and R. Pellat, Phys. Rev. Lett. **31**, 203 (1973).
[6] E. Z. Gusakov and A. D. Piliya, Fiz. Plazmy **6**, 509 (1980) [Sov. J. Plasma Phys. **6**, 277 (1980)].
[7] J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, Phys. Rev. **127**, 1918 (1962).
[8] A. D. Piliya, Zh. Eksp. Teor. Fiz. **64**, 1237 (1973) [Sov. Phys. JETP **37**, 629 (1973)].
[9] S. P. Novikov, V. E. Zakharov, S. V. Manakov, and L. P. Pitaevskii, Theory of Solitons: The Inverse Scattering Method Consultants Bureau, New York (1984).
[10] A. Reiman, A. Bers, and D. Kaup, Phys. Rev. Lett. **39**, 245 (1977).
[11] A. A. Andreev, Pis'ma Zh. Tekh. Fiz. **5**, 377 (1979) [Sov. Tech. Phys. Lett. **5**, 152 (1979)].
[12] V. E. Zakharov and S. V. Manakov, Zh. Eksp. Teor. Fiz. **69**, 1654 (1975) [Sov. Phys. JETP **42**, 842 (1975)].
[13] A. Reiman, Rev. Mod. Phys. **51**, 311 (1979).
[14] V. E. Zakharov and S. V. Manakov, Zh. Eksp. Teor. Fiz. **71**, 203 (1976) [Sov. Phys. JETP **44**, 106 (1976)].

Translated by S. Chomet