Inhomogeneous stationary self-similar distribution of electrons and anomalous transport in turbulent current-carrying plasma

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The stationary self-similar distribution of electrons in a spatially inhomogeneous plasma with fully developed ion-acoustic turbulence is derived and the anomalous charge and heat fluxes corresponding to this distribution are determined.

1. INTRODUCTION

Spatially inhomogeneous self-similar electron distributions that evolve in the course of time have frequently been discussed in the literature in connection with the theory of anomalous resistance and heating of plasma during the excitation of ion-acoustic turbulence (IAT)[1–3] (see also the reviews in Refs. 4 and 5). These distributions evolve over a time interval of the order of the reciprocal of the effective frequency of electron energy relaxation, and their form does not depend on the form of the initial distribution function, but it is determined by quasilinear interaction with ion-acoustic waves. The quasilinear approach was used in Refs. 1 and 2 to develop a theory of the IAT spectrum and of the ion and electron distribution, while the self-similar distributions were investigated in Ref. 3 on the assumption of a *given* IAT distribution. The approach to IAT theory proposed in Ref. 6 was based on the simultaneous inclusion of quasilinear effects and stimulated scattering of sound by ions, and was used in Refs. 7 and 8 to investigate spatially inhomogeneous electron distributions. In particular, Bychenkov *et al*[8] investigated the relaxation of an initially Maxwellian distribution function to the self-similar distribution derived in Ref. 8.

The electron distribution problem can be formulated for spatially inhomogeneous turbulent plasma in another interesting way that has not yet been discussed. A nonequilibrium stationary self-similar distribution of electrons may be expected to exist in inhomogeneous plasma and be maintained by potential, pressure, or temperature gradients. In this paper, we shall establish the conditions for the existence of this type of distribution, and will determine its form using the approach to IAT theory proposed in Ref. 6. We shall also determine the anomalous electron heat and charge fluxes.

In Sec. 2, we start with the transport equations for electrons and ion-acoustic waves, and derive the equations for the isotropic part of the distribution function in an inhomogeneous plasma with fully developed IAT. In Sec. 3, the stationary inhomogeneous self-similar electron distribution is found analytically, and conditions are determined for its existence. The results of a numerical integration of the equation describing this distribution are also given in Sec. 3. The electron heat and charge fluxes are calculated in Sec. 4 for the self-similar inhomogeneous distribution, and the range of validity of the self-similar distribution is examined in velocity and ordinary space.

2. EQUATION FOR THE ISOTROPIC PART OF THE DISTRIBUTION FUNCTION

Our analysis will be based on the transport equation for the electron distribution function $f=f(v,\xi,t)$, which we shall write down on the assumption that all the gradients and the electric field $\mathbf{E}(z)$ point along the z axis, and the positive values of $E(z)$ correspond to the positive direction of the z axis (v is the magnitude of the velocity of $\mathbf{v}$, $\xi=\cos\theta_v$, $\theta_v=\measuredangle\mathbf{v}$, $\mathbf{R}$, $\mathbf{R}=en_e(z)\mathbf{E}(z)-\nabla p_e(z)$ is the effective force density vector, and e, $n_e(z)=n_e$, $p_e(z)$ are, respectively the electron charge, density, and pressure. We shall assume that the instability excitation threshold has been substantially exceeded. Neglecting binary electron-electron and electron-ion collisions, we have[5,6]

$$\frac{\partial f}{\partial t}+\frac{e}{m_e}E(z)\left(\xi\frac{\partial f}{\partial v}+\frac{1-\xi^2}{v}\frac{\partial f}{\partial\xi}\right)+\xi v\frac{\partial f}{\partial z}=\frac{1}{v^2}\frac{\partial}{\partial v}v^2\left[D_{vv}\frac{\partial f}{\partial v}+\frac{(1-\xi^2)^{1/2}}{v}D_{v\xi}\frac{\partial f}{\partial\xi}\right]+\frac{1}{v}\frac{\partial}{\partial\xi}\left[(1-\xi^2)^{1/2}D_{v\xi}\frac{\partial f}{\partial v}+\frac{1-\xi^2}{v}D_{\xi\xi}\frac{\partial f}{\partial\xi}\right]\quad(1)$$

where m_e is the mass of the electron and the components of the quasilinear diffusion tensor $D_{\alpha\beta}$ are determined by the number distribution $N(k,x)$ ($x=\cos\theta_k$) of the ion-acoustic waves over the angles $\theta_k\,\measuredangle\mathbf{k}$, the vector $\mathbf{R}$ and the acoustic wave numbers k. These components are given by

$$D_{\alpha\beta}=\frac{e^2}{\pi v m_e^2\omega_{Li}^2}\int_{k_{min}}^{k_{max}}dk\,k\omega_s^3\int_{-1}^{1}dx\,N(k,x)\,d_{\alpha\beta}\times\left[(1-\xi^2)(1-x^2)-\left(\frac{\omega_s}{kv}-x\xi\right)^2\right]^{-1/2}\times\eta\left[(1-\xi^2)(1-x^2)-\left(\frac{\omega_s}{kv}-x\xi\right)^2\right],\quad(2)$$

where $\eta(x)=1$, $x\geqslant0$, $x<0$; ω_{Li} is the Langmuir frequency of ions, $K_{\min}$ and $k_{\max}$ are the limits of the range of values of k in which the number density of the ion-acoustic waves is positive, $\omega_s=kv_s(1+k^2r_{De}^2)^{-1/2}$ is the frequency of sound, $v_s=\omega_{Li}r_{De}$, r_{De} is the Debye radius of electrons, and the quantities $d_{\alpha\beta}$ have the form

$$d_{vv}=\left(\frac{\omega_s}{kv}\right)^2,\quad d_{v\xi}=\frac{\omega_s}{kv}\frac{x-\omega_s\xi/kv}{(1-\xi^2)^{1/2}},\quad d_{\xi\xi}=\frac{(x-\omega_s\xi/kv)^2}{1-\xi^2}.\quad(3)$$

We shall seek the solution of (1) in the form of the sum of the isotropic part $f_0=f_0(v,t)$ and the small anisotropic $f_1=f_1(v,\xi,t)\ll f_0$:

$$f=f_0+f_1,\quad f_0=\frac{1}{2}\int_{-1}^{1}d\xi\,f.\quad(4)$$

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Substituting $f = f_0 + f_1$, and integrating with respect to ξ between -1 and 1, we obtain the equation for f_0. Subtracting this equation from (1), we obtain the equation for f_1. Next, for electrons with $v \gg v_s$, we have the following relationship between the components of the diffusion tensor (2):

$$D_{\xi\xi} \gg D_{v\xi} \sim \frac{v_s}{v} D_{\xi\xi} \gg D_{vv} \sim \left(\frac{v_s}{v}\right)^2 D_{\xi\xi}. \tag{5}$$

These inequalities enable us to decouple the equations for f_0 and f_1. The stationary solution of the equation for f_1 then has the form (see, for example, Ref. 6):

$$\frac{\partial f_1}{\partial \xi} = -\frac{v^2}{D_{\xi\xi}}\left\{\frac{eE(z)}{2m_e}\frac{\partial f_0}{\partial v} + \frac{v}{2}\frac{\partial f_0}{\partial z} + \frac{D_{v\xi}}{v(1-\xi^2)^{1/2}}\frac{\partial f_0}{\partial v}\right\}. \tag{6}$$

Substituting this into the equation for f_0, we obtain

$$\frac{\partial f_0}{\partial t} + \frac{eE(z)}{2m_e v^2}\frac{\partial}{\partial v}v^2\int_{-1}^{1}\frac{d\xi}{2}\frac{1-\xi^2}{D_{\xi\xi}}\left[-\frac{eE(z)v^2}{2m_e}\frac{\partial f_0}{\partial v} - \frac{v^3}{2}\frac{\partial f_0}{\partial z}\right]$$
$$-\frac{v}{2}\frac{\partial}{\partial z}\int_{-1}^{1}\frac{d\xi}{2}\frac{1-\xi^2}{D_{\xi\xi}}\left[\frac{eE(z)v^2}{2m_e}\frac{\partial f_0}{\partial v} + \frac{v^3}{2}\frac{\partial f_0}{\partial z} + \frac{vD_{v\xi}}{(1-\xi^2)^{1/2}}\frac{\partial f_0}{\partial v}\right]$$
$$= \frac{1}{v^2}\frac{\partial}{\partial v}w^2\int_{-1}^{1}\frac{d\xi}{2}\left\{\left(D_{vv} - \frac{D_{v\xi}^2}{D_{\xi\xi}}\right)\frac{\partial f_0}{\partial v} - \frac{v^2}{2}(1-\xi^2)^{1/2}\frac{D_{v\xi}}{D_{\xi\xi}}\frac{\partial f_0}{\partial z}\right\}. \tag{7}$$

The coefficients $D_{\alpha\beta}$ that describe the evolution of f_0 depend on the distribution of ion-acoustic waves given by (2). To determine the stationary distribution $N(k, x)$, we can use the condition that the sum of the growth rate of the Cherenkov interaction between sound and electrons and the growth rate of stimulated scattering of sound by ions[1)] is zero:

$$\frac{\pi\omega_{Le}^2\omega_s^5}{n_e\omega_{Li}^2k^2}\left\{\int_{-1}^{1}d\xi\int_0^\infty dv\left[\frac{\omega_s}{k}\frac{\partial f_1}{\partial v} + \left(x - \frac{\xi\omega_s}{kv}\right)\frac{\partial f_1}{\partial \xi}\right]\right.$$
$$\left.\times\frac{\eta[(1-\xi^2)(1-x^2)-(\omega_s/kv - x\xi)^2]}{[(1-\xi^2)(1-x^2)-(\omega_s/kv-x\xi)^2]^{1/2}} - \pi\frac{\omega_s}{k}f_0\left(\frac{\omega_s}{k}\right)\right\}$$
$$+\frac{k^2v_{Ti}^2}{4\pi n_i m_i v_s^2}\left(\frac{kv_s}{\omega_s}\right)^3\frac{\partial}{\partial k}k^4\left(\frac{kv_s}{\omega_s}\right)^3\int_0^1 dx' Q(x, x')N(k, x') = 0, \tag{8}$$

where ω_{Le} is the Langmuir frequency of electrons and n_i, m_i, and v_{Ti}, are, respectively, the density, mass, and thermal velocity of ions. The kernel of the nonlinear wave interaction in (8) is given by

$$Q(x, x') = \int_0^{2\pi}\frac{d\varphi_{k'}}{2\pi}\frac{(\mathbf{k}\cdot\mathbf{k}')^2(\mathbf{k}\times\mathbf{k}')^2}{(kk')^4}, \tag{9}$$

where $\varphi_{k'}$ is the azimuthal of the vector $\mathbf{k}'$.

We assumed that (8) is valid in the range between $k_{\min} r_{De} \ll 1$ and $k_{\max} r_{De} \gg 1$, where $k_{\min}$ is determined by collisional damping of sound by ions and $k_{\max}$ is due to the strong Landau damping of sound by ions. We emphasize that (8) does not take into account the Cherenkov interaction with ions. This assumption is valid when the rate at which turbulent acoustic pulsations are damped by ions is lower than that for electrons.[9] We now turn to the solution of (8) and, following Ref. 10, we assume that $(\omega_s/k) f_0(\omega_s/k, t) \approx v_s f_0(0, t)$. According to Ref. 10, the [illegible] is type of approximation for the last term in braces in [illegible] has very little effect on integrals of the wave distribution $N(k, x)$ encountered in IAT theory. Next, substituting (6) in (8), and neglecting small corrections on the order of $\omega_s/kv \ll 1$, we obtain the solution of (8) in the limit of small turbulent Knudsen numbers ($K_N \ll 1$). We then have $N(k, x) = N(k)\Phi(x)$, where $N(k)$ and $\Phi(x)$ are given by[6,7,11]

$$N(k) = \pi(2\pi)^{1/2}n_i m_i\frac{\omega_{Li}}{\omega_{Le}}U\frac{\omega_s^3}{k^7v_{Ti}^2}\left[\ln\frac{\omega_{Li}}{\omega_s} - 0.5\left(\frac{\omega_s}{kv_s}\right)^2 - 0.25\left(\frac{\omega_s}{kv_s}\right)^4\right], \tag{10}$$

$$\Phi(x) = \frac{4K_N}{3\pi x}\frac{d}{dx}\left(\frac{x^4}{1-x+\varepsilon}\right). \tag{11}$$

where

$$U = (2\pi)^{1/2}n_e^{-1}v_0^3 f_0(0), \quad \varepsilon \approx \frac{8K_N}{3\pi}\ln\frac{3\pi}{8K_N} \ll 1,$$
$$K_N = \frac{3\pi R r_{Di}^2}{\lambda U^2 n_e m_e v_s \omega_{Li} r_{De}^2}, \quad R = en_e(z)E(z) - \frac{\partial}{\partial z}p_e(z) > 0, \tag{12}$$

$v_0 = \omega_{Le} r_{De}$, $\lambda \approx 0.5$, and r_{Di} is the Debye radius of ions. The density $n_e(z)$, pressure $p_e(z)$, and Debye radius of electrons in (12) are functionals of the isotropic part of the distribution function:

$$n_e(z) = \int f_0\, dv, \quad p_e(z) = \frac{m_e}{3}\int v^2 f_0\, dv, \tag{13}$$

$$r_{De}^{-2} = -\frac{\omega_{Le}^2}{n_e}\int\frac{dv}{v}\frac{\partial f_0}{\partial v}. \tag{14}$$

Following (2), and recalling that ω_s is small in comparison with kv, and also the fact that $k_{\max} r_{De} \gg 1 \gg k_{\min} r_{De}$, we obtain the following approximate formulas for the components of the quasilinear diffusion tensor:

$$\begin{vmatrix} D_{vv} \\ D_{v\xi} \\ D_{\xi\xi} \end{vmatrix} = \nu_0\frac{v_0^3}{v}\begin{vmatrix} \frac{v_s^2}{v^2}\chi_0((1-\xi^2)^{1/2}) \\ \frac{v_s}{v}\chi_1((1-\xi^2)^{1/2}) \\ \chi_2((1-\xi^2)^{1/2}) \end{vmatrix}, \tag{15}$$

where $\nu_0 = (9\pi/8)^{1/2}R/n_e m_e v_s U$, and the functions $\chi_n(y)$ ($n = 0, 1, 2$) depend on the function $\Phi(x)$ given by (11):

$$\chi_n(y) = \frac{1}{K_N}\int_0^y\frac{\Phi(x)\,dx}{(y^2-x^2)^{1/2}}\left(\frac{x}{y}\right)^n. \tag{16}$$

Since the diffusion tensor is now known, we can write equations (7) for f_0 in the form

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}[A(z)v^5 + B(z)v^{-1}]\frac{\partial f_0}{\partial v}$$
$$+\frac{1}{v^2}\frac{\partial}{\partial v}[C(z)v^6 - \mathscr{D}(z)v^3]\frac{\partial f_0}{\partial z}$$
$$+\frac{\partial}{\partial z}[C(z)v^4 + \mathscr{D}(z)v]\frac{\partial f_0}{\partial v} + v^5\frac{\partial}{\partial z}\left[\mathscr{E}(z)\frac{\partial f_0}{\partial z}\right], \tag{17}$$

where

$$A(z)=\frac{\beta}{\nu_0 v_0}\left[\frac{eE(z)}{2m_e v_0}\right]^2,\quad B(z)=\frac{\beta_0}{2}\nu_0 v_s^2 v_0^3,\tag{18}$$

$$C(z)=\frac{\beta eE(z)}{4m_e\nu_0 v_0^3},\quad \mathscr{D}(z)=\frac{1-\beta}{2}v_s,\quad \mathscr{E}(z)=\frac{\beta}{4\nu_0 v_0^3},$$

and also $\beta = 0.18$ and $\beta_0 = 0.3$. As in Refs. 7 and 8, we assume in the derivation of (17) that the anisotropic part of the distribution function f_1 is much smaller for most of the particles than the isotropic part f_0. Substituting (15) in (6), we readily see (cf. Ref. 7 and 8) that the inequality $f_1 \ll f_0$ is valid only for velocities that are smaller than the limiting velocity $v_{\max}$ determined from the condition $f_1 \sim f_0$. Equation (17), which describes the space-time evolution of the electron distribution in plasmas with fully developed IAT, can therefore be used to find f_0, but only in the region of low velocities $v \lesssim v_{\max}$. We note that, in the special case of a spatially homogeneous distribution of electrons and $E(z) = E$ = const, equation (17) becomes identical with the equation examined in Refs. 7 and 8 in the self-consistent theory of the electron distribution and fully developed IAT.

3. STATIONARY SELF-SIMILAR DISTRIBUTION OF ELECTRONS

We shall now examine the stationary $\partial f_0/\partial t = 0$ self-similar solution of (17). First, we note that if we seek the self-similar solution in the form

$$f_0=\frac{n_e(z)}{w_0^3(z)}F\left[\frac{v}{w_0(z)}\right],$$

where the unknown velocity is $w_0(z) > 0$, we can verify that the solution exists provided

$$w_0'(z)w_0(z)=\alpha\frac{e}{m_e}E(z),\tag{19}$$

$$n_e(z)=\text{const}\, w_0^l(z),\tag{20}$$

where α and l are numbers. The number α in (19) can have either sign (sign $\alpha = \pm 1$), which corresponds to the two possible orientations of the pressure gradient, i.e., either parallel or antiparallel to the electrostatic force $e\mathbf{E}(z)$. The condition given by (20) signifies that, for the self-similar distribution, the electron density and, hence, other microscopic quantities, must be definite functions of the effective velocity $w_0(z)$.

In view of the foregoing, and substituting $w(z) = w_0(z)/(|\alpha|)^{1/2}$, we seek the self-similar solution of (17) in the form

$$f_0=Cw^a(z)F[v/w(z)],\tag{21}$$

where a and C are unknown constants. The pressure $p_e(z)$, the characteristic electron velocity v_0, the velocity of sound v_s, the electron density $n_e(z)$, and the turbulent electron momentum relaxation frequency ν_0 are uniquely determined by the velocity $w(z)$ and the moments μ_n of the functions $F(y)$:

$$\mu_n=\int_0^{y_m} dy\, y^n F(y),\tag{22}$$

provided the function f_0 is given by (21) and $y_m = v_{\max}/w(z)$, $n = 0,1,2...$ Using (12)–(15) and the identity (19), we have

$$p_e(z)=\frac{4\pi}{3}Cm_e\mu_4 w^{a+5}(z),\quad n_e(z)=4\pi C\mu_2 w^{a+3}(z),\tag{23}$$

$$v_0=v_s\omega_{Le}/\omega_{Li}=w(z)(\mu_2/\mu_0)^{1/2},\quad \mu_0\gg y_m F(y_m),$$

$$\nu_0=\frac{eE(z)}{m_e w(z)}\frac{\omega_{Le}}{\omega_{Li}}\gamma\left(\frac{\mu_0}{\mu_2}\right)^2>0,\tag{24}$$

where the γ-functional of the distribution $F(y)$ is

$$\gamma=\frac{1}{F(0)}\left[\frac{3}{2}\mu_2-\frac{1}{2}(a+5)\mu_4\,\text{sign}\,\alpha\right].\tag{25}$$

According to (24), solutions such as (21) exist if $eE(z)\gamma > 0$. In view of (23)–(25), the coefficients (18) that determine the electron distribution are given by

$$A(z)=\frac{g\beta}{4\gamma}\frac{eE(z)}{m_e w^2(z)},\quad B(z)=\frac{\beta_0}{2}g\gamma\frac{e}{m_e}E(z)w^4(z),$$

$$C(z)=\frac{g\beta}{4\gamma w^2(z)},\quad \mathscr{D}(z)=\frac{1-\beta}{2}gw(z),\tag{26}$$

$$\mathscr{E}(z)=\frac{\beta g}{4\gamma w^2(z)}\frac{m_e}{eE(z)},$$

where $g = \omega_{Li}\omega_{Le}^{-1}(\mu_2/\mu_0)^{1/2}$.

The equation for the function $F(y)$ then follows directly from (17), and can be written in the form

$$\frac{1}{y^2}\frac{d}{dy}\left\{\left[\frac{2\beta_0\gamma^2}{y}+\beta y^5(1-y^2\,\text{sign}\,\alpha)^2\right]\frac{dF}{dy}+[8\gamma(1-\beta)y^3+2\beta(a+2)(y^6-y^8\,\text{sign}\,\alpha)]F\,\text{sign}\,\alpha\right\}+(a+4)F\times\{(a+8)\beta y^5-6\,\text{sign}\,\alpha[\beta y^3+\gamma(1-\beta)]\}=0.\tag{27}$$

Since the original equation (17) is valid for particles with velocities v smaller than the limiting maximum velocity $v_{\max}$, equation (27), which is derived from it, is meaningful only for $y < y_m = v_{\max}/w(z)$. To determine y_m, we use the condition that the anisotropy of the distribution function is small, i.e., $f_1 \ll f_0$. We therefore rewrite (6) for the case where the distribution function is given by (21). Recalling (15), (19), and (23)–(25), we have

$$\frac{f_1}{f_0}=\frac{g}{2\gamma}\left[\left(y\frac{d}{dy}\ln F-a\right)y^4\,\text{sign}\,\alpha-y^3\frac{d}{dy}\ln F\right]\times\int_{(1-\xi^2)^{1/2}}^{1}\frac{y'\,dy'}{(1-y'^2)^{1/2}\chi_2(y')}-g\frac{d}{dy}\ln F\int_{(1-\xi^2)^{1/2}}^{1}\frac{\chi_1(y')\,dy'}{(1-y'^2)^{1/2}\chi_2(y')}.\tag{28}$$

Since, according to (11) and (16), we have $\chi_2(y') \simeq y'^2$ as $y' \to 0$, the first integral with respect to y' in (28) diverges logarithmically as $\xi \to 1$. As in the situation established previously in the self-consistent theory of the electron distribution in plasmas with fully developed IAT,[7,8] this divergence can be removed by taking binary electron-electron collisions into account:

$$\lim_{\xi\to 1}\int_{(1-\xi^2)^{1/2}}^{1}\frac{y'\,dy'}{(1-y'^2)^{1/2}\chi_2(y')}\to\ln K_{st}\gg 1,$$

where K_{st} is the excess above the threshold for IAT.[6] Equating the expression given by (28) to unity, we obtain the equation that determines the value of y_m for which the ratio

f_1/f_0 is less than unity for arbitrary ξ. In particular, when $y_m > 1$, we have

$$y_m \approx \left[\frac{\gamma}{g \ln K_{st}}\right]^{1/4}.$$

In general, when the distribution given by (21) is valid, the excess K_{st} over the threshold may depend on position. However, since the dependence of y_m on K_{st} is slower than the logarithmic dependence, we shall assume throughout that y_m is approximately independent of position. Before we solve (27) on the interval $0 \leqslant y \leqslant y_m$, we make one further remark about the shape of the function F, which will enable us to restrict, and at the same time simplify, our discussion. Integrating the original equation (1), we obtain the following continuity equation for F:

$$\frac{\partial}{\partial t} n_e + \operatorname{div}\left(\int d\mathbf{v}\, \mathbf{v} f\right) = 0. \qquad (29)$$

It follows from this that the stationary distribution $\partial n_e/\partial t = 0$ exists provided the electron current density

$$\mathbf{j} = e \int d\mathbf{v}\, \mathbf{v} f \equiv e \int d\mathbf{v}\, \mathbf{v} f_1, \qquad (30)$$

is either generally zero ($\mathbf{j} = 0$) or, if $\mathbf{j} = \mathbf{j}_0 \neq 0$, it is independent of the position coordinate z. In current-free plasmas, the condition $\mathbf{j} = 0$ leads to an additional equation for the unknown parameter a:

$$a\mu_7 \operatorname{sign}\alpha = 6\mu_2 \frac{1-\beta}{\beta}\gamma + 6\mu_5 - 8\mu_7 \operatorname{sign}\alpha,$$

which, together with (25) and (27), can be used as a basis for studying the existence of the stationary self-similar distribution. However, from now on, we shall concentrate our attention on another special case of current-carrying plasma, which is of interest from the experimental point of view. Substituting (28) in (30), and using the explicit form of f_0 given by (21), we can readily verify that the integral (30) is independent of z for a current-carrying plasma ($\mathbf{j}_0 \neq 0$) only for $a = -4$. This value of a corresponds to $l = -1$ in (20). Consequently, the inhomogeneous stationary self-similar distribution in current-carrying plasma must be sought by putting $a = -4$ in (27). The order of (27) can then be reduced, and its positive solution that vanishes at $y = y_m$ is[2)]

$$F = S \int_y^{y_m} \frac{y''\, dy''}{2\beta_0\gamma^2 + \beta y''^6 (1 - y''^2 \operatorname{sign}\alpha)^2}$$

$$\times \exp\left\{\int_y^{y''} dy' \frac{8\gamma(1-\beta) y'^4 - 4\beta(y'^7 - y'^9 \operatorname{sign}\alpha)}{2\beta_0\gamma^2 + \beta y'^6 (1 - y'^2 \operatorname{sign}\alpha)^2}\right\}, \qquad (31)$$

where, without loss of generality, we can assume that the constant $S > 0$ is equal to unity, which is equivalent to a redefinition of the constant C in (21). The constant γ is then found from the equation that follows from the definition (25) if we substitute (31) into it.

Let us first consider the distribution F_+ described by (31) with sign $\alpha = 1$, so that the direction of the pressure gradient and the direction of the effective force density are the same. From (25) and (21), we then find by numerical integration that $\gamma_+ = 0.077$. The calculation of γ_+ was carried out for $y_m = 3$, 5, 7, and 9, and showed that, to within the precision with which the quantities β and β_0 were specified, the parameter γ_+ did not depend on the variation of y_m. Figure 1 shows an example of F_+ for $\gamma_+ = 0.077$. The function F_+ falls rapidly (by almost eleven orders of magnitude) between $y \sim 0.8$ and $y \sim 1.2$. Consequently, the integrated characteristics of the electron distribution are largely determined by particles with $y < 1$ and can be evaluated by using the following approximate expression instead of (31):

$$F_+ = \frac{S}{\beta} \int_y^{y_m} \frac{y''\, dy''}{0.02 + y''^6} \exp\left[\int_y^{y''} \frac{2.8 y'^4\, dy'}{0.02 + y'^6}\right].$$

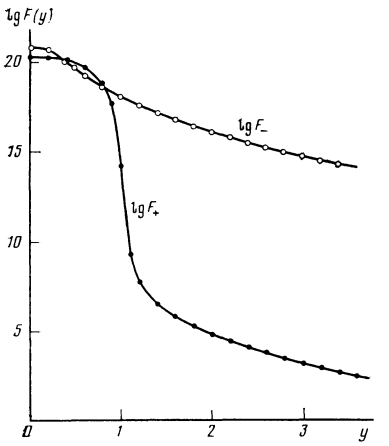

FIG. 1. Dependence of identically normalized F_+ and F_- on y.

Let us now consider the distribution F_- described by (31) with sign $\alpha = -1$. The resulting numerical value of γ_- depends on y_m, i.e., the upper limit of the range of validity of the distribution (31) (see Fig. 2). As y_m increases, there is a tendency for γ_- to reach the limiting value $\gamma_- = 0.024$. While all this is happening (see Fig. 1), the function F_- falls much more slowly than F_+, so that its higher moments depend significantly on y_m, i.e., they are determined by fast particles ($y > 1$). We note that, whatever, the sign of α, the functional γ is always positive. Moreover, as noted earlier [see (24)], the self-similar solution (21) exists when $eE(z)\gamma > 0$. Thus, since $e < 0$ and $\gamma > 0$, the solution defined by (21) and (31) exists when the electric field $\mathbf{E}(z)$ is antiparallel to the z axis.

4. ANOMALOUS TRANSPORT OF HEAT AND CHARGE. DISCUSSION

The above electron distribution can be used to find the electric current $\mathbf{j}$ and the electronic heat flux $\mathbf{q}$. Using (28),

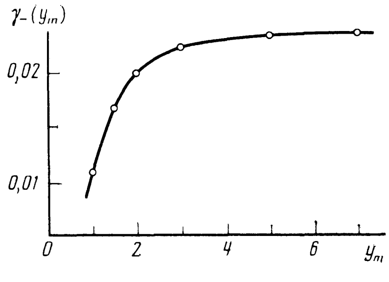

FIG. 2. Dependence of γ_- on y_m.

we obtain

$$j_z=2\pi e\int_{-1}^{1}d\xi\,\xi\int_0^{v_{max}}v^3f_1\,dv$$

$$=en_ev_s\left\{\frac{3}{2}(1-\beta)+\frac{3\beta\mu_5}{2\gamma\mu_2}-\frac{\beta\mu_7}{\gamma\mu_2}\operatorname{sign}\alpha\right\}\equiv Ien_ev_s, \qquad (32)$$

$$q_z=\pi m_e\int_{-1}^{1}d\xi\,\xi\int_0^{v_{max}}v^5\,dv\,f_1$$

$$=n_em_ev_sv_0^2\frac{\mu_0}{\mu_2^2}\left\{\frac{\beta}{\gamma}\mu_7+\frac{5}{4}(1-\beta)\mu_4-\frac{3\beta}{4\gamma}\mu_9\operatorname{sign}\alpha\right\}$$

$$\equiv Qm_en_ev_sv_0^2, \qquad (33)$$

where the electron density n_e and the velocities v_s and v_0 can be expressed in terms of $w(z)$ by (23) with $a = -4$. Formulas (32) and (33) take into account the fact that $F(y_m) = 0$ and $a = -4$. Since the electron distribution is known, the constants I and Q that determine the anomalous transport can be calculated. When sign $\alpha = 1$, we have $I_+ \approx 1.60$, $Q_+ \approx 3.07$. Like γ_+, the quantities I_+ and Q_+ are independent of y_m to the accuracy of the calculations. In the second case, when sign $\alpha = -1$, the strong dependence of the moments of the function F_- on y_m ensures that $I_-(y_m)$ and $Q_-(y_m)$ depend significantly on y_m. The functions $I_-(y_m)$ and $Q_-(y_m)$ are plotted against y_m in Figs. 3 and 4. It is clear that the limiting electron flux increases with increasing y_m, so that it is largely determined by fast particles with $y > 1$ (or $v > w$).

The fact that the function I is positive, whatever the sign of α, means that the electric current $\mathbf{j}$ is antiparallel to the z axis, since $e < 0$, i.e., it is parallel to the electric field $\mathbf{E}(z)$. In its turn, the fact that the function Q is positive for any sign α means that, whatever the direction of the pressure gradient, the electronic heat flux is parallel to the z axis, i.e., it points in the direction of the effective current density generating the instability.

We must now determine the constant $C' = CS$ in (21) and (31), which determines the electron distribution, and to interpret in greater detail the characteristic velocity w. Expressing C' in terms of the density n_e, and introducing the concept of the mean energy of random motion

$$\langle\varepsilon\rangle=\int dv\,m_ev^2f_0/3n_e=m_e\mu_4w^2/3\mu_2,$$

we obtain the following electron distribution:

$$f_0=\frac{n_e}{4\pi\mu_2}\left[\frac{m_e\mu_4}{3\mu_2\langle\varepsilon\rangle}\right]^{3/2}F\left[v\left(\frac{m_e\mu_4}{3\mu_2\langle\varepsilon\rangle}\right)^{1/2}\right]. \qquad (34)$$

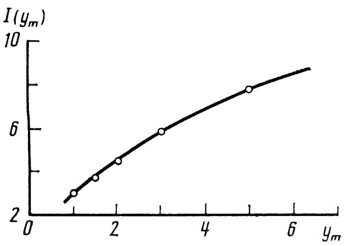

FIG. 3. Dependence of I_- on y_m.

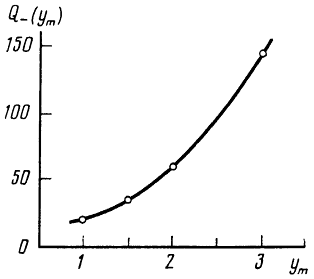

FIG. 4. Dependence of Q_- on y_m.

In accordance with the necessary condition for the existence of the inhomogeneous self-similar stationary distribution [see (19)], and using the relation $E(z) = -d\Phi(z)/dz$, we find that the mean energy $\langle\varepsilon\rangle$ is determined by the potential $\Phi(z)$ in the plasma:

$$\langle\varepsilon\rangle+\frac{2\mu_4}{3\mu_2}e\Phi(z)\operatorname{sign}\alpha=\text{const}. \qquad (35)$$

If we look upon $\Phi(z)$ as a given function, and demand that the current (32) be independent of position, then equation (35) provides a complete description of the variation of the functions n_e and $\langle\varepsilon\rangle$ in space and, hence, of the electron velocity distribution in the current-carrying plasma.

Several assumptions were made in the derivation of the self-similar distribution given by (34). First, we used the solution of the equation for the ion-acoustic waves for $K_N \ll 1$, which gives

$$\nu_0\ll\frac{U}{4(2\pi)^{1/2}}\omega_{Li}\frac{r_{De}^2}{r_{Di}^2}. \qquad (36)$$

Second, the electron-electron collision integral was ignored in the equation for f_0, which is possible if [7]

$$\frac{\beta_0}{2}\nu_0\frac{\omega_{Li}^2}{\omega_{Le}^2}\gg\nu_{ee}, \qquad (37)$$

where the electron-electron collision frequency is $\nu_{ee} \sim 4\pi e^4 n_e \Lambda/m_e^{1/2}\langle\varepsilon\rangle^{3/2}$, and Λ is the Coulomb logarithm. Inequality (37) shows that the collisional instability threshold has been substantially exceeded and that the effect of ion-ion collisions on the distribution of resonance ions with $v \sim v_s$ can be neglected.[9] The distribution of resonance ions is then characterized by a high effective temperature $\varkappa T_h \gg Z\langle\varepsilon\rangle$, where Z is the valence of the ion and $\varkappa$ is the Boltzmann constant. Third, when $N(k, x)$ was derived, we neglected the Cherenkov interaction between sound and ions, which means that, when $\varkappa T_h \gg Z\langle\varepsilon\rangle$, we must satisfy the inequality

$$\delta\approx\frac{n_h\omega_{Le}}{n_i\omega_{Li}}\left[\frac{Z\langle\varepsilon\rangle}{\varkappa T_h}\right]^{3/2}\ll 1, \qquad (38)$$

where n_h is the density of resonance ions. Since, in plasmas with fully developed IAT, $n_h \ll dn_i$ and $\varkappa T_h \gg Z\langle\varepsilon\rangle$, inequality (38) is satisfied relatively simply.

In deriving the equation for f_0, we used the solution $N(k,x)$ for the case where most thermal ions were described by the Maxwell distribution. This assumption is definitely valid, for example, for plasmas with frequent ion-ion collisions[9]

$$\frac{10}{3}\nu_0\frac{\omega_{Li}}{\omega_{Le}}\ln\frac{1}{K_N}\gg\nu_{ii}\approx Z\nu_{ee}\frac{\omega_{Li}r_{De}^3}{\omega_{Le}r_{Di}^3}\gg\frac{4\cdot 2^{1/2}U}{9\pi^{1/2}}\nu_0\frac{\omega_{Li}^2r_{De}^2}{\omega_{Le}^2r_{Di}^2},$$

if we can neglect the distortion of the ion distribution by stimulated scattering.[3] Moreover, under real conditions, there are factors (other than ion-ion collisions) which ensure that most of the ions are described by the Maxwell distribution. These factors include collisions of ions with nuclear particles.

Next, assuming that the ion distribution has the necessary properties, let us examine in greater detail the restrictions (36) and (37). In view of (19), (23), and (24), and using (32) to express $n_e(z)$ in terms of the given current density j_0, we can write (36) and (37) in the form

$$C_1(em_ej_0)^{1/2}\left(\frac{\omega_{Li}}{\omega_{Le}}\right)^{7/2}\frac{w^{11/2}}{v_{Ti}^2}\gg eE(z)w^3\gg C_2\frac{j_0e^3}{m_e}\frac{\omega_{Le}^2}{\omega_{Li}^2},\qquad(39)$$

where

$$C_1=[U/\gamma(8I)^{1/2}](\mu_2/\mu_0)^{11/4},\quad C_2=(8\pi\Lambda/\beta_0\gamma I)(3\mu_2^2/\mu_0\mu_4)^{3/2}.$$

In particular, for the distribution f_0 corresponding to sign $\alpha=1$, for which we have $F_+(0)/\mu_0\approx1.95$, $(\mu_0/\mu_2)^{1/2}\approx3.02$, $(3\mu_2/\mu_4)^{1/2}\approx3.48$, $\gamma_+\approx0.077$, $I_+\approx1.60$, $U_+\approx0.81$ independently of y_m, the constants are $C_1\simeq6.7\times10^{-3}$ and $C_2\simeq10^3\ \Lambda$. We shall assume, without loss of generality, that inequalities (39) are satisfied at $z=0$, so that, using (19), we can write $w(z)$ in the form

$$w^2(z)=w^2(0)+\frac{2}{m_e}\int_0^z dz'\,eE(z')\,\mathrm{sign}\,\alpha.\qquad(40)$$

Since $eE(z)>0$, the function $w(z)$ increases monotonically with increasing z for sign $\alpha=1$, but decreases monotonically for sign $\alpha=-1$. We shall illustrate the consequences of (39) and (40) by considering two examples. First, we take $E(z)=\mathrm{const}<0$ and assume the z dependence of v_{Ti} to be such that

$$r_{De}/r_{Di}=(\mu_2/\mu_0)^{1/2}\omega_{Li}w/\omega_{Le}v_{Ti}$$

so that w/v_{Ti} is constant. According to (39), $w(z)$ is then greater than $w_{\min}$:

$$w(z)\geqslant w_{min}=\max\left\{\left(\frac{C_2e^2j_0\omega_{Le}^2}{m_eE_0\omega_{Li}^2}\right)^{1/3},\ \frac{eE_0^2v_{Ti}^4\omega_{Le}^7}{C_1^2m_ej_0w^4\omega_{Li}^7}\right\}>0,$$

so that, according (40), $w(z)=w(0)[1+2eE_0z\,\mathrm{sign}\,\alpha/w^2(0)]^{1/2}$. If we compare $w(z)$ with $w_{\min}$, we see that, when sign $\alpha=1$, inequality (39) is satisfied for z between $z_{\min}<0$ and $\pm\infty$, where $z_{\min}$ is found from the condition $w_+(z_{\min})=w_{\min}$, whereas for sign $\alpha=-1$, the inequality is satisfied between z and $-\infty$ up to $z_{\max}>0$, where $z_{\max}$ is determined from $w_-(z_{\max})=w_{\min}$.

The other example involves the nonuniform distribution of the field $E(z)=E_m\cosh^{-2}(z/L)$, where $E_m<0$, L is the size on the inhomogeneity, and the potential $\Phi(z)=\Phi(0)-E_mL\,\mathrm{th}(z/L)$ can to some extent be realized experimentally.[12,13] Inequalities (39) then have the form

$$C_1'\left(1+\rho\,\mathrm{sign}\,\alpha\tanh\frac{z}{L}\right)^{1/4}\gg\cosh^{-2}\left(\frac{z}{L}\right)\gg C_2'\left(1+\rho\,\mathrm{sign}\,\alpha\tanh\frac{z}{L}\right)^{-1/2},$$

where

$$C_1'=C_1(j_0m_ew(0)/eE_m^2)^{1/2}(\omega_{Li}/\omega_{Le})^{7/2}(w/v_{Ti})^2,$$
$$C_2'=C_2e^2j_0\omega_{Le}^2/m_eE_mw^3(0)\omega_{Li}^2,\quad\rho=2eE_mL/m_ew^2(0)>0,$$

and, to be specific, the ratio w/v_{Ti} is assumed constant. Let us examine these inequalities in relation to a d.c. discharge in a finite gap of length $d>0$. Suppose that the "anode" is at $z=0$ and is held at $\Phi(0)>0$, and the "cathode" is at $z=-d$, so that the potential $\Phi(z)$ falls from $\Phi(0)$ to $\Phi(-d)$. The velocity $w(z)$ is then found to decrease for sign $\alpha=1$ and to increase for sign $\alpha=-1$ as z varies from 0 to $-d$, and the product $eE(z)$ decreases with increasing z. When $\rho\tanh(d/L)\ll1$, the function $w(z)$ is essentially constant over the length of the discharge gap, and $eE(z)$ decreases. The result is that, if $C_2'\cosh^2(d/L)>1$, the right-hand inequality in (39) breaks down at large distances from the "anode." On the other hand, when $C_2'\cosh^2(d/L)<1$, the above distribution f_0 is established over the entire length of the discharge gap. When $\rho\tanh(d/L)>1$, the function $w(z)$ increases for sign $\alpha=-1$, and the region in which f_0 (34) is established is wider than that for $\rho\tanh(d/L)\ll1$. On the contrary, when using $\alpha=1$ and $\rho\tanh(d/L)>1$, the velocity $w(z)$ decreases and may vanish with the interval $[-d,0]$, which violates the right-hand inequality in (39). The distribution f_0 (34) is then definitely *not* established near the "cathode." These examples demonstrate the extent to which the region in which this distribution is established in coordinate space depends on the structure of the field that generates the IAT.

Finally, let us consider the range of validity of the distribution defined by (31) and (34) in velocity space. Since (31) was obtained for $y\leqslant y_m$, the distribution (34) occurs for $v<v_{\max}=wy_m$. To be specific, let us suppose that $K_{st}=10^2-10^3$, so that when sign $\alpha=1$, we have $y_m\approx(1.1-1.2)(A/Z)^{1/8}$, where $Z=e_i|e|^{-1}$, e_i is the charge on the ion, and $A=m_i/m_p$ is the ratio of the mass of the ion m_i to the mass of the proton m_p. Hence, it would seem at first sight that, for realistic K_{st} and A/Z, the region in which the distribution (34) is established for sign $\alpha=1$ is relatively small. Actually, it corresponds to the velocity range $v\lesssim(4-7)\,(\langle\varepsilon\rangle/m_e)^{1/2}$. However, it is clear from Fig. 1 that, because of the rapid fall in F_+ near $y\sim1$, the electron currents established in this case are essentially independent of y_m (or $v_{\max}$). This corresponds to the situation where the region in which the self-similar distribution (34) is established covers practically the entire region occupied by the electron velocity distribution. In the opposite case, when $\alpha=-1$, the characteristic values y_m have the same order of magnitude. However, in this case, the values of γ_- and of the dimensionless electronic fluxes I_- and Q_- (see Figs. 2–4) depend significantly on y_m, which imposes more stringent conditions on the precision with which the range of validity of (31) must be defined.

APPENDIX

The electron distribution defined by (31) and (34) is established when the particle flux in velocity space is nonzero. In particular, the flux differs from zero near $v=0$ and leads to the outflow of particles from the low-velocity region. To appreciate the extent to which this outflow is significant from the establishment of the above electron distribution, let us determine the electron flux near the point $v = 0$. We shall do this by considering (17), retaining only the leading term [proportional to $B(z)$ for $v \to 0$] in the right-hand side:

$$\frac{\partial f_0}{\partial t} \approx \frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{B(z)}{v}\frac{\partial f_0}{\partial v}\right]. \tag{A1}$$

Integrating (A1) with the weight $4\pi v^2$, we find that the outflow of particles near $v = 0$ is characterized by the current density.

$$J=-4\pi \lim_{v\to 0}\frac{B(z)}{v}\frac{\partial f_0}{\partial v}. \tag{A2}$$

Next, substituting for v_0 from (31) and (34) with $S = 1$ in (A2), we find that

$$J=\frac{B(z)}{2\beta_0\gamma^2}\frac{n_e(z)}{\mu_2 w^5(z)}. \tag{A3}$$

Using (24) and (27), we can rewrite the last expression in the form

$$J=\frac{n_e(z)}{4\gamma\mu_0}\left[\frac{eE(z)}{m_e v_e}\frac{\omega_{Li}^2}{\omega_{Le}^2}\right]\equiv\frac{n_e(z)}{4\gamma\mu_0}\tau_e^{-1}, \tag{A4}$$

where τ_e is the characteristic time for turbulent heating of electrons. Substituting $\gamma \approx 0.077$ and $\mu_0 \approx 2.1 \cdot 10^{13}$ in (A4), which corresponds to the value of the integral (22) of the function (31) for $S = 1$, we finally have

$$J \approx 1.5 \cdot 10^{-13} n_e(z)\tau_e^{-1}. \tag{A5}$$

This formula leads to an anomalously long characteristic time for the depletion of the main mass of electrons, namely,

$$t_R \sim 6 \cdot 10^{12}\tau_e. \tag{A6}$$

Since, in typical experiments, $\tau_e \sim 10^{-5}$–10^{-7} s, the time given by (A6) exceeds by several orders of magnitude the duration of the experiments. We note that the large value of t_R as compared with τ_e is to an extent analogous to that found by Galeev and Sagdeev[5] for electron drag as compared with heating in homogeneous plasma with IAT.

[1)]The terms $(\partial\omega_s/\partial k)(\partial N/\partial z) - (\partial\omega_s/\partial z)(\partial N/\partial k)$ are omitted from (8). This is justified when the characteristic plasma scale L is relatively large $L \gg r_{De}(3\omega_{Le}/8v_0) r_{De}^2 r_{Di}^{-2} \ln K_N^{-1})^{-1}$.

[2)]Equation (27) also has a solution with $S = 0$, $F(y_m) \neq 0$. Analysis of this solution for the most interesting case sign $\alpha = 1$ shows that the corresponding values of γ and the electronic fluxes j_z (32) and q_z (33) are equal (to within the accuracy of the calculations) to the corresponding values for $S \neq 0$, and the function $F_+(y)$ has the form shown in Fig. 1. The reason for the similarity between solutions with $S = 0$ and $S \neq 0$ is the smallness of the electronic flux near $v = 0$ for $S \neq 0$ (see Appendix).

[3)]The left-hand inequality means that we can ignore the influence of ion-ion collisions on the IAT spectrum for $kr_{De} \sim 1$.

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