Dynamics of the formation and destruction of phase-slip centers in narrow superconducting channels

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Numerical methods are used to study the system of dynamic equations of superconductivity theory describing the resistive state of a narrow superconducting channel. The appearance of new phase-slip centers due to a gradual increase in the current is shown to result from the onset of a local instability in the distribution of the order parameter. The new phase-slip centers arise halfway between existing centers. The time scale for this process is estimated. The decay of a phase-slip center which occurs as the current is reduced stems from a global redistribution of the order-parameter profile due to the onset of a diffusion-drift instability. The decaying phase-slip center moves slowly toward its nearest neighbor and merges with this neighbor in the final stage of the process. The time scale for the onset of this instability is significantly greater than the time scale for the onset of the local instability during the appearance of new centers. The numerical calculations reveal hysteresis effects, which are evidence of the existence of several stable states, differing in the number of phase-slip centers, at a given value of the current. The role played by fluctuations is discussed on the basis of the time-dependent Ginzburg-Landau equations. When fluctuations are taken into account, the most probable event is the appearance of a structure with the smallest number of phase-slip centers possible at the given value of the current. This structure is at the boundary of the stability region for structures of this type (a marginally stable structure). This most probable structure is simultaneously the structure with the minimum dissipation.

INTRODUCTION

A resistive state may arise in a narrow superconducting channel, with transverse dimensions small in comparison with the coherence length ξ, at sufficiently high densities of a current flowing through the channel.[1,2] In this state, phase-slip centers arise in the superconducting channels. These centers are points at which the modulus of the complex order parameter (the wave function of the condensate), Δ, periodically vanishes, and its phase χ jumps by a multiple of 2π. In such a situation, a nonzero normal current j_n flows along with the superconducting current j_s in the sample, so a nonzero dissipation arises. For this reason, the distribution of the order parameter which arises in a superconductor in a resistive state is an example of the dissipative structures which have recently attracted active research interest in connection with various problems in microkinetics and macrokinetics (Ref. 3, for example).

The resistive state of superconductors has been the subject of several studies, which are discussed in some detail in Refs. 1 and 2. On the basis of the data in the literature the structure and dynamics of the development of an isolated phase-slip center can be regarded as completely understood. With regard to the dynamics of the development of a resistive state in a process governed by the interaction of different phase-slip centers with each other, in contrast, we have only some crude estimates and qualitative arguments, with rare exceptions.

For example, a study of the dynamic equations of superconductivity theory shows that at a given current density ($j = j_n + j_s = \mathrm{const}$) a periodic chain of phase-slip centers may arise in a sufficiently long sample. The spatial period of this chain, L, however, is not given unambiguously by the existing theory.[1)] It follows from experiments, in contrast, that the period of the chain has a completely definite value at a fixed value of j. A systematic theory of the resistive state must therefore explain just what determines the choice of L at a fixed j.

Analysis of the experimental data also leads to the assertion that as the current density is varied gradually there exist certain intervals of j, whose boundaries form an ordered sequence $j^{(1)} < j^{(2)} \ldots < j^{(M)}$, within which the number of phase-slip centers (i.e., the value of L) does not change as j is varied. When j crosses a boundary between intervals, however, there is either a change in L or a decay of the resistive state as such, in a process accompanied by a transition to a normal state (if $j > j^{M}$) or to a homogeneous superconducting state (if $j < j^{(1)}$). There are several questions to be answered here.

What is the time evolution of the resistive state as the current density crosses a bifurcation value? Is the sequence of bifurcation values $\{j^{(n)}\}$ which the system traces out with increasing j the same as the sequence which is traced out as j is reduced? In other words, can there be hysteresis effects in this system as a result of the existence of several stable states at a given value of j? What role is played by random fluctuations of the parameters in this problem?

Questions of this type are typical of the theory of dissipative structures in distributed systems. They arise in research on convection in liquids,[4,5] Couette flow,[6] and several other hydrodynamic flows.[7] They also arise in problems of the propagation of flames,[8] laser evaporation waves,[9] the crystallization of a supercooled melt,[10,11] etc. The specific dynamic system must be analyzed in order to solve the problem in each specific case, since there are no universal anwers to these questions.

In the present paper we discuss these problems in connection with the problem of the resistive state of superconductors. The paper is organized as follows: In Sec. 1 we state

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the problem and write the dynamic equations of superconductivity theory which are the basis of the rest of the study. Section 2 discusses the results of a numerical solution of this system of equations. In Sec. 3 we discuss the role played by fluctuations in the establishment of the most probable value L in a chain of phase-slip centers in the time-dependent Ginzburg-Landau equation approximation. The results of the study are summarized in the Conclusion.

1. DYNAMIC EQUATIONS

The behavior of a superconductor in an electric field is definitely time-dependent and must be described by dynamic equations. Unfortunately, the system of dynamic equations for superconductors is exceedingly complicated in the general case. In the present paper we use a comparatively simple system of equations, which can be extracted from the miroscopic theory in a certain narrow temperature interval near the critical temperature of the superconducting transition, T_c (Refs. 12 and 13). These equations are applicable only under the conditions $Dk^2, \omega \ll \tau_{\rm ph}^{-1}$, where D is the electron diffusion coefficient, $\tau_{\rm ph}$ is the inelastic electron-phonon relaxation time, and k and ω are typical wave vectors and frequencies of the problem. These conditions are equivalent to restrictions on the temperature difference $T_c - T$ (Ref. 19).

We introduce some dimensionless variables. We express distances in units of $\xi(T)$, times in units of $\tau_{\rm GL}$, order parameters in units of $\Delta_{\rm GL}$, and current densities in units of $\pi\sigma\Delta_{\rm GL}^2/4eT\xi$, where σ is the conductivity of the metal in its normal state, e is the electron charge, $\xi(T)$ is the coherence length, defined by

$$\xi(T)=[\pi D\hbar/8(T_c-T)]^{1/2}, \tag{1}$$

and the parameters $\tau_{\rm GL}$ and $\Delta_{\rm GL}$ are given by

$$\tau_{\rm GL}=\frac{2T\hbar}{\pi\Delta_{\rm GL}^2}, \quad \Delta_{\rm GL}=\left[\frac{8\pi^2T(T_c-T)}{7\zeta(3)}\right]^{1/2}, \tag{2}$$

where $\zeta(z)$ is the Riemann zeta function.

We also introduce some dimensionless gradient-invariant potentials: a scalar potential

$$\Phi=\varphi+\partial\chi/\partial t, \tag{3}$$

which is expressed in units of $\hbar/2e\tau_{\rm GL}$, and a vector potential

$$\mathbf{Q}=\mathbf{A}-\nabla\chi, \tag{4}$$

expressed in units of $\hbar c/2e\xi$, where φ and $\mathbf{A}$ are the dimensionless scalar and vector potentials of the electromagnetic field.

In terms of these variables, the dynamic equations of superconductivity theory are[12,13]

$$-u\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{1/2}\frac{\partial\Delta}{\partial t}+\nabla^2\Delta+(1-\Delta^2-\mathbf{Q}^2)\Delta=0, \tag{5}$$

$$u\Delta^2\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{-1/2}\Phi+\operatorname{div}(\Delta^2\mathbf{Q})=0, \tag{6}$$

$$\mathbf{j}=\mathbf{j}_n+\mathbf{j}_s, \tag{7}$$

$$\mathbf{j}_n=-\partial\mathbf{Q}/\partial t-\nabla\Phi, \quad \mathbf{j}_s=-\Delta^2\mathbf{Q}, \tag{8}$$

where we have introduced a depairing factor, Γ, defined by

$$\Gamma=\frac{\hbar}{2\tau_{ph}\Delta_{\rm GL}}=\frac{\pi}{8u^{1/2}}\frac{\hbar}{T_c\tau_{ph}}\left(\frac{T}{T_c-T}\right)^{1/2}, \tag{9}$$

and the numerical parameter $u=\pi^4/14\zeta(3)\approx 5.79$.

We will ignore the self-magnetic field of the current since the sample is so narrow; i.e., we will assume $\mathbf{Q}\approx -\nabla\chi$. In this case, system (5)–(8) is conveniently rewritten in the form[13]

$$-u\left(\frac{|\psi|^2}{\Gamma^2}+1\right)^{-1/2}\left[\frac{\partial\psi}{\partial t}+i\varphi\psi+\frac{1}{2\Gamma^2}\psi\frac{\partial|\psi|^2}{\partial t}\right] +\nabla^2\psi+\psi-|\psi|^2\psi=0, \tag{10}$$

$$\mathbf{j}_n=-\nabla\varphi, \quad \mathbf{j}_s=-{}^1\!/_2 i(\psi^*\nabla\psi-\psi\nabla\psi^*), \tag{11}$$

where the complex order parameter ψ is defined by

$$\psi=\Delta\exp(i\chi). \tag{12}$$

We also write the value of the critical Ginzburg-Landau current in the dimensionless units which we have adopted here:

$$j_c=2\cdot 3^{-3/2}\approx 0{,}385. \tag{13}$$

The typical values of $T_c\,\tau_{\rm ph}/\hbar$ for various superconductors lie in the interval 10–10^3. Accordingly, in the typical experimental situation the value of Γ satisfies the condition $\Gamma \ll 1$. Let us briefly review the results which have already been obtained in this limiting case and which we will need for the analysis below; we refer the reader interested in a more detailed discussion to Refs. 1 and 2.

It can be seen from (5) that under the condition $\Gamma\ll 1$ the time scale $\tau_{\rm GL}$ is joined by another time scale, τ_Δ, which is the relaxation time of the modulus of the order parameter, and which is related to $\tau_{\rm GL}$ by

$$\tau_\Delta\sim\tau_{\rm GL}u/\Gamma\gg\tau_{\rm GL}. \tag{14}$$

In addition, there is a length scale l_E in this problem, which represents the depth to which the electric field penetrates into the superconductor. In the case $\Gamma \ll 1$, this length scale can be estimated from

$$l_E\sim\xi(u\Gamma)^{-1/2}\gg\xi. \tag{15}$$

In the structure of an isolated phase-slip center, on the other hand, one can distinguish several nested regions in this case. The largest of these regions, with a size

$$l_{ns}\sim\xi(u\Gamma)^{-1/4}\gg\xi, \tag{16}$$

is the region in which the superconducting current j_s undergoes oscillations at the Josephson frequency,

$$\omega_J\sim 1/\tau_{\rm GL}(u\Gamma)^{1/2}\gg 1/\tau_{\rm GL}. \tag{17}$$

These oscillations are associated with a change in the phase of the order parameter, χ. The modulus of the order parameter, Δ, in contrast, is independent of the time and depends only weakly on position, remaining of order unity all the way to the boundary of the following region, whose size is on the order of ξ. In this region Δ is sharply suppressed to values $\Delta \ll 1$, and simultaneously temporal oscillations appear at the frequency ω_J. The amplitude of these oscillations, however, is small in comparison with the average value of Δ everywhere outside the immediate neighborhood of the core of the phase-slip center, whose size is

$$l_c\sim\xi(u\Gamma)^{1/2}\ll\xi. \tag{18}$$

Inside this core, the amplitude of the oscillations in Δ reaches a value of order Δ. At the center of the core there is a point (the phase-slip center proper) at which Δ periodically vanishes; at such times the phase χ undergoes a jump of 2π.

The range of applicability of these results is defined by the inequalities

$$\left(\frac{\hbar}{T_c\tau_{ph}}\right)^2 \ll \frac{T_c-T}{T_c} \ll \left(\frac{\hbar}{T_c\tau_{ph}}\right)^{6/5}. \qquad (19)$$

The inequality on the left follows from the condition $u\Gamma \ll 1$, and that at the right from the condition $\omega_J\,\tau_{\rm ph} \ll 1$, which limits the range of applicability of our original system of equations (5)–(8).

In studying the problem, we integrated equations (7), (10), and (11) numerically over a segment of length $\mathscr{L}/2$ with cyclic boundary conditions

$$\frac{\partial|\psi|^2}{\partial x}=0, \qquad \varphi+\frac{\partial}{\partial t}(\arg\psi)=0 \qquad (20)$$

at $x=0$ and $x=\mathscr{L}/2$. The origin of coordinates for the scalar potential φ is the point $x=0$; i.e., we assume

$$\varphi(0,\ t)=0. \qquad (21)$$

Boundary conditions of this sort lead to spatially periodic distributions $\Delta(x,t)$, $\Phi(x,t)$, $\mathbf{Q}(x,t)$, $\mathbf{j}_n(x,t)$ and $\mathbf{j}_s(x,t)$ with a period $\mathscr{L}$.

The numerical value of the depairing factor Γ is taken to be 0.1, so we have $u\Gamma=0.579$. In this case, ω_J is on the order of $\tau_{\rm GL}$, and we have $l_c \sim l_{ns} \sim l_E \sim \xi$ [see (15)–(18)]. At the same time, we have $\tau_\Delta \sim 60\tau_{\rm GL}$, and a definite hierarchy of time scales is preserved in the problem, if there is no hierarchy of length scales. It thus becomes possible to find a chain of several phase-slip centers if the computation interval is not too long. The effect is to substantially reduce the expenditure of computer time in the numerical calculations.

Calculations were carried out under the condition $j=$ const. As the initial condition we specified the value of this constant and some distribution of the complex order parameter $\psi(x,0)$. We then used equations (7) and (11) to determine the corresponding distribution of the scalar potential, $\varphi(x,0)$. With these initial conditions, we integrated the problem until all the transients died out, and definite steady-state asymptotic distributions $\psi(x,t)$ and $\varphi(x,t)$ were established. When a resistive state arose, these distributions corresponded to a stable limit cycle in the corresponding functional space. We then varied the value of j. As the new initial condition on ψ, we adopted the asymptotic distribution found at the previous value of j. We then repeated the process.

For the calculations we used a Crank-Nicholson difference scheme[14] of second-order accuracy in the time and in the spatial variable. The nonlinear terms in Eqs. (10) and (11) were approximated in such a way that the difference analogs of the conservation laws for Δ were satisfied. To keep the algorithms for calculating Re ψ and Im ψ equivalent, we solved the system of difference equations which arose by the matrix tridiagonal inversion method. The ends of the computation interval were kept equivalent by alternating runs in different directions.

To conclude this section of the paper we would like to point out that a numerical integration of equations (7), (10), and (11) or of an equivalent system of equations has also been carried out previously.[13,15,16] However, those earlier studies focused on an isolated phase-slip center, i.e., on verifying the very existence of phase-slip centers and thus the validity of the overall picture of a resistive state. We would also like to mention a recent paper by Butler and Hsiang[17] who used numerical methods to study the response of a superconductor to a current pulse above the critical value. They also noted that when the superconducting channel is long the change in the distribution of the order parameter is a slow process, with a time scale considerably greater than $\tau_{\rm GL}$ (cf. the results of Sec. 2 of the present paper). In the present paper we are interested in the dynamic behavior of a chain of phase-slip centers, which is why the present paper differs from Refs. 13 and 15–17.

2. DYNAMICS OF THE FORMATION AND DECAY OF PHASE-SLIP CENTERS

System of equations (7), (10), and (11) with boundary conditions (20), (21) was integrated numerically over the segment $\mathscr{L}/2=4\pi$ at the following values of j [the top row ($N_\rightarrow$) corresponds to the situation which arises as j is increased from 0.240 to 0.500, while the bottom row ($N_\leftarrow$) corresponds to the situation which arises as j is reduced from 0.470 to 0.240]:

$N_\rightarrow$	0	1	1	1	1	2	4	4	4	0
j	0.240	0.270	0.300	0.330	0.370	0.395	0.420	0.440	0.470	0.500
$N_\leftarrow$	0	2	2	2	2	4	4	4	4	—

The calculations show that when a uniform initial condition is chosen ($\Delta=$ const, $\nabla\chi=$ const), no phase-slip centers form, regardless of the value of j. In such a case, with $j<j_c$ ($j_c=2\times3^{-3/2}\approx0.385$), the system evolves toward a uniform superconducting state, while for $j>j_c$ there is a transition to a normal state.

The only way to obtain phase-slip centers was to specify a nonuniform initial distribution of the order parameter (nonuniform along x). For this distribution we selected

$$\psi(x,0)=\left(\frac{2x}{\mathscr{L}}\right)^{1/2}\exp\left(4\pi i\frac{x}{\mathscr{L}}\right), \qquad 0\leqslant x\leqslant\frac{\mathscr{L}}{2}. \qquad (22)$$

Even with this initial condition, however, phase-slip centers form only if $j\geqslant0.270$ (Fig. 1). At $j=0.240$, the initial distribution (22), like a uniform distribution, evolves toward an equilibrium superconducting state.

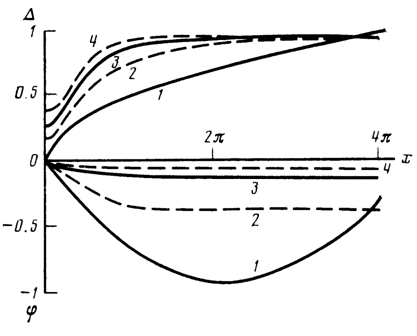

FIG. 1. Profiles of absolute value of the order parameter, Δ, and of the scalar potential of the electromagnetic field, φ at various time ($\Gamma=0.1$; $\mathscr{L}=8\pi$; $j=0.270$). Formation of phase-slip centers from initial conditions (22): 1—$t=0$; 2—$t=60$; 3—$t=120$; 4—$t=240$.

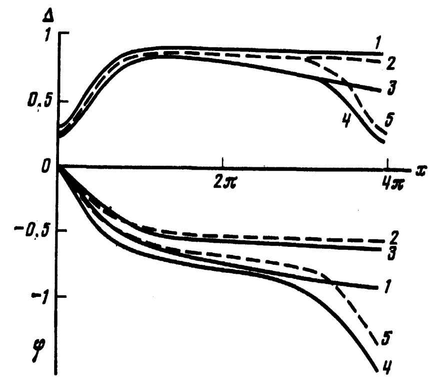

FIG. 2. Appearance of a new phase-slip center at the transition from $j = 0.370$ to $j = 0.395$. 1—$t = 0$; 2—$t = 120$; 3—$t = 360$; 4—$t = 420$; 5—$t = 660$ ($\Gamma = 0.1$; $\mathscr{L} = 8\pi$).

The formation of new phase-slip centers on the segment $\mathscr{L}$ with increasing j is illustrated by Fig. 2 (the transition from $N = 1$ to $N = 2$) and Fig. 3 ($N = 2 \to N = 4$). We see that the new phase-slip centers form in the same way in the two cases. Over a time on the order of hundreds of units after the application of a current pulse exceeding the corresponding bifurcation value, a dip forms halfway between two neighboring phase-slip centers, i.e., at the point at which the superconducting current j_s reaches its maximum value $j_{\rm sm}$, on the $\Delta(x)$ profile. Later on, after a time on the order of a few units, the value of Δ at this point decreases sharply to values much smaller than unity and becomes oscillatory. This event is the termination of the process by which a new phase-slip center forms.

Analysis of the results of the numerical calculation shows that the value of $j_{\rm sm}$ near bifurcation values of j is close to j_c ($j_{\rm sm} = 0.370$ at $j = 0.370$ and $j_{\rm sm} = 0.381$ at $j = 0.395$). This observation is of assistance in reaching an understanding of the reason for the occurrence of this instability. Specifically, in the steady resistive state the value of j_s outside the immediate neighborhood of a phase-slip center is essentially independent of the time. The existence of a steady-state distribution of the superconducting current, however, is possible only under the condition $j_{\rm sm} < j_c$ (the condition of local stability). Since $j_{\rm sm}$ increases with increasing j if the structure has a fixed period L (Refs. 1 and 2), it is clear that for each L there must exist a definite critical value $j^{(n)}$ at which the relation $j_{\rm sm}(j^{(n)}) = j_c$ holds. If j exceeds $j^{(n)}$, an instability will occur, and it will occur at the point at which the local stability of the structure was disrupted, i.e., halfway between two phase-slip centers.

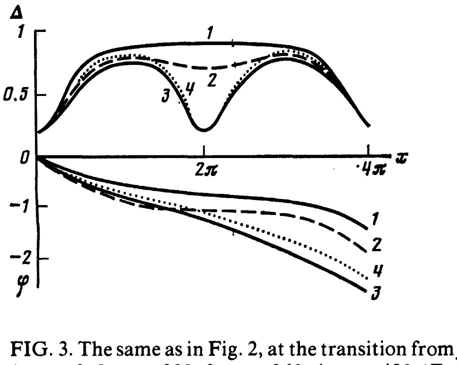

FIG. 3. The same as in Fig. 2, at the transition from $j = 0.395$ to $j = 420$. 1—$t = 0$; 2—$t = 300$; 3—$t = 360$; 4—$t = 420$ ($\Gamma = 0.1$; $\mathscr{L} = 8\pi$).

Using these arguments and the known solution describing the distribution of the order parameter in the static region,[1,2] we can construct a simple approximate expression for the bifurcation values $j^{(n)}$, which correspond to the appearance of new phase-slip centers in a periodic chain as j is increased. Setting $j_{sm} = j_c$ in this solution, we find that the period of the chain at the time of the local instability must be related to $j^{(n)}$ by[18]

$$L = \frac{2}{u\Gamma}\int_{\sqrt{2/3}}^{1} \frac{3\Delta^2-2}{\Phi(\Delta)(1-\Delta^2)^{1/2}}\, d\Delta, \tag{23}$$

where

$$\Phi(\Delta) = \left\{\frac{2}{u\Gamma}\int_{\sqrt{2/3}}^{\Delta} \frac{3\Delta'^2-2}{(1-\Delta'^2)^{1/2}}\left[j^{(n)} - \Delta'^2(1-\Delta'^2)^{1/2}\right] d\Delta'\right\}^{1/2}. \tag{24}$$

The value of Δ in the integral (24) is at most no greater than unity; i.e., we have $\Delta - \sqrt{2/3} \leqslant 0.22\sqrt{2/3}$. It is thus convenient to expand (24) in a power series in $\Delta - \sqrt{2/3}$. Restricting the analysis to the first nonvanishing terms of this series, and substituting the functional dependence $\Phi(\Delta)$ calculated with the help of this expansion into (23), we find

$$j^{(n)} - j_c \approx \frac{\sqrt{2}}{u\Gamma L^2}\left\{1 + \sqrt{2}\left[\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{2}{3}}\right)\right]\right\}^2 \approx \frac{4.9}{u\Gamma L^2}. \tag{25}$$

It follows from (25) that we have $j^{(n)} > j_c$ at any value of L. In other words, bifurcations leading to the appearance of new phase-slip centers in the system are possible only at values of the current greater than the Ginzburg-Landau critical current. We wish to stress that we are talking here exclusively about the bifurcations which occur in a chain of phase-slip centers as the current density is increased gradually—not the possible existence of the chain itself, which may be stable even under the condition $j < j_c$ (Table I). Substituting $u\Gamma = 0.579$ into (25), and taking $L = 8\pi$, we find $j^{(1)} = 0.40$; taking $L = 4\pi$ we find $j^{(2)} = 0.44$, in good agreement with the results of the numerical calculations.

The time scale for a change in the profile Δ during the onset of an instability of this sort must be on the order of a few times τ_Δ, i.e., on the order of $\tau_{\rm GL}$ in this case—again in agreement with the results of the numerical solution of the problem. This time is actually the retardation time which is observable when the superconductivity of narrow channels is destroyed by a strong current pulse.[19] As can be seen from Figs. 2 and 3, the same time scale determines that change in the potential difference across the ends of the sample, which is related to the appearance of new phase-slip centers.

This picture of the loss of stability suggests that in an ideal periodic chain of phase-slip centers each bifurcation which occurs as the current is increased gradually (in each step, $j \to j + \delta j$, where $\delta j \ll j$) leads to a halving of the period of the structure; i.e., $L \to L/2$, $N \to 2N$. The total number of phase-slip centers after the nth bifurcation is $N_n = 2^n N_0$, where N_0 is the initial number of centers.

This process of doubling of the number of centers comes to a halt when the period of the structure becomes comparable to the size of an individual center (tentatively, at $L \sim \xi$), so the region of a steady-state distribution of the order parameter essentially disappears. In such a situation, there is a collapse to a normal state, just as we see in the numerical calculations as $j = 0.500$ (Fig. 4).

In those cases in which the chain of centers is not strictly periodic, because of (for example) a nonuniform distribution of defects, which perturb the chain in a random fashion, the bifurcations with a doubling of the number of centers may be replaced by $N \to N + 1$ transitions. In this situation, the creation of a new center would occur between those two neighboring centers which are farthest apart. A disruption of this sequence may also occur when there are large jumps in the current density, when a local loss of stability occurs simultaneously over a large part of a spatial period of the structure.

Let us examine the bifurcations which occur during a gradual decrease in the current (in each step, $j \to j - \delta j$, where $\delta j \ll j$) and which lead to a destruction of the phase-slip centers. Figure 5 shows an example of such a process: the destruction of a phase-slip center as j is reduced from 0.395 to 0.370 [in view of the parity of the distribution $\Delta(x,t)$ with respect to the point $x = 0$, we will discuss the behavior of Δ only at $0 \leqslant x \leqslant \mathscr{L}/2$, i.e., only over half of the overall spatial period of the structure]. We see that the destruction of the phase-slip center results from a global change in the profile of the order parameter over distances on the order of $2L$, where L is the period of the original structure. The overall process can be broken up somewhat arbitrarily into three stages.

First comes the latent-change stage ($0 \leqslant t \leqslant 200$), during which there is no significant redistribution of the order parameter.

Second is the diffusion-drift stage ($2000 \leqslant t \leqslant 3500$), in which the central phase-slip center moves slowly toward the edge of the computation interval, to a distance on the order of ξ.

Third is the coalescence stage ($3500 \leqslant t \leqslant 3750$), during which two phase-slip centers separated by a distance on the order of ξ merge with each other. The region between the two merging centers goes to the normal state ($\Delta = 0$). The size of this region then decreases, and the region converts into an ordinary phase-slip center. The process by which the central center is destroyed is completed here, and the system reaches a new stable state with fewer phase-slip centers.

We thus see that, in contrast with the appearance of new phase-slip centers as j is increased, the decay of the centers which occurs as the current is reduced is a very slow process, with a time scale considerably longer than both τ_{GL} and τ_D. It is easy to understand that in the latent-change stage some

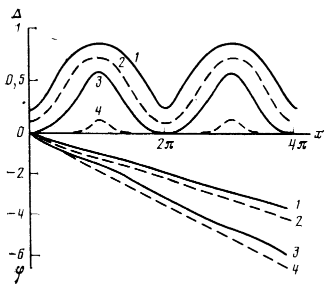

FIG. 4. Collapse to a normal state at the transition from $j = 0.470$ to $j = 0.500$. 1—$t = 0$; 2—$t = 240$; 3—$t = 480$; 4—$t = 600$ ($\Gamma = 0.1$; $\mathscr{L} = 8\pi$).

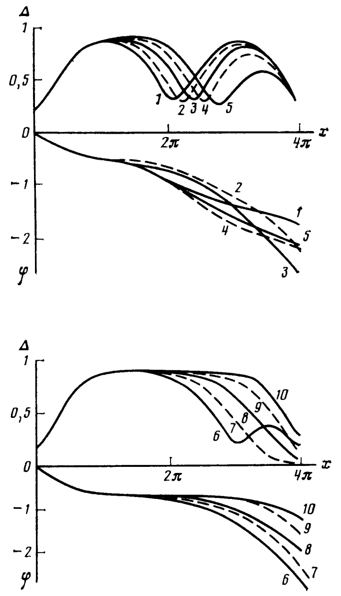

FIG. 5. Destruction of a phase-slip center at the transition from $j = 0.395$ to $j = 370$. The curves are drawn for times corresponding to local maxima of the dependence $\Delta(t)$ for the central phase-slip center. 1—$t = 0$; 2—$t = 2040$; 3—$t = 2640$; 4—$t = 3180$; 5—$t = 3480$; 6—$t = 3600$; 7—$t = 3660$; 8—$t = 3720$; 9—$t = 3780$; 10—$t = 4020$ ($\Gamma = 0.1$; $\mathscr{L} = 8\pi$).

initial perturbations which result from the numerical noise grow to an amplitude of order unity. The duration of this process is determined not only by the small values of the corresponding growth rates but also by the small value of the initial amplitude of the noise. In other words, the duration of this process depends on the initial conditions and is therefore not a characteristic parameter of the problem.[2)]

With regard to the stage of the diffusion drift, we note that its duration, τ_D, can be estimated easily when we note that, as follows from Eq. (5), the time scales and length scales which describe the changes in the profile $\Delta(x,t)$ should be related in this case by

$$\tau \sim \left(\frac{l}{\xi}\right)^2 \tau_\Delta \sim \frac{u}{\Gamma}\left(\frac{l}{\xi}\right)^2 \tau_{GL}. \tag{26}$$

For the situation shown in Fig. 5, the typical value of l is $\pi\xi$, which leads us to the estimate $\tau_D \sim 10^3 \tau_{GL}$, which agrees in turn with the numerical calculations. In contrast, the duration of the coalescence stage, which is not associated with a redistribution of the order parameter over large distances, can be estimated to have the value τ_Δ.

The validity of these arguments is confirmed by Fig. 6, which shows the formation of a chain of phase-slip centers from the initial condition (22) for the case $\mathscr{L}/2 = 8\pi$ (as for Fig. 5, we will discuss here only the part of the structure which corresponds to values of x in the interval $0 \leqslant x \leqslant \mathscr{L}/2$). In this case the initial distribution of the order parameter is not small, and there is no latent-change stage. At the initial time, the Δ relief shifts rapidly to the right (over a time $t \approx 120$), forming a region at $0 \leqslant x \leqslant 4$ which has converted into a normal state. This process amounts to fast relaxation of a highly nonequilibrium initial distribution $\Delta(x,0)$. A relaxation of this sort occurs immediately over the entire length of the computational interval, so the duration of this process should be on the order of τ_Δ, regardless of the value of $\mathscr{L}$.

The next event is the beginning of the formation of a dip,

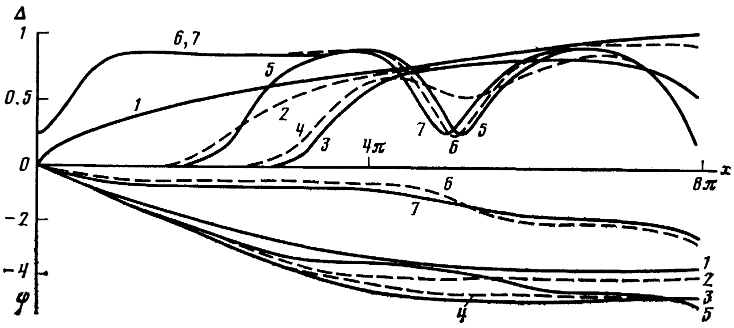

FIG. 6. Formation of a periodic chain of phase-slip centers from initial condition (22). 1—$t = 0$; 2—$t = 40$; 3—$t = 160$; 4—$t = 340$; 5—$t = 600$; 6—$t = 1500$; 7—$t = 3500$ ($\Gamma = 0.1$; $\mathscr{L} = 16\pi$; $j = 0.395$).

over a time order of τ_D, at the upper part of the Δ profile at $x = 16$. This dip develops into a phase-slip center. At the same time, another center appears at the right end of the segment, at $x = 8\pi$, and the left-hand front of the Δ profile begins to move to the left. The formation of the phase-slip center at $x = 0$ terminates by $t \approx 1200$. By this time, the central phase-slip center has already acquired the structure typical of a phase-slip center, but its position is still not the middle of the computation interval. The last stage of the establishment process is accordingly a slow diffusion-drift motion of the central phase-slip center toward the point $x = 4\pi$. We were not able to pursue this process to its completion because of the large amount of computer time required, but the corresponding tendency (toward the establishment of a chain of phase-slip centers with a period $L = 4\pi$) is obvious. An estimate based on (26) with $l = 4\pi\xi$ puts the duration of this establishment process at $\tau_D \sim 9000\tau_{\rm GL}$. As well as we can judge from an extrapolation of the results in Fig. 6, this estimate agrees with the data of the numerical calculations.

As j is reduced further, below $j = 0.370$, the state with two phase-slip centers persists to $j = 0.270$ (Table I), going over a uniform superconducting state at $0.240 \leqslant j < 0.270$ (Fig. 7). The apparent explanation for this pronounced hysteresis is the circumstance that the positions of the phase-slip centers at $x = 0$ and $x = \mathscr{L}/2$ are fixed by virtue of the boundary conditions, so a diffusion-drift instability cannot occur. In this case, the destruction of a phase-slip center with decreasing j may result from the onset of an instability of a different nature, due to oscillations in j_s. In this example, the amplitude of these oscillations is fairly high, since the product $u\Gamma$ cannot, strictly speaking, be regarded as a small quantity. If this is indeed the case, then this mechanism should disappear in the limit $u\Gamma \to 0$.

We would like to conclude this section of the paper by pointing out that our numerical calculations show that the various phase-slip centers in a chain are not equivalent to each other, even after a stable asymptotic regime has been completely established in the system. The differences among the centers are extremely slight, but they exist. In particular, each center has its own frequency for the Josephson oscillations of Δ. For example, in the case in which there are two phase-slip centers over a segment of length $\mathscr{L} = 8\pi$, with a current $j = 0.370$, the period of the Josephson oscillations of each center is close to $5.6\tau_{\rm GL}$. In this case there is a phase shift between the oscillations in Δ at the left edge (i.e., at $x = 0$) and those at the right edge ($x = \mathscr{L}/2$) of the computational interval. This phase shift disappears approximately every 600 time units. Since the oscillations in Δ at each phase-slip center lead to corresponding oscillations in the voltage across the sample, the resultant effect of two phase-slip centers with slightly different values of ω_J gives rise to beats at the difference frequency. In other words, the voltage oscillations acquire a low-frequency modulation. In the example which we have been discussing, the frequency of these beats is on the order of $10^{-2}\omega_J$. It is possible that these beats are associated with the low-frequency oscillations which are observed[20] in the resistive state, and whose origin has yet to be explained.

3. MOST PROBABLE VALUE OF THE PERIOD OF A CHAIN OF PHASE-SLIP CENTERS IN THE PRESENCE OF FLUCTUATIONS

The hysteresis effects observed in the numerical analysis of the dynamic equations (7), (10), (11) show that at a given value of j a system may have not just one but several different solutions which are stable with respect to small perturbations (in an infinitely long sample, there may be a continuous family of such solutions). These solutions describe periodic chains of phase-slip centers which differ in the value of the spatial period. In such a case, transitions between different stable structures may be caused by perturbations of finite amplitude of the nature of first-order transitions. The amplitude of such a perturbation (the height of the barrier) decreases as we approach the stability boundary of the original structure. When this boundary is reached, a transition to the new state occurs when there is any arbitrarily small perturbation (without a barrier). Barrier-free transitions of precisely this type have been observed in the numerical calculations.[2)]

The role played by fluctuations in the initiation of such

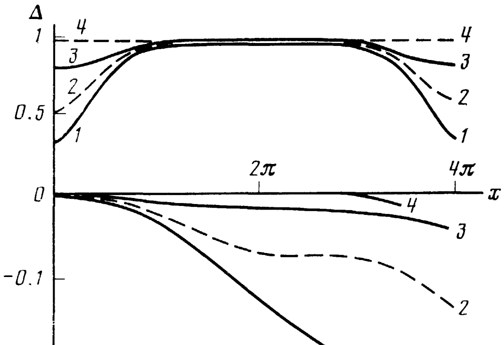

FIG. 7. Collapse to a uniform superconducting state at the transition from $j = 0.270$ to $j = 0.240$. 1—$t = 0$; 2—$t = 30$; 3—$t = 60$; 4—$t = 120$ ($\Gamma = 0.1$; $\mathscr{L} = 8\pi$).

transitions and the existence of a certain most probable state of the dynamic system become particularly important questions in this situation. These questions are the subject of the present section of this paper. We will describe the behavior of the system not by means of the exact microscopic equations, (5)–(8), but by means of the model equations which are found from (5)–(8) through the substitution $u(\Delta^2/\Gamma^2+1)^{\pm 1/2}\to\gamma=\text{const}$, where the upper sign in the exponent corresponds to Eq. (5), and the lower to Eq. (6). It is easy to see that in a gapless situation ($\Gamma\gg 1$) model equations of this type are the same as equations (5)–(8) (with $\gamma=u$). We will not assume this equality, however; we treat γ as a parameter of the problem, which can take on any positive value. A similar approach was taken to the description of the resistive state of a superconductor in Refs. 15 and 16. It was shown that these model equations are, at a qualitative level, completely equivalent to the system (5)–(8). In particular, the entire hierarchy of time scales and length scales is preserved, and for these scales relations (15)–(18) continue to hold, with the formal substitution $u\Gamma\to\gamma$.

It follows from the results of Ref. 16 and from numerical analysis that all the time scales of the problem which are associated with the onset of various instabilities in the chain of phase-slip centers are long in comparison with ω_J^{-1}. For this reason, it is natural to transform to a "slow" time, by taking an average of the dynamic equations over the fast Josephson oscillations. Since the oscillation amplitude is small everywhere outside the immediate vicinity of a phase-slip center, this averaging procedure does not alter the dynamic equations. It reduces to the formal replacement of the oscillating quantities by their average values.

In taking an average of this sort we are actually adopting the approximation of point phase-slip centers, at which l_{ns} and all of the smaller length scales associated with the structure of the phase-slip center are assumed to be zero. It should be assumed here that the conditions[1,2]

$$\Delta=1,\quad j_s=0,\quad Q=0 \tag{27}$$

must hold at the point at which a phase-slip center is localized, and it should be assumed that the potential Φ is discontinuous at this point.

Since we are dealing exclusively with average quantities everywhere in this section of the paper, we have retained for them the same notation we were using earlier for unaveraged variables. Later on, it will be convenient to transform to a gauge for the potentials of the electromagnetic field in which we have φ, so we have

$$\Phi=\partial\chi/\partial t,\quad Q=A-\nabla\chi. \tag{28}$$

This transformation can be made since the problem involves no space charge.

It is not difficult to see that in this case, when we make use of electrical neutrality, which gives us

$$\operatorname{div}\mathbf{j}=0, \tag{29}$$

we can write the dynamic equations as

$$\eta_i\frac{\partial q_i}{\partial t}+\frac{\delta V}{\delta q_i}=0, \tag{30}$$

where $i=1, 2, 3$; $q_i(x,t)=\{\Delta,\ \chi,\ A\}$; $\eta_i=\{\eta_\Delta,\ \eta_\chi,\ \eta_A\}=\{\gamma,\ \Delta^2\gamma,\ 1\}$ and the "potential" V is given by

$$V\{q_i\}=\int dx\left\{-\frac{\Delta^2}{2}+\frac{\Delta^4}{4}+\frac{1}{2}(\nabla\Delta)^2+\frac{1}{2}\Delta^2(\mathbf{A}-\nabla\chi)^2+\mathbf{j}(\mathbf{A}-\nabla\chi)\right\} \tag{31}$$

and represents the free energy of the superconductor at a given total current j.

We now incorporate fluctuations in the value of Δ, χ, and $\mathbf{A}$, putting equations (30) in the form of Langevin equations, i.e., adding random "forces" (a noise) $\zeta_i(x,t)$ to the right side. These random forces satisfy the correlation relation

$$\langle\zeta_i(x,\ t)\zeta_k(x',\ t')\rangle=(2\eta_i/\beta)\delta_{ik}\delta(x-x')\delta(t-t'), \tag{32}$$

where β^{-1} is a dimensionless temperature of the noise.

A generalization of this sort can be justified at the microscopic level by means of (for example) the model of a heat reservoir consisting of a system of equilibrium oscillators.[21]

If we now introduce a functional $\rho\{q_i\}$, which describes the probability density of the states of this dynamic system, then we find a Fokker-Planck equation for $\rho\{q_i\}$ from Eqs. (30)–(32):

$$\frac{\partial\rho}{\partial t}=\int dx\sum_i\frac{\delta}{\delta q_i}\left\{\frac{1}{\eta_i}\left[\frac{\delta V}{\delta q_i}\rho+\frac{1}{\beta}\frac{\delta\rho}{\delta q_i}\right]\right\}. \tag{33}$$

We know quite well that the right side of this equation vanishes identically when we substitute into it a Gibbs distribution

$$\rho\{q_i\}=C\exp(-\beta V\{q_i\}), \tag{34}$$

which gives the probability for the system to be in a state characterized by any time-independent set of coordinates $\{q_i(x)\}$.

In our case, the situation is complicated by the circumstance that in a state corresponding to a steady-state periodic chain of phase-slip centers the variables χ and $\mathbf{A}$ depend on the time explicitly. To see this, we note that after an average is taken over the Josephson oscillations the gradient-invariant potentials Φ and $\mathbf{Q}$ are time-independent. It follows immediately that we have

$$\chi=\Phi t,\quad \mathbf{A}=\mathbf{Q}+\nabla\chi=\mathbf{Q}+t\nabla\Phi \tag{35}$$

[see (28)]. In this case we have $\Phi\neq 0$, since in the resistive state there is an electric field $\mathbf{E}$ in the superconductor, which is related to Φ by $\mathbf{E}=-\nabla\Phi$ in the case $\partial\mathbf{Q}/\partial t=0$.

However, the quantities $\mathbf{A}$ and χ enter the expression for V only in the combination $\mathbf{A}-\nabla\chi=\mathbf{Q}$. The quantity $\mathbf{Q}$, on the other hand, does not depend on the time. As a result, when we substitute a solution corresponding to a steady-state chain of phase-slip centers into (34), we find that the time-dependent terms cancel out, and the functional $\rho\{q_i\}$ turns out to be independent of t and thus satisfies Eq. (33). It follows that the most probable state of a chain of phase-slip centers is determined by the absolute minimum of the functional $V\{q_i\}$.

By virtue of the conditions $\partial\mathbf{A}/\partial t\neq 0$ and $\partial\chi/\partial t\neq 0$, the solutions of Eqs. (30) which correspond to resistive states are extrema of the functional V, since an extremal path would have to satisfy the conditions $\delta V/\delta q_i=0$. Nevertheless, a periodic chain of phase-slip centers turns out to be stable with respect to small perturbations because of the stabilizing effect of the external forces which keep j constant, despite the presence of dissipative processes in the system.

How does the quantity $V\{q_i\}$ depend on the chain period L? To answer this question, we write $V\{q_i\}$ in the form

$$V\{q_i\}=F\{\Delta,\ Q\}+\int jQ\,dx, \tag{36}$$

where $Q=A-\partial\chi/\partial x$; the x axis is oriented parallel to the vectors **A**, **Q**, and **j**; and

$$F\{\Delta, Q\}=\int dx\Big[-\frac{\Delta^2}{2}+\frac{\Delta^4}{4}+\frac{1}{2}\Big(\frac{\partial\Delta}{\partial x}\Big)^2+\frac{\Delta^2Q^2}{2}\Big]. \qquad (37)$$

The steady resistive state is then described by the equations

$$\delta F/\delta\Delta=0, \qquad (38)$$

$$\eta_\chi\partial\chi/\partial t+\delta F/\delta\chi=0, \qquad (39)$$

$$\eta_A\partial A/\partial t+\delta F/\delta A+j=0. \qquad (40)$$

For a solution corresponding to a chain of phase-slip centers we have $Q=Q(x,L)$ and $j_s=j_s(x,L)$, where the period of the structure, L, is an adjustable parameter (instead of L we could take this parameter to be $j_{\rm sm}$, the value of Δ halfway between two phase-slip centers, etc.). The functional dependences $Q(x,L)$ and $j_s(x,L)$ can be found through an approximate analytic solution of the averaged equations.[1,2] The only point of importance to the discussion below, however, is that the derivatives $\partial Q/\partial L$ and $\partial j_s/\partial L$ satisfy the inequalities

$$\partial Q/\partial L<0, \quad \partial j_s/\partial L>0 \qquad (41)$$

for all values of x, as can easily be seen by analyzing this solution. We turn now to the calculation of the derivative of the functional V with respect to L under the condition that equations of motion (38)–(40) hold, i.e., along the path of the system. We have

$$\Big(\frac{dV}{dL}\Big)_{tr}=\int\Big[\Big(\frac{\delta F}{\delta Q}+j\Big)\frac{\partial Q}{\partial L}+\frac{\delta F}{\delta\Delta}\frac{\partial\Delta}{\partial L}\Big]_{tr}dx$$
$$=\int\Big[(j-j_s)\frac{\partial Q}{\partial L}\Big]_{tr}dx, \qquad (42)$$

since we have $\partial F/\delta\Delta=0$ by virtue of (38), and $-\delta F/\delta Q=-\Delta^2Q$ is the superconducting current density j_s.

From this point on, there are two possible cases. If $j<j_c$, Eqs. (38)–(40) have the trivial solution with $j=j_s=\Delta^2(1-\Delta^2)^{1/2}$. This solution is an extremum of the functional V, which in this case is the ordinary free energy of a current-carrying superconductor. For $T<T_c$, this free energy reaches its minimum value, as it should, as the system goes into a homogeneous superconducting state.

For $j>j_c$, (in the resistive state), we have a positive difference $j-j_s>0$, so it follows from (41) that the sign of $(\partial V/\partial L)_{\rm tr}$ is the same as the sign of $(\partial Q/\partial L)_{\rm tr}$, i.e., negative.

We have thus found that for a periodic chain of phase-slip centers the quantity $V\{q_i\}$ falls off monotonically with increasing L. Since, on the other hand, at a fixed value of j the quantity L cannot exceed the value $L_{\max}(j)$, determined from the condition for a disruption of the local stability of the structure, according to (25), it follows $L_{\max}(j)$ is the most probable period of the chain. Strictly speaking, these arguments hold only for superconducting channels whose length is great enough that the boundary conditions at the ends of the sample have no significant effect on the structure of the resistive state.

In other words, when there are fluctuations in this system the most probable event is the establishment of a dissipative structure at the boundary of the stability region for structures of this sort (marginal stability). In this connection we would like to call attention to Langer's derivation[10] of a corresponding result for spatially periodic structures formed during the crystallization of a melt of eutectic composition.

We can now show that a minimum of the functional V corresponds to a minimum of the dissipation function W. By definition, the density of the dissipation function is equal to the product jE. Using (30) and (36), we then find

$$W=\int jE\,dx=-\int j\frac{\partial A}{\partial t}=\int\frac{j}{\eta_A}\Big(\frac{\delta F}{\delta A}+j\Big)dx=\int j(j-j_s)\,dx \qquad (43)$$

(we recall that we have $\eta_A=1$). We find $W=0$ at $j<j_c$ (in the homogeneous superconducting state) and $W>0$ at $j>j_c$ (in the resistive state), as we should.

Differentiating (43) with respect to L along the path, we find

$$\Big(\frac{dW}{dL}\Big)_{tr}=-\int j\Big(\frac{\partial j_s}{\partial L}\Big)_{tr}dx<0$$

[see (41)]. Consequently, the dissipation function, like the functional $V\{q_i\}$, falls off monotonically with increasing period of the chain of phase-slip centers. Accordingly, the most probable value of L corresponds not to a minimum of V but to a minimum of W, justifying the application of the principle of minimal dissociation to this problem.

It is pertinent here to consider the model of time-dependent Ginzburg-Landau equations which we have used in this section of the paper instead of the actual system of equations, (5)–(8), to describe the dynamics of the phase-slip centers. It can be seen that Eqs. (5)–(8) can also be put in form (30). If we do, we find that the viscosity coefficients η_Δ and η_χ are complicated functions of Λ. To the best of our knowledge, however, the question of whether a microscopic foundation can be laid for the Langevin equations when the viscosity coefficients depend on the coordinates q_i in this way has yet to be resolved. For this reason, we restrict the discussion to the simplest model. The final results of this section—expressions (34), (31) and (42), (43)—do not contain the viscosity coefficients η_Δ and η_χ. This fact suggests that these results are of more general applicability than the simple time-dependent Ginzburg-Landau equations which we have used.

To conclude this section we would like to point out the following circumstance: A dissipative structure in a so-called two-dimensional mixed state, which arises at the inner surface of a hollow superconducting cylinder when the superconductivity is destroyed by a current, was studied in Ref. 22. It was pointed out there that the dissipative structure in this case has an adjustable parameter and that the free energy and the dissipation reach minimum values as a function of this parameter simultaneously, in a state which lies at the boundary of the stability region of the dissipative structure. The results of the present study furnish an explanation for this circumstance.

CONCLUSION

1. The formation of phase-slip centers from initial perturbations of the order parameter requires a fairly pronounced irregularity in the initial profile $\Delta(x)$. We would therefore expect that in an actual experiment various disruptions of the uniformity of the sample (impurities, defects, etc.) should play an exceedingly important role in the formation of a resistive state.

2. The sequence of bifurcation values $\{j^{(n)}\}$ which corresponds to the appearance of new phase-slip centers in the superconducting channel as the current density is increased is not the same as the sequence $(j^{(n)}\}$ which corresponds to the destruction of the phase-slip centers as j is reduced: Hysteresis effects are observed in the system.

3. As j is increased, the appearance of new phase-slip centers is associated with the development of a local instability of the Δ profile because the superconducting current exceeds the Ginzburg-Landau critical current. This instability develops in a region in which the value of j_s is at a maximum, i.e., halfway between two existing phase-slip centers. The time scale over which the new profile of phase-slip centers forms is found to be on the order of a few times τ_Δ [see (2) and (14)]. The bifurcation values $j^{(n)}$ are given by (25).

4. The process by which the phase-slip centers are destroyed as j is reduced is a slow drift of a phase-slip center toward one of its nearest neighbors, accompanied by global redistribution of the order-parameter profile. This process terminates in the merging of the phase-slip centers and the formation in their place of a single new phase-slip center. The time scale for the onset of this diffusion-drift instability is $\tau_D \sim (L/\xi)^2\tau_\Delta$, where L is the initial distance between two neighboring centers. By virtue of the condition $L \gg \xi$, this time is significantly longer than the rise time of the local instability which we discussed in the preceding section of this paper.

5. In the steady state, the various phase-slip centers in a chain can have slightly different Josephson oscillation frequencies. As a result, beats at the difference frequency appear; i.e., there is a low-frequency modulation of the voltage drop across the sample.

6. Within the framework of the time-dependent Ginzburg-Landau equations, one can introduce a functional which is an analog of the free energy for a very dissipative resistive state of a superconductor. Incorporating fluctuations in this case has the consequence that at $j > j_c$ the most probable state of the system is a chain of phase-slip centers whose spatial period is the largest of all the values possible at the given value of j. This most probable value of the period corresponds to the boundary for local stability of the chain of centers. This functional reaches its absolute minimum. At the same time, the dissipation function of the system reaches its absolute minimum, justifying our use of the minimum-dissipation principle as the criterion for choosing the most probable state of the superconductor.

We note in conclusion that in all versions of the numerical calculations we only observed phase-slip centers of such a nature that the phase jump at the time at which Δ vanished was equal to 2π. We did not observe phase-slip centers with phase jumps which were higher multiples of this value.

We wish to thank B. I. Ivlev for a discussion of this work and L. I. Usanova for assistance in the numerical calculations.

[1)]The criterion which was used in Refs. 1 and 2 for choosing L, and which is based on the principle of a minimum dissipation (a minimum entropy production), can be regarded only as heuristic, since the application of that principle to such a highly nonequilibrium situation requires specific justification (Sec. 3 of the present paper).

[2)]We note, however, that the very fact that there is a stage of latent changes plays an important role in reaching an understanding of the physics of the phenomenon. The presence of this stage is evidence that the initial state of the system is unstable against small perturbations. In other words, the unstable state is disrupted in a process which does not involve a barrier, through the development of a soft mode.

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