Fractional topological charge, torons and breaking of discrete chiral symmetry in the supersymmetric $O(3)$ σ model

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A novel class of self-dual solutions in the $O(3)$ σ model with half-integer topological charge $Q = 1/2$ is considered. The contribution of the corresponding fluctuations to $\langle\bar{\psi}\psi\rangle$ is calculated. The result turns out to be finite, indicating spontaneous chiral symmetry breaking. Various possible descriptions of such fluctuations are considered, such as analytic continuation to complex space and the alternative description on orbifolds.

1. INTRODUCTION

At this time the best-known example of a nonperturbative fluctuation is the instanton.[1,2] The integral nature of the topological charge Q is in that case related to the compactification of the space to a sphere, i.e., with the identification of all infinitely distant points. The choice of other boundary conditions could result in fractional topological charges. In particular, in gluodynamics with the $SU(N)$ gauge group the introduction of so-called twisted boundary conditions[3] permits solutions of the classical equations—torons[4]—with $Q = k/N$, $k = 0,1,...$ and with action $S = (8\pi^2/g^2)Q$.

The possible physical manifestations of such fluctuations will be discussed below, for now we note[3,4] that the admissibility of fractional topological charge in $SU(N)$ gluodynamics is related to the existence of elements of the center $Z_N = \exp(2\pi ik/N)$, which belong to the group and leave invariant the fields of the adjoint representation, i.e., gluons: $A'_\mu = Z_N^{-1}A_\mu Z_N = A_\mu$. Thus, in effect, the group is $SU(N)/Z_N$, and the nontriviality of the map $\pi_1[SU(N)/Z_N] = Z_N$ implies the existence of new (non-instanton) solutions of the classical equations.

Introduction of the fields in the fundamental representation (quarks) naively violates the $SU(N)/Z_N$ symmetry. However, in Ref. 5 the hypothesis was presented that for a class of theories fluctuations with fractional topological charge may be significant also in this case. It so happens that the $O(3)$ σ model belongs to this class of theories, and presents therefore a perfect theoretical laboratory to help understand the role of fractional Q in the analysis of more complex gauge models.

This is precisely the purpose of the current work—to find a means for the description of solutions with fractional Q in the $O(3)$ σ model and to calculate the contribution of the corresponding fluctuations to the functional integral.

What physical effects arise due to fluctuations with fractional Q? These effects appear most glaringly in supersymmetric variants of the theory. In particular, in supersymmetric Yang-Mills theory (SYM) with the $SU(2)$ gauge group, torons[4] ensure spontaneous breaking of discrete chiral symmetry. Indeed, the model possesses naive $U(1)$ chiral symmetry with respect to the transformations $\lambda^a \to \exp(i\alpha)\lambda^a$ (λ^a is the gluino field), which is broken by the anomaly $\partial_\mu a_\mu \sim G_{\mu\nu}\tilde{G}_{\mu\nu}$. However, under this transformation the discrete symmetry Z_4 is conserved. The torons generate the condensate $\langle\lambda^2\rangle$ and break this symmetry down to Z_2 ($\lambda \to -\lambda$).[6] Let us note that instantons give zero contribution to $\langle\lambda^2\rangle$ and can ensure nonzero values only for the correlator $\langle\lambda^2(x),\lambda^2(0)\rangle$.[7]

As will be shown below, analogous behavior occurs in the supersymmetric $O(3)$ σ model. In that case, as in SYM, there is present naive chiral symmetry $\psi \to \exp(i\alpha\gamma_5)\psi$, which is broken by the anomaly

$$\partial_\mu a_\mu = 4\frac{1}{4\pi}\varepsilon_{\mu\nu}F_{\mu\nu}, \quad Q = \frac{1}{4\pi}\int \varepsilon_{\mu\nu}F_{\mu\nu}\, d^2x. \tag{1}$$

Here the discrete symmetry Z_4 is conserved. The toron-like[1)] solutions described in Sec. 4 ensure nonzero values for $\langle\bar{\psi}\psi\rangle$, thus spontaneously breaking the discrete symmetry down to Z_2: $\psi \to -\psi$. This agrees with the value of the Witten index,[8] which equals two.

We note that in this case, too, the instanton can only ensure nonzero value for the correlator $\langle\bar{\psi}\psi(x),\bar{\psi}\psi(0)\rangle$.[9,10]

The paper is organized as follows. In Sec. 2 the $O(3)$ σ model is formulated in terms of various fields: the unit vector field n^a, $a = 1,2,3$, $n^a n^a = 1$; the complex field φ, the unit complex spinor u_α, $\alpha = 1,2$, $u^+u = 1$. The various formulations help to understand different aspects of the fluctuations with fractional Q. Section 3 carries a double load. On the one hand, we describe in it the well-known instanton calculation within the context of interest to us. On the other hand, we formulate (needed for later analysis) the criteria for selection of zero modes. Moreover, the connection between different descriptions of zero modes will turn out to be useful in the corresponding analysis of the toron calculation.

2. THE $O(3)$ σ MODEL

Before describing the toron solution we discuss the duality equations and the Lagrangian for the ordinary (not supersymmetric) $O(3)$ σ model. The modification due to introduction of fermions will be considered in later Sections. The action, the topological charge and the equations of duality have the following form in terms of the fields n^a [1]:

$$S = \frac{1}{4f}\int d^2x(\partial_\mu n^a)^2, \quad n^a n^a = 1, \quad a = 1, 2, 3, \quad \mu = 1, 2,$$
$$Q = \frac{1}{8\pi}\int d^2x\, \varepsilon^{abc}\varepsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c, \tag{2}$$
$$\partial_\mu n^a = -\varepsilon^{abc}n^b\varepsilon_{\mu\nu}\partial_\nu n^c.$$

Here f is the bare coupling constant.

As usual, for the quasiclassical calculation it is necessary to expand the field n^a in the neighborhood of $n^a_{\rm cl}$, keeping only the quadratic terms. Then the problem of diagona-

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lizing the resultant bilinear form reduces to the following equation for the eigenvalues[11,12]:

$$\mathfrak{M}^2 q_l=\lambda_l q_l, \quad q_l^a n_{\rm cl}^a = 0,$$
$$\mathfrak{M}^2 = -\frac{2}{(\partial_\mu n_{\rm cl}^a)^2}\,\partial_\nu^2-2, \tag{3}$$

where λ_l is the lth eigenvalue, and q_l the corresponding eigenfunction, orthogonal to the classical solution and normalized by the condition

$$\int d^2x\, q_l^b q_l^b (\partial_\mu n_{\rm cl}^a)^2= 1. \tag{4}$$

We note that the supplementary condition $q_l^a n_{\rm cl}^a = 0$ is due to the constraint $n^2 = (n_{\rm cl}^a + q^a)^2 = 1$.

As usual, the transition amplitude is normalized relative to vacuum, for which relations (3) and (4) are valid, the only difference being that the operator $\mathfrak{M}^2_{\rm vac}$ in (3) does not contain the constant term (-2).

To avoid the complications due to the constraint $n^2 = 1$, one often introduces (see, e.g., the review, Ref. 9) in place of the three fields n^a, which live on the unit sphere, two independent fields, φ_1 and φ_2, by means of stereographic projection:

$$\varphi_1=n_1/(1+n_3), \quad \varphi_2=n_2/(1+n_3), \tag{5}$$
$$n_1=\frac{2\varphi_1}{1+\varphi_1^2+\varphi_2^2}, \quad n_2=\frac{2\varphi_2}{1+\varphi_1^2+\varphi_2^2}, \quad n_3=\frac{1-\varphi_1^2-\varphi_2^2}{1+\varphi_1^2+\varphi_2^2}.$$

Next one combines φ_1 and φ_2 into one complex field $\varphi = \varphi_1 + i\varphi_2$ and introduces the complex variable $z = x_1 + ix_2$ and then reformulates Eqs. (2) as follows:

$$S=\frac{2}{f}\int\frac{d^2x}{(1+\bar\varphi\varphi)^2}\left[\left|\frac{\partial\varphi}{\partial z}\right|^2+\left|\frac{\partial\varphi}{\partial \bar z}\right|^2\right],$$
$$Q=\frac{1}{\pi}\int\frac{d^2x}{(1+\bar\varphi\varphi)^2}\left[\left|\frac{\partial\varphi}{\partial z}\right|^2-\left|\frac{\partial\varphi}{\partial \bar z}\right|^2\right], \tag{6}$$
$$\partial\varphi_{\rm cl}/\partial\bar z=0, \quad z=x_1+ix_2, \quad \bar z=x_1-ix_2,$$
$$\varphi=\varphi_1+i\varphi_2, \quad \bar\varphi=\varphi_1-i\varphi_2.$$

In the φ-field language, taking into account quadratic deviations is connected with the problem of diagonalizing the following operator:

$$-\frac{\partial}{\partial z}\frac{1}{(1+\varphi_{\rm cl}\bar\varphi_{\rm cl})^2}\frac{\partial}{\partial\bar z}\varphi_l=\lambda_l\frac{1}{(1+\varphi_{\rm cl}\bar\varphi_{\rm cl})^2}\varphi_l,$$
$$\int\bar\varphi_l\varphi_l\frac{d^2x}{(1+\varphi_{\rm cl}\bar\varphi_{\rm cl})^2}=1. \tag{7}$$

Any supplementary requirements on the modes φ_l, due to constraints, are now absent.

As can be seen from Eq. (6), the duality equations have their simplest form in the φ language, while, as will be shown below, the quantum modes are most naturally described in terms of the original fields n^a. Regarding topological ideas, it turns out that for the closest analogy with gauge theories we need another formulation of the $O(3)$ σ model in which local gauge invariance is present.

Namely, we define the action of a CP^1-theory, equivalent to the $O(3)$ σ model, as follows[13,14]:

$$S=\frac{1}{f}\int d^2x|D_\mu u|^2, \quad D_\mu=\partial_\mu+iA_\mu, \quad A_\mu=-i\bar u\partial_\mu u, \tag{8}$$

$$Q=\frac{1}{4\pi}\int d^2x\,\varepsilon_{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \quad \bar u u=1.$$

Here u_a is a two-component complex spinor, A_μ is an auxiliary gauge field. In terms of (8) local gauge invariance has the obvious form:

$$u'=e^{i\alpha}u, \quad A_\mu'=A_\mu+\partial_\mu\alpha. \tag{9}$$

Equivalence with the original formulation is verified with the help of the relations

$$n^a=\bar u\sigma^a u, \quad \varphi=\frac{u_2}{u_1}, \quad u=\binom{u_1}{u_2}, \tag{10}$$

where σ^a are the usual Pauli matrices. We note that the duality equations are written in the spinor language very simply:

$$\frac{\partial}{\partial\bar z}\left(\frac{u_2}{u_1}\right)=0. \tag{11}$$

In the following Section we shall give a brief description of the instanton calculation[9–12] in the context of interest to us. Particular attention will be paid to zero modes and the requirements that these modes must satisfy. The discussion of this question will help us obtain in Sec. 5, practically without any calculations, the expression for the toron density in the supersymmetric $O(3)$ σ model.

3. INSTANTONS IN THE $O(3)$ σ MODEL. ZERO AND NONZERO MODES

As is well known, the instanton solution[1]

$$\varphi_{\rm cl}=\rho/(z-z_0) \tag{12}$$

characterized by two complex parameters ρ and z_0, has $Q = 1$ and action $S = 2\pi/f$. Without loss of generality we take in what follows $\rho = 1$, $z_0 = 0$. Here we have taken the boundary conditions in the form $\varphi(z\to\infty) = 0$, which corresponds to directing n^a $(z\to\infty)$ strictly along the third axis: $n^3(z\to\infty) = 1$.

To analyze quadratic deviations from the classical solution (12) it is convenient to turn in Eq. (3) and make the following change of variables:

$$\theta=\arccos\eta=(1-x_1^2-x_2^2)/(1+x_1^2+x_2^2), \quad \alpha={\rm arctg}\,(x_2/x_1),$$
$$-1\leqslant\eta\leqslant1, \quad 0\leqslant\alpha\leqslant2\pi. \tag{13}$$

The meaning of θ and α is obvious—they are the corresponding coordinates of the sphere obtained with the help of stereographic projection of the x_1x_2 plane. In this notation the instanton solution has a particularly clear form—the unit vector field $n^a(\theta,\alpha)$ has precisely the direction specified by the angles θ,α.

In terms of the variables η,α the operator $\mathfrak{M}^2$ in (3), subject to diagonalization, is expressible through the standard angular momentum operator L^2:

$$\mathfrak{M}^2=L^2-2,$$
$$L^2=-(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\eta\frac{\partial}{\partial\eta}-\frac{1}{1-\eta^2}\frac{\partial^2}{\partial\alpha^2}. \tag{14}$$

It follows that the eigenvalues λ_l of the operator $\mathfrak{M}^2$ are $l(l+1)-2$.[11,12] We note that the weight with respect to which the eigenfunctions q_l are normalized in (4) is precisely the correct measure for spherical harmonics Y_{lm}:

$$d\Omega = (\partial_\mu n^a_{\rm cl})^{2,2} d^2x \sim dx_1 dx_2/(1+x_1^2+x_2^2)^2 \sim d\alpha \sin\theta d\theta. \qquad (15)$$

As was alrady discussed, similar calculations should be carried out for the vacuum field, i.e., for $\varphi = 0$. In that case the operator $\mathfrak{M}^2$ in (14) does not have the term (-2). Therefore the vacuum eigenvalues are $\lambda_l = l(l+1)$, and the degree of degeneracy is $g_l = 2(2l+1)$. The factor $2l+1$ needs no explanation; the additional factor 2 is connected with the two possible orientations of q^a_l, $a = 1,2$, orthogonal to the classical vacuum solution $n^a_{\rm cl} = \delta^{a3}$, directed along the third axis.

Thus the supplementary requirement $q^a_l n^a_{\rm cl} = 0$, due to the constraint, is easily satisfied in empty space with $n^a = \delta^{a3}$. This requirement can be satisfied somewhat less trivially in the case of the instanton. Since the instantons $n^a_{\rm cl}$ describe a vector directed along the radius **r**, the existence of two independent vectors lying in the plane orthogonal to **r** is totally obvious. Consequently the degree of degeneracy is $g_l = 2(2l+1)$ in the case of the instanton as well. As regards the explicit form of the eigenfunctions, they are easily constructed out of the Y_{lm},[11,12] which are eigenfunctions of (14):

$$\mathbf{q}^1_{l,m} = \nabla Y_{lm}, \quad \mathbf{q}^2_{l,m} = \mathbf{L} Y_{lm}. \qquad (16)$$

As expected, there are two types of independent modes orthogonal to the instanton solution $\propto \mathbf{r}$:

$$\mathbf{r}\mathbf{q}^1 \propto \mathbf{r}\nabla Y_{lm} = 0, \quad \mathbf{r}\mathbf{q}^2 \propto \mathbf{r}[\mathbf{r}\nabla] Y_{lm} = 0. \qquad (17)$$

We are now ready to count the zero modes. For the instanton this corresponds to the value $l = 1$, $g = 2(2l+1) = 6$, $\lambda = l(l+1) - 2 = 0$. We note that to $l = 0$ correspond the modes (16), which vanish identically.

For empty space the zero modes correspond to $l = 0$ and have degree of degeneracy $g = 2$. These two modes correspond to the freedom in the choice of boundary conditions. Analogous two modes are present also in the instanton field and they were included in the $g = 6$ calculation. Consequently the number of nontrivial zero modes in the instanton field equals $6 - 2 = 4$. What do these modes look like in the φ-field [Eq. (5)] language? Upon substitution of $\mathbf{n} = \mathbf{n}_{\rm cl} + f^{1/2}\mathbf{q}_i$ and $\varphi = \varphi_{\rm cl} + f^{1/2}\delta\varphi_i$, where $\mathbf{q}_i$ is any of the zero modes (16) with $l = 1$, $i = 1,6$, we arrive with the help of (5) at the following connection between the modes $\mathbf{q}_i$ in terms of the **n**-field (3), (4) and the modes $\delta\varphi$ in terms of the φ-field (7).

$$\delta\varphi = [q^1 + iq^2 - q^3\varphi_{\rm cl}(z)]/(1 + n^3_{\rm cl}). \qquad (18)$$

When we substitute explicit expressions for **q** from (16) for $l = 1$, we readily see the four nontrivial modes are associated with $\delta\varphi \propto 1/z$, $1/z^2$; the two remaining modes, connected with changes in boundary conditions, are associated with the non-normalizable function $\delta\varphi = c$. How can these results be understood directly from Eq. (7)? Looking at (7), one sees that to the solution $\lambda = 0$ corresponds an arbitrary analytic function. By means of what criteria do we choose only $\delta\varphi \propto 1/z, 1/z^2$?

The answer is that we require that the modes be single-valued and the topological charge fixed.[15] This is satisfied only for the functions $\delta\varphi \propto 1/z, 1/z^2$. Anticipating events, we also formulate requirements for fermionic zero modes[15]:

$$|\psi| \leqslant c/|z|, \quad |z| \to \infty,$$
$$|\psi| \leqslant c|\varphi_{\rm cl}|^2 \sim c/z^2, \quad z \to 0. \qquad (19)$$

In the $O(3)$ σ model this requirement is satisfied by precisely two complex modes[9,10]:

$$\psi = 1/z, \quad \psi = 1/z^2. \qquad (20)$$

We also note that the normalization integral (7) diverges logarithmically for the mode $\delta\varphi \propto 1/z$ for large z. However, as noted in Ref. 15, this fact has no effect on the physical content of the theory. We shall run into analogous behavior also in the case of torons.

We pause briefly to mention the contribution of the nonzero modes. To calculate this accurately requires knowledge of the eigenvalues $l(l+1) - 2$, the degrees of their degeneracy $2(2l+1)$, taking into account the regulator fields, etc.[11,12] However, up to logarithmic accuracy, the total contribution of the nonzero modes can be easily calculated with the help of the usual Feynman diagrams, as was done for gauge theories in Ref. 16. In particular, for the $O(3)$ σ model in terms of the φ-field (6) the effective addition to the action is determined by Fig. 1 and equals

$$S = \frac{2\pi}{f} + \Delta S,$$

$$\Delta S = \frac{2}{(2\pi)^2}\int_1^{M_0^2}\frac{d^2k}{k^2}\frac{|\partial_\mu\varphi|^2}{(1+\bar\varphi\varphi)^2}d^2x = \frac{\ln M_0^2}{2\pi}S_{\rm cl} f. \qquad (21)$$

Here the factor 2 in front of the integral is connected with the four-point vertex in the Lagrangian $2\partial_\mu\varphi\partial_\mu\bar\varphi(\bar\varphi\varphi)$; the upper cut-off is determined by the regulator M_0^2, the lower by characteristic field dimension ~ 1. Further, the operator $|\partial_\mu\varphi|^2$ is complementary to the appropriate $O(3)$-invariant expression $|\partial_\mu\varphi|^2/(1+\bar\varphi\varphi)^2$ to within higher-order corrections in the coupling constant f. Substituting in (21) $fS_{\rm cl} = 2\pi$ we arrive at the well-known expression[11,12] for the contribution of the nonzero modes:

$$\Delta S_{\rm nonzero} = \ln M_0^2 \qquad (22)$$

Collecting all factors together we obtain the following expression for the instanton density in the $O(3)$ σ model[11,12]:

$$Z = \exp(-2\pi/f) M_0^4 d^2z_0 d^2\rho \exp(-\ln M_0^2)/\rho^2. \qquad (23)$$

Here the factor $\exp(-2\pi/f)$ is connected with the classical action; $d^2z_0 d^2\rho$ corresponds to integration over the four collective variables (12); M_0^4 is the regulator contribution, corresponding to the four zero modes mentioned above; $1/\rho^2$ is reconstructed by dimensional arguments.

For the supersymmetric variant of the $O(3)$ σ model we have additionally the two complex fermion zero modes (20), so that the corresponding instanton density equals[9,10]

$$Z(SUSY) \propto \exp(-2\pi/f) M_0^4 d^2z_0 d^2\rho (d^2\varepsilon_1/M_0)(d^2\varepsilon_2/M_0). \qquad (24)$$

In obtaining (24) we took into account the fact that the nonzero contributions cancel between bosons and fermions.

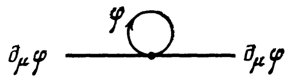

FIG. 1.

Further, each complex fermion zero mode is accompanied by the corresponding collective integral $d^2\varepsilon$ and regulator contribution M_0^{-1}. As was to be expected, there appears in Eq. (24) the renormalization-invariant combination:

$$m^2=M_0^2\exp[-2\pi/f(M_0)]. \qquad (25)$$

We have on purpose analyzed in detail the instanton zero modes and the requirements applicable to them. In the following Sections the corresponding criteria will help us choose the "correct" zero modes in the case of the toron. Moreover, the various formulations of the $O(3)$ σ model described above, will help us to understand substantially different aspects of the toron solution.

4. TORONS IN THE $O(3)$ σ MODEL

We begin with the formulation of the $O(3)$ σ model in terms of the spinor field (8). It is not hard to see in that case that the action is invariant not only with respect to global $SU(2)$ transformations, but also with respect to local $U(1)$ transformations (9). However, as remarked in Ref. 5, the group of transformations is not simply $SU(2)\times U(1)$, but $G=SU(2)\times U(1)/Z_2$, so that $\pi_1(G)\sim Z_2$. This last circumstance is connected with the fact that a simultaneous transformation from the $SU(2)$ group of the form $\exp(i\pi\sigma_3)$ and rotation $\exp(i\pi)$ by angle π from the $U(1)$ group leaves the form of the fields unchanged. Consequently the corresponding transformations should be identified with unity. This means, in turn,[5] that the theory admits $Q=1/2$ and consequently (as will be seen below) multivalued functions appear in the description of classical solutions. A geometric interpretation of this fact is given in the Appendix.

Before beginning a consistent description of the toron solution we remark on some connections with other work. The existence of fluctuations with fractional Q in $2d$-theories was first noted in Ref. 17 (see also Ref. 18) in solving $U(1)$ gauge theories with fermions in the fundamental representation of the $SU(N)$ group. Since, just as in the case described above, for the model of Ref. 17 $G=SU(N)\times U(1)/Z_N$ and $\pi_1(G)\sim Z_N$,[18] this provides a formal argument in favor of existence of $Q\sim 1/N$.

A second analogy is purely technical[2)] and connected with the recently discussed twisted states in string models (see, e.g., Refs. 19 and 20). In that case, too, multivalued functions appear in the theory. Single-valuedness is achieved by the introduction of covering spaces, analogously to the way in which the faction $z^{1/2}$ is single-valued on two Riemann sheets.

As is easily verified, the toron solution with $Q=1/2$, to whose description we now pass, is a double-valued function. Indeed, it is easy to see from (8), that the topological charge is determined by the phase acquired by the spinor upon completing a circle of large radius:

$$Q=\frac{1}{4\pi}\int d^2x\,\varepsilon_{\mu\nu}F_{\mu\nu}=\frac{1}{2\pi}\oint_{|x|\to\infty}A_\mu\,dx_\mu=\frac{1}{2\pi}\oint d\varphi,$$
$$u(x\to\infty)=u_0e^{i\varphi},\quad u_0=\text{const},\quad A_\mu(x\to\infty)=\partial_\mu\varphi. \qquad (26)$$

To the standard instanton with $Q=1$ corresponds the single-valued function

$$u=(|a-z|^2+|b-z|^2)^{-1/2}\begin{pmatrix}a-z\\ b-z\end{pmatrix}.$$

Indeed, since u_2/u_1 depends only on z, then, according to (11), the duality equation is satisfied. Further, after traversing a large contour the spinor acquires the phase 2π, which in accordance with (26) corresponds to $Q=1$.

As was discussed above and in more detail in Ref. 5, we admit a larger class of solutions. Namely, upon completion of the contour we allow the appearance of an overall factor (-1). Taking into account that a factor (-1) arises due to analytic functions of the type $z^{1/2}$, we arrive at the following form of toron solution:

$$u=\lim_{a\to b}(|z-a|+|z-b|)^{-1/2}\begin{pmatrix}(z-a)^{1/2}\\ (r-b)^{1/2}\end{pmatrix} \qquad (27)$$

This solution is defined on two Riemann sheets; real physical space corresponds to but one of them. Further, it is easily seen, that the duality equation $\partial(u_2/u_1)/\partial\bar z=0$ is automatically satisified; moreover, upon completion of the large circle in the physical space the spinor acquires a phase π, which according to (26) corresponds to $Q=1/2$.

We note further that the solution (27) is defined in the sense of the limit $a\to b$. This ensures that a factor (-1) common to the entire spinor appears when either of the points a,b is enclosed. In terms of the field $\varphi=u_2/u_1$, Eq. (10), the solution (27) corresponds to the function

$$\varphi=\lim_{a\to b}[(z-b)/(z-a)]^{1/2}$$

with a cut, tending to zero as $a\to b$, i.e, in terms of the field φ the limit $a\to b$ means reestablishment of single-valuedness on one physical sheet. [We note that geometrically the passage $(a\to b)$ is interpreted as regularization ("blowing up") of the fixed points of the orbifold, see Appendix.]

If one sets $a=b$ from the beginning, then $\varphi=1$, corresponding to the empty vacuum solution. At first sight this suggests that such a solution can lead to no physical effects. The analysis carried out above shows, however, that this is not so. We shall convince ourselves that in the supersymmetric $O(3)$ σ model the solution (27) ensures nonzero value of the chiral condensate. Analogous behavior arises in the calculation of the toron contribution to the gluino condensate in supersymmetric gluodynamics. Although to the toron solution[4] corresponds a field intensity $G_{\mu\nu}{}^a\propto 1/L^2$, which tends to zero everywhere with increasing system size, $L\to\infty$, the condensate turns out to be finite.[6]

We return to the analysis of the solution (27). To this end, instead of

$$\varphi=\lim_{a\to b}[(z-b)/(z-a)]^{1/2},$$

corresponding to the boundary conditions $\varphi\to 1$ $(n^1\to 1)$ as $|z|\to\infty$, we consider the solution

$$\varphi=\lim_{\Delta\to 0}[\Delta/(z-a)]^{1/2},$$

satisfying the standard boundary conditions $\varphi(z\to\infty)=0$, $n^3=1$ and differing from the original by an overall rotation. Now the evaluation of the action (6) is quite simple:

$$\varphi=\lim_{\Delta\to 0}\left(\frac{\Delta}{z-a}\right)^{1/2},\quad S_{\rm cl}=\frac{2}{f}\int\frac{d^2x}{(1+\bar\varphi\varphi)^2}\left[\left|\frac{\partial\varphi}{\partial z}\right|^2+\left|\frac{\partial\varphi}{\partial\bar z}\right|^2\right]$$
$$=\frac{\Delta}{2f}\int\frac{d^2x}{|z-a|[\Delta+|z-a|]^2}=\frac{\pi\Delta}{f}\int_0\frac{\rho\,d\rho}{\rho(\Delta+\rho)^2}=\frac{\pi}{f},$$

$$Q=\frac{1}{\pi}\int\frac{d^2x}{(1+\bar{\varphi}\varphi)^2}\left(\left|\frac{\partial\varphi}{\partial z}\right|^2-\left|\frac{\partial\varphi}{\partial\bar{z}}\right|^2\right)=\frac{1}{2},\quad S_{cl}=\frac{2\pi}{f}Q. \tag{28}$$

As expected, the classical action decreased in comparison with the instanton value (21) by a factor 2. The next stage in the calculation of toron density consists, as usual, of an analysis of quantum fluctuations. However, as was clarified in Sec. 3, the resultant contribution of nonzero modes can be written down to logarithmic accuracy immediately, without detailed analysis [see (21)]:

$$S=S_{cl}+\Delta S,\quad \Delta S_{\text{nonzero}}=\frac{\ln M_0^2}{2\pi}fS_{cl}=\frac{1}{2}\ln M_0^2. \tag{29}$$

Thus, in passing from the instanton solution to the toron solution not only does the classical action decrease (as is natural), but also the contribution of nonzero modes is smaller by a factor of two.

In order to reach a deeper understanding of this important[3)] fact, a detailed analysis is needed of the operator (3), responsible for quadratic fluctuations. We consider this problem next.

5. EVALUATION OF TORON MEASURE

We pass now to the analysis of quantum fluctuations, surrounding the classical solution (28). To this end it is necessary to solve the eigenvalue problem (3). Without loss of generality we set, as in the instanton case, $a=0$, $\Delta=1$. In the final relations the corresponding dependence can be easily restored by dimensional considerations.

For further analysis the following change of variables is critical:

$$\tilde{\eta}=(1-|z|)/(1+|z|),\quad \alpha=\operatorname{arctg}(x_2/x_1),$$
$$-1\leqslant\tilde{\eta}\leqslant1,\quad 0\leqslant\alpha\leqslant2\pi. \tag{30}$$

In terms of the variables $\tilde{\eta},\alpha$ the operator (3) of interest to us becomes the much-studied equation for Legendre polynomials:

$$\mathfrak{M}^2=L^2-2,$$
$$L^2=-(1-\tilde{\eta}^2)\frac{\partial^2}{\partial\tilde{\eta}^2}+2\tilde{\eta}\frac{\partial}{\partial\tilde{\eta}}-\frac{4}{1-\tilde{\eta}^2}\frac{\partial^2}{\partial\alpha^2}. \tag{31}$$

The difference as compared to the instanton case is two-fold: in the first place the expression for $\tilde{\eta}$ in terms of the physical coordinates x_1,x_2 in (30) differs from (13). In the second place, there appears in (31) an additional factor 4 in front of the term $\partial^2/\partial\alpha^2$. This last remark turns out to be of principal importance in the explanation of the additional factor of 1/2, noted at the end of the previous Section. Indeed, the eigenvalues of the operator $\mathfrak{M}^2$ (31), as in the instanton case, equal $\lambda_l=l(l+1)-2$. However the degree of degeneracy differs from the instanton case. Thus, the single-valuedness of the eigenfunctions fixes the polar angle dependence as $\sim\exp(im\alpha)$.[4)] After that the eigenvalue problem reduces to an analysis of the operator L^2, which contains $2m$ (where $-l\leqslant2m\leqslant l$) in place of the usual integer. Roughly speaking, for large l this reduces the degree of degeneracy by a factor 2 as compared to the instanton case. But $\ln M_0^2$ arises precisely due to summing over large l. Therefore the decrease in the degree of degeneracy by two for $l\gg1$ gives rise to the appearance of the factor 1/2 in front of $\ln M_0^2$. Indeed, integration over quadratic fluctuations gives rise to the following general relation (vacuum and regular contributions are understood):

$$\exp(-\Delta S_{\text{nonzero}})\propto\prod_l\lambda_l^{-g_l/2}, \tag{32a}$$

$$\Delta S_{\text{nonzero}}\propto\frac{1}{2}\sum_l g_l\ln\lambda_l. \tag{32b}$$

Since the values of λ_l for the toron and instanton coincide, it is seen from (32b) that when g_l is reduced by a factor of two the logarithmic part of ΔS is reduced by the same factor relative to the instanton calculation. This confirms the result (29), previously obtained by simpler means.

Our problem, however, was not the derivation of Eq. (29) by yet another method. Rather we wanted to demonstrate the important criterion for selection of modes, which must be taken into account in relations of the type (32). Namely, only single-valued modes φ_l must be included in expansions of the form

$$\varphi=\varphi_{cl}+f^{1/2}\sum_n c_n\varphi_n.$$

It is precisely this requirement that ensures the correct result (29). In the instanton case the problem does not arise—in the single-valued classical field all modes are automatically single-valued. For the toron this is not so. If we were to admit in (31) functions of the type $\exp(im\alpha/2)$, then the degree of degeneracy relative to the instanton case would be unchanged and we would obtain the wrong answer.

How should one interpret the fact that $\varphi_{cl}=\lim_{\Delta\to0}(\Delta/z)^{1/2}$ is not single-valued on the physical sheet, while we require the fluctuations φ_n to be single-valued? The point is that φ_{cl} is defined in the sense of a limit and we arbitrarily make it single-valued as $\Delta\to0$. The quantum fluctuations cannot be influenced in this manner; they are created and annihilated with definite probability regardless of any outside knowledge. The geometric treatment of this requirement is contained in the Appendix.

We pass to the analysis of the zero modes. They correspond to the value $l=1$, and their number equals 4 (see footnote 4) rather than 6 as for the instanton. Since there are two zero modes in empty space, the number of nontrivial toron modes equals $4-2=2$ (for the instanton $6-2=4$). Making use of relation (18) we write the toron zero modes in terms of the φ-field:

$$\delta\varphi^0\propto1/z. \tag{33}$$

Although this mode, as in the case of the instanton, is logarithmically divergent for $z\to\infty$, this fact has no bearing on the physical content of the theory.[15] Infrared regularization is usually achieved by introduction of the factor[10] $\Omega=(1+x^2/R^2)^{-1}$, $R\to\infty$, or by cutting the integral off from above[9]:

$$\int\frac{|\delta\varphi|^2}{(1+\varphi_{cl}\bar{\varphi}_{cl})^2}\Omega^2d^2x=1,\quad \int_0^R\frac{|\delta\varphi|^2}{(1+\varphi_{cl}\bar{\varphi}_{cl})^2}d^2x=1. \tag{34}$$

In the following relations infrared regularization is understood, although not explicitly indicated.

Were we to attempt to find zero modes directly in terms of the φ-field, as in the instanton case, we would obtain the result that any analytic function satisfies Eq. (7). However just one function, namely (33), satisfies the additional requirements discussed in Sec. 3.

We pass to the discussion of the supersymmetric variant of the $O(3)$ σ model. As is known, supersymmetric models differ conveniently from ordinary ones in that only zero modes need be considered. In the bosonic sector we found two modes, written in the form of one complex mode (33). In the fermionic sector any spinor of the form $\psi_2^0=0$, $\psi_1^0=f(z)$, automatically satisfies the equation for zero modes.[9] However only one complex fermion zero mode satisfies the requirement (19), namely:

$$|\psi|\leqslant|\varphi_{\rm cl}|^2=1/z, \quad z\to 0,$$
$$\delta\psi_2{}^0=0, \quad \delta\psi_1{}^0=\varepsilon f, \quad f=c/z, \tag{35}$$
$$\int\frac{|f|^2}{(1+\varphi_{\rm cl}\overline{\varphi}_{\rm cl})^2}d^2x=1.$$

Here ε is some Grassmann number, 1 and 2 are spinor indices and c is a constant.

With the above considerations taken into account the toron measure acquires the following form:

$$Z\sim M_0{}^2d^2a\frac{d^2\varepsilon}{M_0}\exp\left[-\frac{\pi}{f(M_0)}\right]=md^2a\,d^2\varepsilon,$$
$$m=M_0\exp[-\pi/f(M_0)]. \tag{36}$$

Here the factor $M_0^2d^2a$ is due to the single complex bosonic zero mode; d^2a is the corresponding integral over the collective variable; the factor $d^2\varepsilon/M_0$ is connected with the single complex fermion zero mode (35); lastly, $\exp(-\pi/f)$ is the contribution of the classical toron action.

As in the case of the instanton (25), the expression (36) for the toron measure has precisely the renormalization-invariant form. It is easy to trace this phenomenon: While the action decreased by a factor two, the number of zero modes decreased by the same factor, which exactly restored the correct renormalization-invariant relation.

Now all is ready for the calculation of the chiral condensate in the $O(3)$ σ model. Following Refs. 9 and 10 we introduce to this end the appropriate operator

$$O=\overline{\psi}\psi/(1+\varphi_{\rm cl}\overline{\varphi}_{\rm cl})^2, \tag{37}$$

which is invariant with respect to $O(3)$ rotations and noninvariant with respect to chiral rotations. Substituting in place of ψ their zero modes (35), and recalling that integration over the collective fermionic variables exactly satisfies $\int\varepsilon\overline{\varepsilon}d\varepsilon d\overline{\varepsilon}=1$, we verify that

$$\langle O\rangle\propto m\int d^2a\frac{|f|^2}{(1+\varphi_{\rm cl}\overline{\varphi}_{\rm cl})^2}=m. \tag{38}$$

In the last step we used the value of the normalization integral (35). As is well known,[8-10] the nonvanishing of the condensate (38) indicates spontaneous breaking of discrete chiral symmetry: $\psi\to\pm\gamma_5\psi$, which does not take place in any order of perturbation theory. We note that the instanton can only ensure a nonvanishing value for the correlator $\langle O(x),O(0)\rangle$,[9,10] in accordance with the fact that the solution with $Q=1$ changes the chiral charge ΔQ_5 by four units [the four zero modes (20) express this fact]. The toron solution with $Q=1/2$ changes the chiral charge by two units and has two zero modes (35). Therefore the corresponding vacuum transition is necessarily accompanied by the production of a $\overline{\psi}\psi$ pair, as the explicit calculation of (38) also demonstrated.

Conceptually, this calculation is analogous to the evaluation[6] of the gluino condensate $\langle\lambda^2\rangle$ in supersymmetric gluodynamics. In both cases the presence of the condensate reflects the violation of just the discrete symmetry. The difference is that in Ref. 6 only fields in the adjoint representation occur; in our formulation (27) spinor fields transforming according to the fundamental representation of $SU(2)$ are present. Moreover, in the calculation in Ref. 6 use was made of the standard quasiclassical approximation, which is not valid when the cell size L is increased: $L\to\infty$, $g(L)\to\infty$. In our calculation the characteristic scales are: $z\sim\Delta\to 0$, $g^2(z)\to 0$, and the quasiclassical calculation is under control.

Now a few words about the choice of the value $Q=1/2$ as compared to other fractional values. As was already explained, in the formulation (8), which includes local $U(1)$ gauge invariance, the value $Q=1/2$ is already singled out at the classical level. Thus we note that a superposition of transformations from the center of $SU(2)$ and rotation by the angle $\exp(i\pi)$ from $U(1)$ leaves the fields invariant. Consequently the corresponding transformations should be identified with unity, and this results in $Q=1/2$.[5]

In terms of the φ- and n^a-fields this singling out is not apparent, since any solution $(\Delta/z)^Q$ becomes single-valued as $\Delta\to 0$ for any Q. Therefore, in terms of the n^a-fields the special nature of $Q=1/2$ only appears at the quantum level in solving the eigenvalue problem (3). It turns out that the single-valued zero mode exists only for $Q=1/2$. In other cases neither the zero mode, satisfying the single-valuedness criterion, nor the integral over the collective coordinate describing the location of the toron, exists.

In terms of the field φ the special nature of $Q=1/2$ does not appear even when Eqs. (7) are solved for the zero modes. Any analytic function is a solution of (7) with $\lambda=0$. Only the additional requirements of the type of (19) single out the value $Q=1/2$ an thus ensure the existence of a single-valued zero mode.

We also note that only for $Q=1/2$ is the correct renormalization-group dependence restored. The geometrical treatment of the special nature of $Q=1/2$ is described in the Appendix.

6. CONCLUSION

The main point of this work is an analysis of the physical consequences of the existence of fractional charge $Q=1/2$ in the supersymmetric variant of the $O(3)$ σ model. It is shown that the corresponding fluctuations ensure spontaneous breaking of discrete chiral symmetry and give a nonzero contribution to the chiral condensate. From our point of view this is a new independent contribution, which should be taken into account along with the instanton calculations.[9,10] This point of view does not contradict the old idea that the calculation of any quantity requires the summation over all topological classes (although it is not clear with what weight). An alternate point of view is also possible, going back to Ref. 21, according to which the instanton is the superposition of two objects with half-integer topological charge. In Ref. 21 such an object with $Q=1/2$ was the meron,[22] possessing infinite action. In a certain sense our solution is similar to the meron: both have zero measure. There is also a difference: the toron has finite action, the meron infinite. We also note that the solution proposed here with $Q = 1/2$, which is self-dual and minimizes the action $S = 2\pi Q/f = \pi/f$, admits a simple generalization to a CP^{N-1} theory with $Q = k/N$ and action $S = (2\pi/f)(k/N)$.

Further, it turns out, that the corresponding construction can also be generalized to gauge theories. For the supersymmetric variant of the Yang-Mills theory one can also calculate the gluino condensate in complete analogy with the discussed above calculations in the $O(3)$ σ model.

In conclusion the author expresses gratitude to P. B. Vigman, A. I. Vainshtein, A. Yu. Morozov, V. L. Chernyak and M. A. Shifman for useful discussions and critical remarks.

APPENDIX

Geometrical interpretation of fractional charge: orbifolds

We wish to make clear at the outset that this Appendix has a purely auxiliary character and contains no new assertions. Thus the aim of this Appendix is strictly illustrative, namely a geometrical description of the ideas given in the text.

We compactify the complex plane z into the sphere S_2 in accord with transformation (30):

$$\cos\tilde{\theta}=\tilde{\eta}=(1-|z|)/(1+|z|), \quad \alpha=\operatorname{arctg}(x_2/x_1). \quad (A1)$$

We note (even though this is irrelevant from the topological point of view) that in comparison with the standard projective transformation (13) the quantity $|z|$ enters to first order, and not to second, so that the projection lines are not straight lines, see Fig. 2. Thus $\tilde{\theta},\alpha$ have the meaning of coordinates of the sphere S_2. Next we make a cut in the z-plane from 0 to ∞ corresponding to the double-valued nature of our classical solution $(\Delta/z)^{1/2}$, Eq. (28). On the sphere S_2 this cut joins the north and south poles. With the help of the cut we open up the sphere and uniformly squeeze it to a half-sphere. We then glue to the half-sphere a second copy, corresponding to the second Riemann surface. In this manner we construct the sphere $\tilde{S}$ (Fig. 3), which may be understood as the compactification of the two Riemann sheets, or, in other words, as the compactification of the complex plane $\tilde{z}$, where

$$z=\tilde{z}^2, \quad \tilde{\alpha}={}^1/_2\alpha. \quad (A2)$$

We are now ready to give a geometric interpretation of the toron solution ($\varphi_{\rm tor} = z^{-1/2}$). But first let us recall the situation for the instanton, for which $\varphi_{\rm inst} = 1/z$. If this solution is expressed in terms of n^a, Eq. (5), then this is "hedgehog" with the directions $n^a(\theta,\alpha)$ specified by the angles θ, α obtained by the compactification (13) of coordinate space.

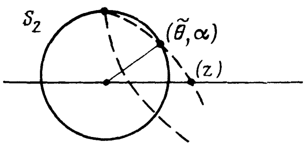

FIG. 2.

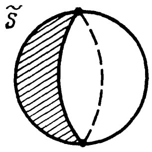

FIG. 3.

The toron solution $\varphi_{\rm tor} = z^{-1/2} = \tilde{z}^{-1}$ looks in terms of $n^a(\tilde{\theta},\tilde{\alpha})$ precisely like the instanton but in terms of another sphere, $\tilde{S}(\tilde{\theta},\tilde{\alpha})$. Thus the toron is a hedgehog defined on the above constructed sphere $\tilde{S}$.

We have discussed the geometry of the classical solution. We now discuss the geometrical interpretation of the single-valuedness criterion of the quantum fluctuations [see the text following formula (32)]. But first we give several formal definitions.[19,20] Let there be a certain manifold $\tilde{S}$, on which the action of a discrete group G is defined. We consider the factor-space

$$S=\tilde{S}/G. \quad (A3)$$

Suppose that we want to describe the system on the space S. We may proceed in two ways: first, we may directly describe states on S, second, we may describe states on the larger space $\tilde{S}$, and demand that physical states be invariant with respect to the transformations G, i.e.

$$G\left|\begin{matrix}\text{physical}\\ \text{state}\end{matrix}\right\rangle = \left|\begin{matrix}\text{physical}\\ \text{state}\end{matrix}\right\rangle. \quad (A4)$$

Then states, defined on $\tilde{S}$ but satisfying the requirement (A4), are acceptable states on S.

We note that the selection criterion of single-valued modes in (32) means that the eigenfunctions on the two Riemann sheets coincide. Therefore values of the functions at opposite points of the sphere $\tilde{S}$ (Fig. 3) coincide, i.e., under the transformation

$$G\colon \tilde{z}\to-\tilde{z} \quad (A5)$$

single-valued functions remain unchanged and satisfy (A4). In particular, the zero modes $1/z = 1/\tilde{z}^2$ satisfy the requirement (A5) and are therefore acceptable states on S.

We note that the transformation $\tilde{z}\to-\tilde{z}$ has two fixed points—the north and south poles. These points remain in place under $\tilde{z}\to-\tilde{z}$. Such manifolds $\tilde{S}/G$, possessing fixed points, are called orbifolds.[19,20]

At each fixed point there is a conic singularity with angular defect equal to π. That is precisely the angle (26), ensuring $Q = \pi/2\pi = 1/2$. For this value of the angle the two copies of the cone exactly cover the plane, so states symmetric with respect to (A5) are acceptable states on the cone. This is precisely the content of our criterion of mode selection.

In this fashion the manifold that we are dealing with is, in essence, an orbifold. Such manifolds are singular at the fixed points. To regularize them one usually introduces a free parameter (measure). In our case this role is played by the parameter Δ, Eq. (28).

We also note that the nontriviality of the homotopic group π_1 in superstring theories[19] and in our case (see Ref. 5 and the text at the beginning of Sec. 4) is due to the existence of the fixed points. It is precisely when they are enclosed that the Wilson line integral acquires a nontrivial value. In superstring theories this guarantees spontaneous breaking of E_6-symmetry and yields a reasonable number of generations.[19] In the case of the $O(3)$ σ model it gives rise to the existence of the condensate.

We should like to call attention with this Appendix to the close analogy between the description of solutions with fractional topological charge and the twisted states in string theories.

[1)]We keep the term "toron", introduced in Ref. 4, for the self-dual solution in the $O(3)$ σ model as well. By this means we emphasize the fact that the solution minimizes the action $S=(2\pi/f)Q$ and carries topological charge $Q=1/2$, i.e., possesses all of the characteristics ascribed to the toron.[4]

[2)]Which does not prevent one from intensely exploiting the indicated analogy, see Appendix.

[3)]The importance of the indicated fact has to do with the circumstance, that it is precisely the additional factor 1/2 in (29) that automatically ensures the correct renormalization-group dependence (25), see Sec. 5.

[4)]The explanation given here is somewhat simplified. The correct requirement consists of single-valuedness of the physical modes (16), orthogonal to the classical solution. The conceptual aspect of the problem is not affected, however, by this simplification and therefore, omitting technical details, we state the results. For the toron $\lambda_l = l(l+1)-2$; $g_l = 2(l+1)$ for odd l, $g_l = 2l$ for even l; for the vacuum $\lambda_l = l(l+1)$; $g_l = 2(l+1)$ for even l, $g_l = 2l$ for odd l. As was explained in the text for $l \gg 1$ we have $g_l = 2l$, which is two times smaller than in the instanton case, where $g_l = 2(2l+1)$.

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